GAINS AND LOSSES IN NONADDITIVE EXPECTED UTILITY*

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1. INTRODUCTION

In a seminal paper, Schmeidler (1989) proposed a nonadditive expected utility theory, called Choquet expected utility (CEU). For decision under uncertainty CEU provides a greater flexibility in predicting choices than Savage's subjective expected utility (SEU). The key feature of Schmeidler's theory is that the probability of a union of two disjoint events is not required to be the sum of the individual event probabilities. Schmeidler's theory and its subsequent developments (e.g., see Gilboa, 1987, Wakker, 1989, Chapter VI) do not, however, make a distinction between gains and losses with respect to the status quo. These theories typically assume that the consequence of a given decision alternative is described by the final wealth position.

In recent years, a body of empirical literature (see Kahneman and Tversky, 1979) has convincingly demonstrated that people's attitudes towards gains and losses are distinctly different. For example, people are risk averse when consequences represent a gain relative to the status quo, and are risk seeking when these represent a loss relative to the status quo. Recently, Kahneman and Tversky (1992) have introduced cumulative prospect theory (CPT), an extension of their original prospect theory, that generalizes nonadditive expected theories to permit differential attitudes towards gains and losses. Similar forms were proposed in Starmer and Sugden (1989) and Luce and Fishburn (1991).

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In Sarin and Wakker (1992a) a simple approach was used to obtain a transparent axiomatization of CEU. The idea of this approach is to use unambiguous events, such as those generated by a random device, to calibrate decision weights for general events. In this paper we show that the same approach can also be utilized to obtain a simple and transparent axiomatizion of CPT. The main result in this paper is obtained by modifying the cumulative dominance axiom P4 of Sarin and Wakker (1992a).

Section 2 presents two decision problems. The first, a variation of the classical Ellsberg paradox, shows the desirability of extending expected utility; it motivated the development of CEU. The second illustrates the desirability of extending CEU to CPT.

Section 3 derives the CEU model, along the lines developed in Sarin and Wakker (1992a). The latter paper gave a fully developed axiomatic derivation of CEU, where for unambiguous acts expected utility was maximized. Here we simplify the derivation by assuming expected utility for unambiguous acts from the start. That leads to a transparent derivation of CEU.

Section 4 presents the main result of this paper, a new derivation of CPT, and Section 5 contains a discussion.

2. EXAMPLE

Consider a "known" urn that contains 50 yellow and 50 white balls, and an "unknown" urn that contains an unknown proportion of yellow and white balls, totaling 100 balls. In a first scenario, the subjects are offered a choice between the following two options.

Option 1: Bet on known urn, win \$200 if yellow ball is drawn;
win \$0 otherwise.

Option 2: Bet on unknown urn, win \$200 if yellow ball drawn;
win \$0 otherwise.

A majority of subjects chooses option 1 in the above scenario. A typical rationale offered for this choice is that the probability of winning in option 1 is 0.5, while the probability of winning in option 2 is vague or ambiguous. The preference for the known urn is observed even when the subjects are given freedom to specify the color of the ball on which they bet, and even with real money awards. This preference for betting on the known urn is termed ambiguity aversion in the literature and has been observed in many empirical studies in a variety of different settings; e.g., see Ellsberg (1961), Curley and Yates (1985), Einhorn and Hogarth (1985), Kahn and Sarin (1988).

Since the subjects exhibit symmetric preferences within the known urn and the unknown urn by being indifferent between betting on white or yellow colored balls, the observed preference for option 1 cannot be explained by a standard application of subjective expected utility theory. This is because in either urn, under SEU, the subjective probability of drawing a yellow ball equals the probability of drawing a white ball, and is 0.5. If psychological attributes such as regret, suspicion, or disappointment are incorporated to describe the consequence, then SEU could indeed explain the observed choice. We do not pursue this line of reasoning in the present paper.

CEU theory is consistent with the observed preference for option 1. This is because CEU permits the decision weight associated with the superior event (yielding \$200) to be less than 0.5 when the ball is drawn from the unknown urn.[1] Here, by symmetry, the decision weight of drawing the yellow ball (if this is the superior event), and for drawing the white ball (if this is the superior event), are both below 0.5, so they add up to less than 1. This is where the nonadditive theories generalize SEU. Thus subjects reveal in scenario 1:

decision weight of yellow ball (superior event) from unknown urn
$< \ 0.5$

In the second scenario, the subjects are offered a choice between the following two options.

Option 1: Bet on known urn, lose \$200 if white ball is drawn;
lose \$0 otherwise.

Option 2: Bet on unknown urn, lose \$200 if white ball drawn;
lose \$0 otherwise.

A majority of subjects now chooses option 2. So here subjects exhibit a preference for betting on the unknown urn, i.e., they show a preference for ambiguity. Under CEU, this preference implies

decision weight of yellow ball (superior event) from unknown urn
$> \ 0.5.$

[1]In general, CEU does allow decision weights to differ from objective probabilities, such as those generated by the known urn. For simplicity we assume here that they coincide. This assumption does not affect our subsequent reasoning.

Since the decision weight for the same event (yellow ball) as superior event cannot simultaneously be smaller and larger than 0.5, the above example demonstrates that CEU is unable to accommodate the observed modal preference pattern. CPT, however, does permit the above preference pattern by allowing decision weights to differ in the gain and loss scenario.

3. CHOQUET EXPECTED UTILITY

To bring to the fore the transparency of the axiomatization method adopted in this paper, we first give an elementary, somewhat informal, derivation of CEU. Subsequently we summarize this more formally. The approach in this paper follows Sarin and Wakker (1992) who use unambiguous events to calibrate ambiguous events.

The decision problem is to choose an act from the available set of acts when the outcome of each act is uncertain. An act is denoted

$$(A_1,x_1;\cdots;A_n,x_n)$$

which means that the act yields an outcome x_i if an event A_i obtains. Events $A_1,\cdots,A_n$ are mutually exclusive and collectively exhaustive. The decision maker does not know which event A_i will eventually obtain and therefore does not know which outcome x_i will result from the act.

Preferences between acts are denoted by the symbol $\succcurlyeq$ (with $\preccurlyeq$, $\sim$, $\succ$, $\prec$ as usual). We assume throughout that preferences are complete (every pair of acts is comparable) and transitive. We also assume that all acts have a finite number of outcomes (i.e., acts are simple) and that outcomes in acts $(A_1,x_1;\cdots;A_n,x_n)$ are rank-ordered so that

$$x_1 \succcurlyeq \cdots \succcurlyeq x_n.$$ [2]

A crucial assumption in our analysis is that there exist physical devices such as roulette wheels, random number tables, dice, etc., that can be used to generate events with known or generally agreed upon probabilities. Such events are called unambiguous events. An act whose outcomes are generated using unambiguous events alone is called an unambiguous act. We assume that the decision maker maximizes expected utility with respect to these unambiguous acts. The unambiguous acts are usually described through the probability distribution they generate over the outcomes. So we write

$$(P(A_1),x_1;\cdots;P(A_n),x_n) \text{ for } (A_1,x_1;\cdots;A_n,x_n), \text{ or}$$
$$(p_1,x_1;\cdots;p_n,x_n),$$

where p_i is the probability of event A_i.

We now turn our attention to general events relevant to the decision problem, such as: interest rates fall, remain the same, or increase. Likelihood comparisons of events are defined in terms of preferences over gambles on the events. That is, an event A is defined as more likely than an event B if a person will prefer to bet on A rather than on B. For unambiguous events, the likelihood relation is represented by probabilities. We use these probabilities to calibrate the likelihood of ambiguous events. This of course requires that for each ambiguous event A, we can find a matching unambiguous event B of equal likelihood, i.e., such that the person is indifferent between betting on A or on B. Then we can assign to event A the number P(B), the probability of event B. We call this assigned number the *capacity* of A, denoted v(A). In this manner we can use the probabilities of unambiguous events to calibrate capacities for ambiguous events. Note that at this stage it cannot be concluded that the capacity satisfies conditions such as additivity ($v(A \cup B) = v(A)+v(B)$), and indeed, in the models of this paper it will not. We will ensure a weaker condition, i.e., *monotonicity* (if $A \supset B$ then $v(A) \geq v(B)$). This is accomplished by simply requiring that a person prefers betting on a larger set (e.g., {cold, warm, hot}) to betting on a smaller set (e.g., {cold, warm}) that is included in the larger set.

Schmeidler and Gilboa introduced a decision model, Choquet expected utility (CEU), that allows for nonadditivity of capacities. In CEU "cumulative" events, i.e., events describing the receipt of an outcome α or a superior outcome, play a central role. For an act $(A_1,x_1;\cdots;A_n,x_n)$, the cumulative events are events of the form

$$A_1 \cup \cdots \cup A_i,$$

for $1 \leq i \leq n$. In CEU, an act is preferred to another if all its cumulative events are more likely, i.e., if the receipt of an outcome α or a superior outcome is at least as likely under the first act as under the second, for all outcomes α. Notice that if each act comes associated with a probability distribution, then this condition merely requires a preference for first order stochastically dominating acts. One could therefore view it as stochastic dominance for the case of uncertainty.

We are now ready to show that the conditions discussed informally so far lead to the CEU representation. Consider an arbitrary ambiguous act $(A_1,x_1;\cdots;A_n,x_n)$. Take an unambiguous act $(p_1,x_1;\cdots;p_n,x_n)$ that is "matching" in the sense that

$$p_1 + \cdots + p_i = v(A_1 \cup \cdots \cup A_i) \text{ for each } i.$$

[2]Preferences over outcomes are derived from acts that have constant outcomes under all events.

This matching unambiguous act is constructed by recursively eliciting the capacities of cumulative events $A_1, A_1 \cup A_2$, etc. By cumulative dominance, the ambiguous and unambiguous acts are equivalent. We know that the evaluation of the unambiguous act is given by its expected utility, $\sum_{i=1}^{n} p_i U(x_i)$. It is useful to rewrite this as[3]

$$\sum_{i=1}^{n}((p_1+\cdots+p_i)-(p_1+\cdots+p_{i-1}))U(x_i),$$

which is identical to

$$\sum_{i=1}^{n}(v(A_1\cup\cdots\cup A_i)-v(A_1\cup\cdots\cup A_{i-1}))U(x_i).$$

The latter value is defined as the *Choquet expected utility* (*CEU*) of the act $(A_1,x_1;\cdots;A_n,x_n)$. Let us compare this formula with EU. Both forms take a weighted mean of the utility values $U(x_i)$. In CEU, the decision weights, which we denote as π_i below, are the marginal capacity contributions of events A_i to the more favorable events $A_1,\cdots,A_{i-1}$. That is, if we define

$$\pi_i = v(A_1\cup\cdots\cup A_i) - v(A_1\cup\cdots\cup A_{i-1})$$

for all i, then we can write CEU as

$$\sum_{i=1}^{n}\pi_i U(x_i).$$

Let us now summarize, somewhat more precisely, the assumptions and results presented so far. The conditions (R1)-(R3), presented first, are not necessary, and mainly concern richness assumptions, plus an expected utility assumption in (R1). The conditions (A1)-(A3), presented next, are all necessary for the CEU representation, and also sufficient, given the other conditions. Throughout, $\succcurlyeq$ denotes a preference relation over a set of "conceivable" acts.

(R1) All simple probability distributions $(p_1,x_1;\cdots;p_n,x_n)$ are conceivable. Preferences over these are evaluated by expected utility

$$\sum_{i=1}^{n}p_i U(x_i).$$

[3]Throughout this paper we adopt the standard mathematical convention that, for i=1, $p_1+\ldots+p_{i-1}=0$ and $A_1\cup\ldots\cup A_{i-1}=\varnothing$.

We formally define $A \succcurlyeq B$ if there exist outcomes $\alpha \succ \beta$ such that

$$(A,\alpha;A^c,\beta) \succcurlyeq (B,\alpha;B^c,\beta)^4,$$

and assume:

(R2) For each ambiguous event A, there exists an unambiguous event B such that $A \sim B$.

We want to be able to compare all pairs of events A,B, so we assume:

(R3) There exist outcomes $\sigma \succ \tau$ such that, for each event A, $(A,\sigma;A^c,\tau)$ is conceivable.

Next we turn to the necessary conditions.

(A1) $\succcurlyeq$ is complete and transitive.

Define, in condition (R2) above, v(A)=P(B).[5] The following condition will imply monotonicity of the capacity with respect to set inclusion

(A2) $A \supset B$ implies $A \succcurlyeq B$.

Note that, in the preference $(A,\sigma;A^c,\tau) \succcurlyeq (B,\sigma;B^c,\tau)$, (A2) can be reformulated as the requirement that the decision maker should appreciate replacement of the outcome τ by the preferred outcome σ in A−B.

Cumulative dominance means:

(A3) If the event of receiving outcome α or a superior outcome is at least as likely under act f as under act g, for all outcomes α, then $f \succcurlyeq g$.

Cumulative dominance implies that $(A,\alpha;A^c,\beta) \succcurlyeq (B,\alpha;B^c,\beta)$ for some outcomes $\alpha \succ \beta$ if and only if this holds true for all outcomes $\alpha \succ \beta$. A consequence of this condition is that the likelihood ordering on events is independent of the particular outcomes that have been chosen, and can be derived from the outcomes $\sigma \succ \tau$ in (R3). Cumulative dominance also

[4]The cumulative dominance axiom, defined formally below, will ensure that the ordering $A \succcurlyeq B$ is independent of the particular choice of outcomes $\alpha \succ \beta$; A^c denotes complement.

[5]The cumulative dominance axiom will also ensure that this number is independent of the particular choice of event B.

implies that v(A) is independent of the choice of the unambiguous event B~A because, if C is another unambiguous event such that A~C, then

$$(B,\alpha;B^c,\beta) \sim (A,\alpha;A^c,\beta) \sim (C,\alpha;C^c,\beta)$$

so that, by transitivity of ~ and EU, P(B) = P(C) = v(A). Further, cumulative dominance implies monotonicity of the capacity, because, for A⊃B, and unambiguous events C,D with C~A, D~B, the following preferences hold:

$$(C,\sigma;C^c,\tau) \sim (A,\sigma;A^c,\tau) \succcurlyeq (B,\sigma;B^c,\tau) \sim (D,\sigma;D^c,\tau).$$

The weak preference above was implied by (A2). Finally, as can be seen from the result below, the capacity v represents the likelihood ordering ≽ on the events.

Now we present the result developed in this section.

THEOREM 3.1. Suppose (R1)-(R3) hold. Then CEU holds if and only if (A1)-(A3) hold. □

This seems to be the most transparent characterization of CEU presently available in the literature. The CEU of an act $(A_1,x_1;\cdots;A_n,x_n)$ is given by

$$\sum_{i=1}^{n}(v(A_1\cup\cdots\cup A_i)-v(A_1\cup\cdots\cup A_{i-1}))U(x_i).$$

Uniqueness results are standard. From the definition of the capacity it immediately follows that the capacity is uniquely determined. The utility function for the CEU form is the same as for the EU form for the unambiguous acts, so is unique up to scale and location.

4. A DERIVATION OF CUMULATIVE PROSPECT THEORY

In prospect theory or its extended version, cumulative prospect theory, outcomes are not described as final states of well-being, but rather as deviations relative to the "status quo" outcome. The latter is denoted as θ. To highlight this interpretation, Tversky and Kahneman use the term prospect instead of act. We follow this terminology from now on. We call outcomes α≻θ gains, and outcomes α≺θ losses. For convenience of notation we shall "collapse" the equivalence class of the status quo, i.e., we assume that all outcomes equivalent to the status quo are in fact identical to the status quo.

The key idea of cumulative prospect theory is that risk attitudes towards gains usually differ from risk attitudes towards losses, and

therefore the capacity for gains, denoted v^+, may differ from the capacity for losses, denoted v^-. This means, obviously, that two different elicitations should be carried out, one with gains involved, and one with losses. So we now write, for events A,B, $A \succcurlyeq^+ B$ if there exists a gain α≻θ such that $(A,\alpha;A^c,\theta) \succcurlyeq (B,\alpha;B^c,\theta)$.

It turns out to be more convenient and natural to elicit the likelihood ordering for losses, denoted $\succcurlyeq^-$, from bets against events instead of bets on events. Further comments on this will be given in Section 5. Thus we define
$A \succcurlyeq^- B$ if there exists a loss α≺θ such that $(A,\alpha;A^c,\theta) \preccurlyeq (B,\alpha;B^c,\theta)$. If a person avoids a loss contingent on event A in favor of a loss contingent on event B, then it seems natural to say that he considers event A as more likely to occur than event B.

It is now natural to proceed as follows, modifying the approach to CEU in order to accommodate the CPT theory. First, we assume that R1 (all simple probability distributions available) and A1 (weak ordering) of the previous section hold. In addition we can, and will, assume throughout that U(θ)=0.

(R2') For each ambiguous event A, there exists an unambiguous event B such that $A \sim^+ B$, and an unambiguous event C such that $A \sim^- C$.

Again, we ensure likelihood comparability of all events by:

(R3') There exist outcomes σ≻θ≻τ such that, for each event A, $(A,\sigma;A^c;\theta)$ and $(A^c,\theta;A;\tau)$ are conceivable.

Now we define $v^+(A) = P(B)$, and $v^-(A) = P(C)$ for B and C as in (R2'). Monotonicity of the capacities will again be implied by the following condition:

(A2') A⊃B implies $A \succcurlyeq^+ B$ and $A \succcurlyeq^- B$.

Cumulative dominance in (A3) is now modified to cumulative gain-loss dominance in a natural way:

(A3') For all prospects f,g we have:

f ≽ g whenever
$\{s\in S: f(s)\succcurlyeq\alpha\} \succcurlyeq^+ \{s\in S: g(s)\succcurlyeq\alpha\}$ for all gains α≻θ
and
$\{s\in S: f(s)\preccurlyeq\beta\} \preccurlyeq^- \{s\in S: g(s)\preccurlyeq\beta\}$ for all losses β≺θ.

This condition says that prospect f is preferred to prospect g if:

a gain α or a superior gain is *at least* as likely under prospect f as under prospect g, for all gains α,

and

a loss β or a greater loss is *at most* as likely under act f as under act g, for all losses β.

Clearly, f is a more attractive prospect than g if under f greater gains are more likely and greater losses are less likely. Again, this condition, when restricted to gains prospects, implies that $\succcurlyeq^+$ and the gains capacity v^+ are well-defined, and that v^+ is monotonic with respect to set inclusion; a similar conclusion holds for its restriction to loss prospects with $\succcurlyeq^-$ and v^- replacing $\succcurlyeq^+$ and v^+.

We now show that the conditions above lead to the CPT representation for a subclass of the prospects, the "prospects that can be matched" with an unambiguous prospect. Consider an arbitrary ambiguous prospect $(A_1,x_1;\cdots;A_n,x_n)$. Suppose that $x_1 \succcurlyeq \cdots \succcurlyeq x_{k-1}$ are gains, $x_k=\theta$ (this can always be obtained by adding $A_k=\varnothing$ if necessary, or by collapsing the events that yield θ) and $x_{k+1} \succcurlyeq \cdots \succcurlyeq x_n$ are losses. That is,

$$x_1 \succcurlyeq \cdots \succcurlyeq x_{k-1} \succ x_k = \theta \succ x_{k+1} \succcurlyeq \cdots \succcurlyeq x_n.$$

Suppose we can find an unambiguous prospect $(p_1,x_1;\cdots;p_k,x_k;\cdots;p_n,x_n)$ that is "matching" in the sense that

$$p_1+p_2+\cdots+p_i \quad = \quad v^+(A_1\cup\cdots\cup A_i) \qquad \text{for each } i<k,$$

and

$$p_n+p_{n-1}+\cdots+p_j \quad = \quad v^-(A_n\cup\cdots\cup A_j) \qquad \text{for each } j>k.$$

Thus for gains, the cumulative events $A_1\cup\cdots\cup A_i$ match with respect to $\sim^+$, but for losses the "decumulative" events $A_n\cup\cdots\cup A_j$ match, now with respect to $\sim^-$. By (A3'), the two prospects are equivalent, so that the evalution of the ambiguous one is given by the expected utility of the unambiguous one, i.e., by $\sum_{i=1}^{n} p_i U(x_i)$. We suppress the term $p_k U(\theta)$ which is 0, and rewrite the sum as

$$\sum_{i=1}^{k-1}((p_1+\cdots+p_i)-(p_1+\cdots+p_{i-1}))U(x_i) \quad +$$

$$\sum_{j=k+1}^{n}((p_n+\cdots+p_j)-(p_n+\cdots+p_{j+1}))U(x_j),$$

which is identical to

$$\sum_{i=1}^{k-1}(v^+(A_1\cup\cdots\cup A_i)-v^+(A_1\cup\cdots\cup A_{i-1}))U(x_i) \quad +$$

$$\sum_{j=k+1}^{n}(v^-(A_n\cup\cdots\cup A_j)-v^-(A_n\cup\cdots\cup A_{j+1}))U(x_j).$$

The latter value is defined as the *Cumulative prospect theory value (CPT)* of the prospect $(A_1,x_1;\cdots;A_n,x_n)$.[6] Defining decision weights

$$\text{for } i<k: \quad \pi_i^+ := v^+(A_1\cup\cdots\cup A_i) - v^+(A_1\cup\cdots\cup A_{i-1})),$$

and

$$\text{for } j>k: \quad \pi_j^- := v^-(A_n\cup\cdots\cup A_j) - v^-(A_n\cup\cdots\cup A_{j+1}),$$

the CPT value is rewritten as

$$\sum_{i=1}^{k-1}\pi_i^+U(x_i) + \sum_{j=k+1}^{n}\pi_j^-U(x_j).$$

Again, like expected utility and CEU, the form resembles a weighted mean of the utility values, with the zero term related to $U(\theta)$ suppressed. Now the decision weights for gains are derived from cumulative events and the gains-capacity, and decision weights for losses are derived from decumulative events and the loss-capacity. Unlike the weights in CEU, the weights in CPT need not sum to one. We have now obtained the main result:

THEOREM 4.1. Suppose R1,R2',R3' are satisfied. Then on the set of prospects that can be matched, CPT holds if and only if A1, A2', and A3' hold. □

We derived CPT above for the matchable prospects. A prospect can be matched if and only if the sum of the decision weights of the nonneutral outcomes is less than or equal to one. A simpler way to behaviorally test that a prospect can be matched is provided by the following condition, that is necessary and sufficient for matchability. Let $(A_1,x_1;\cdots;A_n,x_n)$ be a prospect with $x_{k-1} \succ x_k = \theta \succ x_{k+1}$. Take any unambiguous event B such that $B \sim^+ A_1\cup\cdots\cup A_{k-1}$. Now the prospect can be matched if and only if

[6] Also if the prospect is not matched by an unambiguous prospect, we call this value the CPT value of the prospect.

$B^c \succcurlyeq^- A_n \cup \cdots \cup A_{k+1}$. Note that any prospect yielding only gains, or only losses, can always be matched. This shows in particular that Theorem 4.1 gives a CPT representation for any prospect that only yields gains, and for any prospect that only yields losses. Of course, in the case in which there are only gains, or only losses, the CPT representation reduces to the CEU representation.

It is straightforwardly verified that all prospects are matchable if one assumes the "reflection property[7]" of Tversky and Kahneman (1992) and if furthermore superadditivity[8] of the capacities is satisfied; these are two common properties.

In general, it is possible that some prospects may not be matchable. As an extreme example, think of the case in which CPT holds, but v^+ assigns value 1 to an event A, and v^- assigns value 1 to the complement of A. Then, for $\alpha \succ \theta \succ \beta$, the prospect $(A,\alpha;A^c,\beta)$ yields decision weight $\pi^+ = 1$ for the outcome α, and also decision weight $\pi^- = 1$ for the outcome β. This prospect obviously cannot be matched, since for an unambiguous prospect the decision weights (=probabilities) cannot sum to more than one. In such cases the CPT representation may not hold. An elaborated example is presented in the Appendix.

In the absence of matchability, it is still possible to obtain a CPT representation by assuming an additional condition. To explain one such condition, we write, for any prospect f, f^+ for the prospect that results if all losses in f are replaced by the neutral outcome, and f^- for the prospect that results if all gains in f are replaced by the neutral outcome. Now suppose $f^+ \sim g^+$, and $f^- \sim g^-$, for two prospects f,g. It is a necessary condition for CPT that these two equivalences, separately in the gain and loss domain, imply the equivalence $f \sim g$.[9] This is the *double matching* condition of Tversky and Kahneman (1992). By means of this additional condition, a CPT representation can be obtained for all prospects f for which an unambiguous prospect g can be found such that $f^+ \sim g^+$, and $f^- \sim g^-$. Such prospects f are called *doubly matchable*. Note that all prospects are doubly matchable if the utility function is unbounded from both sides.

PROPOSITION 4.2. Suppose all conditions of Theorem 4.1 (R1, R2', R3', A1, A2', A3') hold. CPT holds for all doubly matchable prospects if and only if double matching is satisfied.

[7]*Reflection* means that $v^+ = v^-$.

[8]A capacity v is *superadditive* if $v(A \cup B) - v(A) \geq v(B) - v(A \cap B)$ for all events A,B.

[9]This follows mainly because $CPT(f) = CPT(f^+) + CPT(f^-)$.

PROOF. Theorem 4.1 has given a CEU representation (denoted CPT^+ below) for prospects yielding merely gains, and a CEU representation (denoted CPT^- below) for prospects yielding merely losses. For unambiguous prospects, CPT^+ and CPT^- coincide with EU. For a doubly matchable prospect f, and g the doubly matching unambiguous prospect, this ensures that $CPT^+(f^+) = EU(g^+)$, and $CPT^-(f^-) = EU(g^-)$. By double matching $f \sim g$, and f can be evaluated by $CPT(g) = EU(g^+) + EU(g^-) = CPT^+(f^+) + CPT^-(f^-) = CPT(f)$. □

5. DISCUSSION

In Theorem 4.1 and in Proposition 4.2, we have derived a CPT representation that relies heavily on the availability of unambiguous prospects. The introduction of unambiguous events simplifies the derivation and the elicitation of the model. If the CPT model holds true but the matchability conditions fail, then the elicitation procedures based on the calibration with respect to unambiguous events can still be carried out. That is, for each ambiguous event A, unambiguous events B and C can be found such that $A \sim^+ B$ and $A \sim^- C$. Thus $v^+(A) = P(B)$, and $v^-(A) = P(C)$ can still be inferred.

There are two differences between the CEU and CPT models. First, the decision weights for gains in the CPT model are independent from those for losses. This provides a greater flexibility in predicting choices. For example, in Section 2 a preference for the known urn in the first (gain) example, in Section 2 a preference for the unknown urn in the second (loss) scenario is consistent with CPT by taking $v^+(Y) < 1/2$ and $v^-(W) < 1/2$; the latter implies that the decision weight for yellow in the second scenario, $(1-v^-(W))$, is greater than 1/2.

The second difference between CEU and CPT concerns the way in which the capacities are used to calculate decision weights. Consider a prospect $(A_1,x_1;\cdots;A_n,x_n)$. In CEU one takes differences $v(A_1 \cup \cdots \cup A_i) - v(A_1 \cup \cdots \cup A_{i-1})$ to obtain the decision weight π_i for event A_i. So here cumulative events are used. In CPT, the computation for decision weights for gains follows a similar scheme, using the gains capacity v^+. In the loss domain however, one uses a difference $v^-(A_n \cup \cdots \cup A_j) - v^-(A_n \cup \cdots \cup A_{j+1})$ to obtain the decision weight π_j^- of event A_j. The decision weight π_j^- is the difference between the capacities of the events "x_j or worse loss" and "strictly worse loss than x_j".[10] So here "decumulative" events are used. This also explains the definition of the likelihood relation $\succcurlyeq^-$ for losses through bets against events, rather than bets on events.

[10]For simplicity of presentation assume here $x_j \succ x_{j+1}$.

We note, however, that CEU is a special case of CPT. This can be seen by setting $v^+(A)=v(A)$, and $v^-(A)=1-v(A^c)$, for all events A. In the latter case, the decision weight for a loss x_j under CPT is

$$v^-(A_n\cup\cdots\cup A_j) - v^-(A_n\cup\cdots\cup A_{j+1}) =$$
$$1-v(A_1\cup\cdots\cup A_{j-1}) - (1-v(A_1\cup\cdots\cup A_j)) =$$
$$v(A_1\cup\cdots\cup A_j) - v(A_1\cup\cdots\cup A_{j-1}),$$

which is exactly the decision weight resulting from CEU. In other words, the CPT model reduces to CEU if $v^-(A)=1-v^+(A^c)$ for all events A.

We now briefly comment on the conditions that we used to derive CPT. We assumed expected utility maximization for unambiguous prospects. Our main results follow, by identical procedures, if one assumes Quiggin's (1982) rank-dependent utility for the unambigious prospects, or even, for CPT, if one assumes CPT for the unambiguous prospects. In other words, if CPT is satisfied on a domain that is sufficiently rich, then by condition (A3'), CPT spreads over the other prospects. Our cumulative gain-loss dominance condition seems to be a generalization of stochastic dominance to the case of uncertainty.

Next we turn to the matchability condition that seems to be the Achilles heel of our development. There is, however, an intuition behind this condition. Recall that, in our example in Section 2, the decision weight for the gain seemed to be reduced for the ambiguous prospect. For an ambiguity-neutral person the decision weight would have been 0.5. But for the modal preference in the example, the decision weight is less than 0.5, i.e., some decision weight is shifted to the neutral outcome, leading to ambiguity aversion. Similarly, in the loss domain the decision weight for the loss is shifted toward the neutral outcome, leading to ambiguity seeking behavior. We conjecture that for a prospect that involves both gains and losses, the shifts of decision weights from tails to the middle occur jointly. Should this occur, then the decision weights for nonneutral outcomes will sum to less than one. In this case the matchability condition will be satisfied.

APPENDIX

EXAMPLE A1. This example shows that the CPT representation in Theorem 4.1 does not necessarily hold for prospects that cannot be matched by an unambiguous prospect.

Suppose outcomes are real numbers, and the utility function U is the identity function. The neutral outcome θ is 0. M is a positive real number. A denotes an ambiguous event (say rain tomorrow). We study the ambiguous prospect $g=(A,M;A^c,-M)$. To meet condition (R3'), we also include the prospects $(A,M;A^c,0)$, $(A^c,M;A,0)$, $(A^c,0;A,-M)$, and $(A,0;A^c,-M)$. The set of all conceivable prospects consists of these five ambiguous prospects, and the set of all simple probability distributions over $\mathbb{R}$.

We elicit

$$v^+(A) = v^+(A^c) = 0.6 = v^-(A) = v^-(A^c)$$

by the equivalences

$$(A,M;A^c,0) \sim (0.6,M;0.4,0) \sim (A^c,M;A,0)$$

and

$$(A,0;A^c,-M) \sim (0.4,0;0.6,-M) \sim (A^c,0;A,-M).$$

So the decision weights for the prospect $(A,M;A^c,-M)$ sum to 1.2, i.e., to more than one, and the prospect is not matchable. Suppose that $g\sim M/10$. We can represent the preference relation by a function V that assigns expected value to all unambiguous prospects, further

$$V(A,M;A^c,0) = V(A^c,M;A,0) = 6M/10,$$
$$V(A,0;A^c,-M) = V(A^c,0;A,-M) = -6M/10,$$

and

$$V(g) = M/10.$$

All values except the last coincide with CPT values. The CPT value of g can be calculated, however, it is 0, so it differs from the V value, and does not provide a proper measure to represent preferences.

The preference relation does satisfy all conditions in Theorem 4.1. The only problem that might be expected concerns condition (A3'). In view of the increase of value of g as compared to the CPT model, it could be feared that there exists an unambiguous prospect f (for all ambiguous prospects f the condition is satisfied) such that f and g satisfy all conditions in (A3'), but still $g\succ f$. However, this never occurs. To satisfy the conditions in (A3'), f should assign a probability of at least 0.6 to an outcome at least as high as M. The remaining probability of 0.4 must be assigned to outcomes at least as good as $-M$. So the CPT value of f is at least $2M/10$, and $f\succcurlyeq g$ follows.

The prospect $(0.5,\frac{6M}{5};\ 0.5,\frac{-6M}{5})$ shows that double matching is violated in the example. If outcomes would be restricted to $[-M,M]$, then also double matching would be satisfied; then g would not be doubly matchable. □

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