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The Axiomatic Basis of Anticipated Utility: A Clarification

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Quiggin (*J. Econ. Behav. Organization* **3** (1982), 323–345) introduced anticipated ("rank-dependent") utility theory into decision making under risk. Questions have been raised about mathematical aspects of Quiggin's analysis. This paper settles these questions and shows that a minor modification of Quiggin's axioms leads to a useful and correct result, with features not found in other recent axiomatizations. *Journal of Economic Literature* Classification Number: D81. © 1994 Academic Press, Inc.

1. INTRODUCTION

This paper discusses mathematical aspects of Quiggin [12], the paper that introduced anticipated utility into decision making under risk.[1] Other terms are "rank-dependent utility," or, less tractable, "expected utility with rank-dependent probabilities." We shall use the term anticipated utility for the special case of rank-dependent utility where the probability transformation function assigns value 1/2 to probability 1/2. The rank-dependent stream is currently the most popular one in nonexpected utility. Independently from [12], essentially the same form was developed by Schmeidler [14—first version 1982], Yaari [18], Luce [8], and Allais [3]. The special case considered by Yaari (with linear utility) had been developed and axiomatized before in welfare theory by Weymark [17]. The importance of the form is based on the possibility to express risk attitudes by ways to deal with probabilities, without violating basic requirements such as stochastic

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[1] The form had already appeared in Quiggin [11].

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dominance or transitivity. A recent account has been given in Quiggin [13]. Tversky and Kahneman [15] adopted the rank-dependent form to obtain a new version of prospect theory.

Given the historical importance of Quiggin [12], a new study of the mathematics in the paper seems appropriate. Examples 4.1–4.3 below show some complications for that mathematics. There have been some discussions and misunderstandings about Quiggin's main theorem, and this paper aims to clarify the issues. As we shall see, only a minor modification of the axioms is needed. Yaari [18, p. 113] already suggested that Quiggin's Axiom 2 should be strengthened. Indeed, it suffices to strengthen Quiggin's Axiom 2 to stochastic dominance or, as we shall do, to a weaker version that only considers two-outcome prospects. The proof of the result will be entirely rewritten and will not invoke continuity with respect to outcomes.

Recently, variations on the axiomatization of Quiggin have been developed. Chew [4] generalized Quiggin's model by deleting the restriction that the probability transformation assign value $\frac{1}{2}$ to probability $\frac{1}{2}$; he still required continuity both in outcomes and in probabilities. In Wakker's [16] axiomatization the probability transformation need not be continuous, while Nakamura [9] relaxed the requirement that the utility function be continuous. So in a structural sense these results are more general than [12]. Still, in a logical sense none of these results is a complete generalization of Quiggin's; i.e., Quiggin's result cannot be obtained as a corollary. First, his independence Axiom 4 uses only $\frac{1}{2}$–$\frac{1}{2}$ mixtures, whereas the bisymmetry/commutativity axioms of Chew and Nakamura use mixtures with other probabilities, and Wakker's tradeoff consistency axiom, in isolation, is logically independent. Second, Quiggin's dominance Axiom 2 and continuity Axiom 3 are only imposed on two-outcome prospects.[2] So Quiggin's result still stands as a useful axiomatization.

An additional advantage of Quiggin's result is that concavity of utility can be characterized as easily as in expected utility: For $\frac{1}{2}$–$\frac{1}{2}$ prospects the model coincides with expected utility. Hence, given the usual continuity conditions, preference of expected values over $\frac{1}{2}$–$\frac{1}{2}$ prospects is necessary and sufficient for concavity of utility, as it is in expected utility.

2. Definitions and Notations, and a Discussion Thereof

The notations and terminology of this paper will as much as possible follow Quiggin [12]. X is a set of *outcomes*, and may at this stage be any

[2] A research question is whether other existing axiomatizations can be generalized by similar weakenings of continuity and dominance.

general set. We shall see, at the end of the Appendix, that X is isomorphic to a connected topological space; the analysis in [12] implicitly used continuity with respect to a connected topology on X at several places. Our analysis will not use such an assumption, and the isomorphism to a connected topological space will be a consequence of the other assumptions rather than a presupposition.

By Y we denote the set of *prospects*, i.e., of all probability distributions over X with finite support. By $\succcurlyeq$ (rather than by P as in [12]) we denote a binary ("preference") relation on Y. We write $\succ$ for strict preferences and $\sim$ for indifferences. Outcomes x are identified with degenerate prospects. This induces a binary relation $\succcurlyeq$ on the outcome set X through the degenerate prospects. By $\{(x_1, ..., x_n); (p_1, ..., p_n)\}$ we denote the prospect assigning probability p_j to outcome x_j, $j = 1, ..., n$. Of course, the p_j's are nonnegative and sum to one; $p_j = 0$ is permitted. We write $\mathbf{x}$ for $(x_1, ..., x_n)$ and $\mathbf{p}$ for $(p_1, ..., p_n)$. In all results of this paper, $\succcurlyeq$ will be a weak order. So we can, and do, assume without further mentioning that $x_1 \preccurlyeq \cdots \preccurlyeq x_n$, i.e., the outcomes are rank-ordered. Let us emphasize that this assumption is essential to the analysis; the rank-ordering of outcomes is central in rank-dependent utility.

A function $V: Y \to \mathbb{R}$ *represents* $\succcurlyeq$ if $\{\mathbf{x}; \mathbf{p}\} \succcurlyeq \{\mathbf{x}'; \mathbf{p}'\} \Leftrightarrow V\{\mathbf{x}; \mathbf{p}\} \geqslant V\{\mathbf{x}'; \mathbf{p}'\}$. If a representing function V exists, then $\succcurlyeq$ is a *weak order*, i.e., it is *complete* (for all $\{\mathbf{x}; \mathbf{p}\}$ and $\{\mathbf{x}'; \mathbf{p}'\}$, $\{\mathbf{x}; \mathbf{p}\} \succcurlyeq \{\mathbf{x}'; \mathbf{p}'\}$ or $\{\mathbf{x}'; \mathbf{p}'\} \succcurlyeq \{\mathbf{x}; \mathbf{p}\}$) and transitive; completeness implies reflexivity. *Rank-dependent utility* holds if there exists a representing functional V of the form

$$V(\{(x_1, ..., x_n); (p_1, ..., p_n)\}) = \sum_{i=1}^{n} \left(f\left(\sum_{j=1}^{i} p_j \right) - f\left(\sum_{j=1}^{i-1} p_j \right) \right) U(x_i) \tag{2.1}$$

for a function $U: X \to \mathbb{R}$ and a nondecreasing function f: $[0, 1] \to [0, 1]$ with $f(0) = 0$, $f(1) = 1$. Let us recall here that $x_1 \preccurlyeq \cdots \preccurlyeq x_n$; $\sum_{j=1}^{0} p_j$ is conventionally defined as 0. *Anticipated utility* (AU) is the special case where $f(\frac{1}{2}) = \frac{1}{2}$.

In the remainder of this section we discuss identities such as $x_i = x_{i+1}$, which we permit in the notation $\{(x_1, ..., x_n); (p_1, ..., p_n)\}$ for a prospect. For instance, the prospect $\{(\alpha, \alpha, \beta); (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\}$ is identical to the prospect $\{(\alpha, \beta); (\frac{1}{2}, \frac{1}{2})\}$. A form applied to $2n$-tuples $\{(x_1, ..., x_n\}; (p_1, ..., p_n)\}$ must respect these identities and assign the same value to $\{(\alpha, \alpha, \beta); (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\}$ as to $\{(\alpha, \beta); (\frac{1}{2}, \frac{1}{2})\}$. From this it can be derived that a form

$$\sum_{j=1}^{n} h_j(p_1, ..., p_n)\, U(x_j), \tag{2.2}$$

with h_j depending only on the vector of "rank-ordered" probabilities, must satisfy the equation

$$h_i(p_1, ..., p_n) = f\left(\sum_{j=1}^{i} p_j\right) - f\left(\sum_{j=1}^{i-1} p_j\right). \tag{2.3}$$

That is, the "decision weight" $h_i(p_1, ..., p_n)$ can only depend on the "cumulative probabilities," so that rank-dependent utility results; the derivation of (2.3) is obtained by considering $\{(x_1, ..., x_n); (p_1, ..., p_n)\}$ with $x_1 = \cdots = x_j$ and $x_{j+1} = \cdots = x_n$ for $j = i-1$ and $j = i$, and is not elaborated here. In [12, Sect. 2] an alternative derivation is given that is discussed in further detail below.

The form (2.2) had already been proposed by Allais [2–first version 1953; see Formula IV in Sect. 41] where, however, no rank-ordering of outcomes was imposed. Note that, in the absence of rank-ordering, this form must reduce to expected utility! Only in Allais [3, Formula (1)] was rank-ordering imposed on the outcomes, and the above result was derived—Allais' form (5) is equivalent to (2.2).

It is easily derived that, in the notation of this paper, the form $\sum_{j=1}^{n} \phi(p_j)\, U(x_j)$ reduces to expected utility. The form was often studied for prospects where all outcomes should be distinct; then it can really deviate from expected utility. See [10; 6; 7, p. 283]. If, however, minimal continuity or dominance conditions are imposed, the form must reduce to expected utility after all. This was discovered relatively late, by Fishburn [6] and some others; see also Kahneman and Tversky [7, p. 283].

Let us give some elucidation to the analysis of Quiggin [12]. In line 6 of Section 2 there, it is suggested that the outcomes in the notation $\{(x_1, ..., x_n); (p_1, ..., p_n)\}$ are to be distinct at that moment; they are not yet rank-ordered at that stage. Thus some "non-rank-ordered" functionals from the literature can be discussed. It is shown in Eq. (1)–(5) there that those functionals violate monotonicity. In that derivation, continuity is used implicitly. Above Eq. (6) the outcomes are assumed to be rank-ordered. Equation (6) defines the functional as in (2.2) above (with $h(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$). Then (2.3) is derived from monotonicity, where again continuity is used implicitly. Given that, the equalities $x_i = x_{i+1}$ can be permitted in the notation. This is indeed done in the remainder of Quiggin's paper. Formally, it was already permitted in the notation introduced above Eq. (6); it is repeated in Assumption R.1. This also shows that the general form ((2.2) above) as found in Quiggin's Proposition 1 is identical to the AU form ((2.1) above) as derived in Quiggin's proof in the Appendix.

3. The Main Theorem

This section presents the modification of Quiggin's [12] axiomatization of anticipated utility. We use the following structural assumption of [12], ensuring that for each prospect there exists a "certainty equivalent":

R.2. For each prospect $\{\mathbf{x}; \mathbf{p}\}$ there exists an outcome x such that $x \sim \{\mathbf{x}; \mathbf{p}\}$.

Now we turn to the axioms:

Axiom 1. The binary relation $\succcurlyeq$ is a weak order.

The dominance axiom of Quiggin will be adapted as follows. Both axioms below are implied by strict stochastic dominance when restricted to two-outcome prospects. The first imposes weak monotonicity with respect to probabilities, the other strict monotonicity with respect to outcomes for fixed probabilities $\frac{1}{2}$, $\frac{1}{2}$. Remember that in our notations for prospects, outcomes are assumed to be rank-ordered.

Axiom 2′a. If $p' \geqslant p$, then $\{(x_1, x_2); (1-p', p')\} \succcurlyeq \{(x_1, x_2); (1-p, p)\}$.

Axiom 2′b. $\{(x_1', x_2'); (\frac{1}{2}, \frac{1}{2})\} \succcurlyeq \{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\}$ whenever $x_2' \succcurlyeq x_2$, $x_1' \succcurlyeq x_1$, where the former preference is strict if one of the latter two is strict.

For the sake of comparison, we give Quiggin's Axiom 2, which is the restriction of Axiom 2′a to the case $p' = 1$:

Axiom 2Q. $x_2 \succcurlyeq \{(x_1, x_2); (1-p, p)\}$ for all p.

Axiom 3 (Continuity). If $x_1, x_2, x_3 \in X$, $x_1 \preccurlyeq x_2 \preccurlyeq x_3$, then there exists p^* such that

$$x_2 \sim \{(x_1, x_3); (1-p^*, p^*)\}.$$

Note that under AU, with $f(\frac{1}{2}) = \frac{1}{2}$, in $(\frac{1}{2}, \frac{1}{2})$ prospects it does not matter which outcome is substituted first in the form (2.1), since each outcome receives weight $\frac{1}{2}$. This suggests that for $(\frac{1}{2}, \frac{1}{2})$ prospects the rank-ordering of outcomes is immaterial. We introduce an additional notation for $(\frac{1}{2}, \frac{1}{2})$ prospects: $\{\{x, x'\}; (\frac{1}{2}, \frac{1}{2})\}$ denotes the prospect $\{(x, x'); (\frac{1}{2}, \frac{1}{2})\}$ if $x \preccurlyeq x'$, and $\{(x', x); (\frac{1}{2}, \frac{1}{2})\}$ if $x' \preccurlyeq x$. The notation is useful in Axiom 4, where the rank-ordering of each pair x_i, x_i', and of x and x', is undetermined. The notation will also be useful in proofs.

Axiom 4 (Independence). See Figure 1. Whenever $x \sim \{\mathbf{x}; \mathbf{p}\}$, $x' \sim \{\mathbf{x}'; \mathbf{p}\}$, $c_i \sim \{\{x_i, x_i'\}; (\frac{1}{2}, \frac{1}{2})\}$ for all i, then $\{\mathbf{c}; \mathbf{p}\} \sim \{\{x, x'\}; (\frac{1}{2}, \frac{1}{2})\}$.

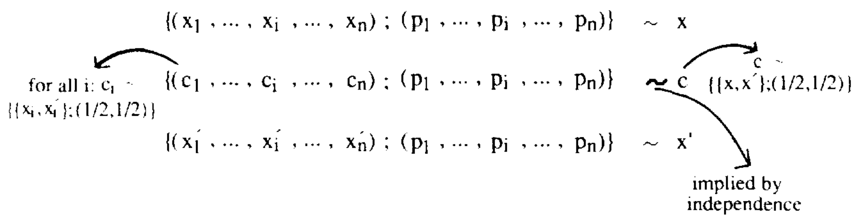

FIG. 1. (*Independence*) Every c-outcome is a "midpoint" between the x-outcome above, and the x'-outcome below. The bold-printed indifference is implied by the other indifferences. In other words, midpoints can be taken as well before as after the taking of certainty equivalents.

The result of the lemma below is implied by Quiggin's [12, top of p. 327] assumption that indifferent outcomes are identical.

LEMMA 3.1. *If* R.2 *holds, as well as Axiom* 1, *2'b, and* 4, *then* $\{\mathbf{x}; \mathbf{p}\} \sim \{\mathbf{x}'; \mathbf{p}\}$ *whenever* $x_i \sim x_i'$ *for all i.*

Proof. By Axiom 2'b, $c_i \sim \{\{x_i, x_i'\}; (\frac{1}{2}, \frac{1}{2})\}$ for both $c_i = x_i$ and $c_i = x_i'$. By R.2, x and x' as in Axiom 4 exist. Now apply Axiom 4 both with $\mathbf{c} = \mathbf{x}$ and with $\mathbf{c} = \mathbf{x}'$. ▮

The following modification of Quiggin's characterization of AU is the main result of the paper; its proof is given in the Appendix.

THEOREM 3.2. *Let* $\succcurlyeq$ *be a binary relation on the set Y of prospects. Then the following two statements are equivalent*:

(i) *Condition* R.2 *and Axioms* 1, 2'a, 2'b, 3, *and* 4 *are satisfied.*

(ii) AU *holds* (*so* $f(\frac{1}{2}) = \frac{1}{2}$), *where f is continuous and nondecreasing, and the range of U is an interval.*

Further, f in (ii) *above is uniquely determined and U is unique up to scale and location.* □

Note that, if X is an interval in the above theorem and U is nondecreasing, as it will be under traditional dominance, then U must be continuous, as its range is an interval.

4. EXAMPLES

The following examples discuss mathematical complications in Quiggin's [12] analysis. The first example is, strictly speaking, a counterexample to Quiggin's Proposition 1. However, it only applies to a degenerate case. The second example shows another complication, i.e., for the general form provided in Quiggin's Proposition 1, the function f may not be monotonic in probability. Finally, in Example 4.3 we discuss an example given in Yaari [18].

EXAMPLE 4.1. Let $X=\{x_1, x_2\}$; i.e., there are only two outcomes. Suppose $x_2 \succ x_1$ and $\{(x_1, x_2); (1-p, p)\} \sim x_1$ for all $p<1$. There does exist a rank-dependent utility representation for $\succcurlyeq$, with $U(x_2)=1$, $U(x_1)=0$, and $f(p)=1$ for all $0<p<1$. Here f is uniquely determined. Thus there does not exist an AU model for $\succcurlyeq$ because $f(\frac{1}{2}) \neq \frac{1}{2}$. The preference relation satisfies all conditions in Statement (i) of Theorem 3.2, with the exception of Axiom 2′b. We only discuss Axiom 4. Nonindifference in the conclusion can only occur if either $\{\mathbf{c}; \mathbf{p}\}$ or $\{\{x, x'\}; (\frac{1}{2}, \frac{1}{2})\}$ is maximal, i.e., is x_2. But this straightforwardly implies that all other prospects are maximal, i.e., are x_2, as well. So Axiom 4 is satisfied. In particular, Axiom 2′a is satisfied, which for the special case $p'=1$ gives Axiom 2Q, i.e., Quiggin's Axiom 2. So all of Quiggin's conditions are satisfied, and formally this is a counterexample to Quiggin's Proposition 1.

EXAMPLE 4.2. Let $X=\mathbb{R}$, let U be the identity, and let AU hold, with one exception: the function f: $[0, 1] \to \mathbb{R}$ is not necessarily nondecreasing; it does satisfy $f(0)=0$, $f(\frac{1}{2})=\frac{1}{2}$, and $f(1)=1$. Neccessary and sufficient for verification of Axiom 2Q, i.e., Quiggin's [12] Axiom 2, is that $f(p) \geqslant 0$ for all p. Necessary and sufficient for verification of Axiom 3, is that $f([0, 1]) \supset [0, 1]$. Condition R.2 and Axiom 4 are satisfied. Thus f does not have to be nondecreasing and may even take values larger than 1.

EXAMPLE 4.3. Yaari [18] suggested, for $X=\mathbb{R}$ and U the identity, the form $\sum w(p_j)\, x_j$ with w continuous and $w(p)+w(1-p) \leqslant 1$, as a counterexample to Quiggin's [12] characterization of AU in his Proposition 1. Yaari did not make explicit which notational conventions he followed. Under the notational conventions of this paper, Yaari's form must be identical to expected value maximization, which obviously would not provide a counterexample to Quiggin's result.

If his form is only to be applied to prospects $\{(x_1, ..., x_n); (p_1, ..., p_n)\}$ with distinct outcomes, then a violation of Axiom 4 can be derived unless w is linear (substitution of $n=3$, $0<x_1<x_2<x_3<x'_1=x'_2<x'_3$ shows that $w(p_1+p_2)=w(p_1)+w(p_2)$, etc.), so again the form does not provide a counterexample to Quiggin's result. Finally, if not only this form, but also Quiggin's axioms and form, would only be applied to prospects $\{(x_1, ..., x_n);\ (p_1, ..., p_n)\}$ with distinct outcomes, then Yaari's form (with $w(\frac{1}{2})=\frac{1}{2}$, to satisfy Axiom 4) is a special case of the form in Quiggin's Proposition 1. So then it still does not provide a counterexample to

Quiggin's result. Note that, under such a notational convention, the general form in Quiggin's Proposition 1 would be strictly more general than AU. This, finally, shows that, if outcomes in the notation would be required to be distinct, then the axioms in Theorem 3.2 would not imply AU and would not discard the form $\sum \phi(p_j)\, U(x_j)$; the observation that this form violates stochastic dominance, motivated Quiggin to develop AU.

5. Conclusion

The aim of this paper has been to clarify some discussions in relation to the classic paper by Quiggin [12]. We have shown that, by a strengthening of the stochastic dominance Axiom 2 in [12], a characterization of anticipated utility can be obtained. The main restriction of this characterization in comparison to later characterizations by Chew [4], Nakamura [9], and Wakker [16] is that the probability transformation function f should assign value $\frac{1}{2}$ to probability $\frac{1}{2}$. This restriction, however, has some advantages. First, the independence Axiom 4 only invokes $\frac{1}{2}$–$\frac{1}{2}$ mixtures. Second, the dominance Axioms 2′, as well as the continuity Axiom 3, need only be imposed on two-outcome prospects. Because of this, Quiggin's axiomatization continues to be of interest to date and still offers features not found in other axiomatizations.

APPENDIX: Proof of Theorem 3.2

Necessity of the conditions is straightforwardly verified; we only mention that R.2 is implied by the assumption that the range of U is an interval. So we assume Statement (i), and derive Statement (ii). In the major part of the proof we make the following assumption; only at the end of the proof, the assumption will be relaxed.

Assumption A1. There exists a best outcome x^1 and a worst outcome x^0; $x^1 \succ x^0$.

Stage 1 (Construction of Binary Values of U). Define $U(x^1)=1$, $U(x^0)=0$. By R.2, there exists $x^{1/2} \sim \{(x^0, x^1); (\frac{1}{2}, \frac{1}{2})\}$. By Axiom 2′b, $x^1 \succ x^{1/2} \succ x^0$. Define $x^{1/4} \sim \{(x^0, x^{1/2}); (\frac{1}{2}, \frac{1}{2})\}$, $x^{3/4} \sim \{(x^{1/2}, x^1); (\frac{1}{2}, \frac{1}{2})\}$, and similarly define $x^{1/8}$, $x^{3/8}$, ..., and inductively every $x^{a/2^n}$. To each $x^{a/2^n}$ we assign U value $a/2^n$. By repeated application of Axiom 2′b and transitivity, U is representing on the set of all $x^{a/2n}$.

Stage 2 (An Application of Axiom 4). We derive the following condition:

$$\{(x^{a/2^n}, x^{b/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a/2^n + b/2^m)/2}. \tag{A1}$$

By multiplying by a large $2^{m'}$, it suffices to derive the result only for $n = m$ and $a - b$ even. For such n, m, a, b the result is derived by induction with respect to m. For $m = 1$ it holds true. Suppose, as induction hypothesis, that it holds true for $1, ..., m-1$, where $m \geqslant 2$. We show, for all appropriate a, k, that

$$\{(x^{(a-k)/2^m}, x^{(a+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{a/2^m}. \tag{A2}$$

Below, Axiom 4 is applied several times. The indifferences needed for that always follow from the induction hypothesis (and the definition of the $x^{c/2^n}$). To verify that, it must be checked that several integers, and differences of these integers divided by 2, are even. This will not be made explicit. In each application of Axiom 4, the left prospect in (A2) plays the role of $\{\mathbf{c}; \mathbf{p}\}$ in Axiom 4, and $x^{a/2^m}$ the role of the outcome c in Fig. 1.

Case 1. a and k are even. Then the indifference follows from the induction hypothesis.

Case 2. a is odd, k is even. Then, by Axiom 4, (A2) follows from the two indifferences below:

$$\{(x^{(a-1-k)/2^m}, x^{(a-1+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a-1)/2^m}$$

$$\{(x^{(a+1-k)/2^m}, x^{(a+1+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a+1)/2^m}.$$

Case 3. a is odd, k is odd. Then either $a - k \geqslant 2$ or, if $a = k$, then $a + k \leqslant 2^m - 2$, given that $m \geqslant 2$ and $a + k = 2a$ is not a multiple of 4. If $a - k \geqslant 2$ then, by Axiom 4, (A2) follows from the two indifferences below:

$$\{(x^{(a-2-k)/2^m}, x^{(a+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a-1)/2^m}$$

$$\{(x^{(a+2-k)/2^m}, x^{(a+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a+1)/2^m}.$$

If $a + k \leqslant 2^m - 2$ then, by Axiom 4, (A2) follows from the two indifferences below:

$$\{(x^{(a-k)/2^m}, x^{(a-2+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a-1)/2^m}$$

$$\{(x^{(a-k)/2^m}, x^{(a+2+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{(a+1)/2^m}.$$

Case 4. a is even, k is odd. Then, by Axiom 4, (A2) follows from the two indifferences below:

$$\{(x^{(a-1-k)/2^m}, x^{(a+1+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{a/2^m}$$

$$\{(x^{(a+1-k)/2^m}, x^{(a-1+k)/2^m}); (\tfrac{1}{2}, \tfrac{1}{2})\} \sim x^{a/2^m}.$$

Stage 3 (Definition of U on Entire X). We define

$$U: x \mapsto \sup\{U(x^{a/2^m}): x^{a/2^m} \preccurlyeq x\}.$$

This is indeed a true extension of U, and it follows straightforwardly that

$$x' \succcurlyeq x \Rightarrow U(x') \geqslant U(x). \tag{A3}$$

This implies in particular that U is constant on $\sim$ indifference classes of the outcome set, which will be crucial for several definitions below. We cannot conclude at this stage that U would represent $\succcurlyeq$ on outcomes, as the implication $x' \succ x \Rightarrow U(x') > U(x)$ has not yet been derived. This implication will only be established below, and its derivation will invoke the definition of f below and Axioms 3 and 4. For a prospect $\{\mathbf{x}; \mathbf{p}\}$, we define $V\{\mathbf{x}; \mathbf{p}\}$ as the U value of an outcome x for which $x \sim \{\mathbf{x}; \mathbf{p}\}$; by R.2 such an x exists and, by constancy of U on $\sim$ outcome indifference classes, $V\{\mathbf{x}; \mathbf{p}\}$ is independent of the particular x that we choose. Obviously, by (A3),

$$\{\mathbf{x}'; \mathbf{p}'\} \succcurlyeq \{\mathbf{x}; \mathbf{p}\} \Rightarrow V(\{\mathbf{x}'; \mathbf{p}'\}) \geqslant V(\{\mathbf{x}; \mathbf{p}\}). \tag{A4}$$

Next we derive the following variation of (A2), for all $x_2 \succcurlyeq x_1$:

$$V\{(x_1, x_2); (\tfrac{1}{2}, \tfrac{1}{2})\} = \tfrac{1}{2}U(x_1) + \tfrac{1}{2}U(x_2). \tag{A5}$$

First we prove this for $x^0 \prec x_1 \preccurlyeq x_2 \prec x^1$. By the implication $U(x') < U(x) \Rightarrow x' \prec x$ as following from (A3), the inequalities $a/2^m < U(x_1) < a'/2^m$ and $b/2^m < U(x_2) < b'/2^m$ imply the preferences $x^{a/2^m} \prec x_1 \prec x^{a'/2^m}$ and $x^{b/2^m} \prec x_2 \prec x^{b'/2m}$. Hence, by Axiom 2′b, $\{(x^{a/2^m}, x^{b/2^m}); (\frac{1}{2}, \frac{1}{2})\} \prec \{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\} \prec \{(x^{a'/2^m}, x^{b'/2^m}); (\frac{1}{2}, \frac{1}{2})\}$. By (A4), $V\{(x^{a/2^m}, x^{b/2^m}); (\frac{1}{2}, \frac{1}{2})\} \leqslant V\{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\} \leqslant V\{(x^{a'/2^m}, x^{b'/2^m}); (\frac{1}{2}, \frac{1}{2})\}$. So, by (A1), $V\{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\}$ lies between $(a/2^m + b/2^m)/2$ and $(a'/2^m + b'/2^m)/2$ for all m, a, b, a', b' as above. This can only be if (A5) holds true.

Next we consider the case in which $x_1 \sim x^0$ or $x_2 \sim x^1$. The result is immediate if $x_1 \sim x^0$ and $x_2 \sim x^1$. For the case where $x_1 \sim x^0$ and $x_2 \prec x^1$, $b/2^m < U(x_2) < b'/2^m$ implies $x^{b/2^m} \prec x_2 \prec x^{b'/2^m}$. By Axiom 2′b, $\{(x_1, x^{b/2^m}); (\frac{1}{2}, \frac{1}{2})\} \prec \{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\} \prec \{(x_1, x^{b'/2^m}); (\frac{1}{2}, \frac{1}{2})\}$. By (A4), $V\{(x_1, x^{b/2^m}); (\frac{1}{2}, \frac{1}{2})\} \leqslant V\{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\} \leqslant V\{(x_1, x^{b'/2^m}); (\frac{1}{2}, \frac{1}{2})\}$. So, by (A1), $V\{(x_1, x_2); (\frac{1}{2}, \frac{1}{2})\}$ lies between $(0 + b/2^m)/2$ and $(0 + b'/2^m)/2$ for all m, b, b' as above. This can only be if (A5) holds true. The case $x_1 \succ x^0$, $x_2 \sim x^1$ is treated similarly.

Stage 4 (Construction of Continuous and Nondecreasing f). For every $0 \leqslant p \leqslant 1$ we define $f(p) := 1 - V\{(x^0, x^1); (p, 1-p)\}$. Obviously, $f(0) = 0$, $f(1) = 1$, and, by definition of $x^{1/2}$, $f(\frac{1}{2}) = \frac{1}{2}$. Further f is nondecreasing, by Axiom 2′a and (A4). Also f is continuous: For every $x^{a/2^n}$ there exists, by

Axiom 3, a p such that $\{(x_0, x_1); (p, 1-p)\} \sim x^{a/2^n}$, i.e., $f(p) = 1 - a/2^n$. This shows that the range of f is dense in $[0, 1]$. The nondecreasing f cannot make "jumps," and must be continuous.

Stage 5 (Surjectivity of U). We show now that $U(X) = [0, 1]$. Let $\mu \in [0, 1]$. Take p such that $1 - \mu = f(1-p) = 1 - V\{(x^0, x^1); (1-p, p)\}$; so $V\{(x^0, x^1); (1-p, p)\} = \mu$. Then, by R.2, there exists $x \sim \{(x^0, x^1); (1-p, p)\}$. By (A4), $V(x) = \mu$; so $U(x) = \mu$.

Stage 6 (U and V are Representing). The derivation in this stage will not be elementary. Of course, if U is representing for $\succcurlyeq$ on X, then V is representing for $\succcurlyeq$ on Y, so we only derive the former. By (A3), it suffices to assume that there are x', x'' such that $x'' \succ x'$ and $U(x'') = U(x')$, and derive a contradiction. We define $\mu = U(x'') = U(x')$.

Note that there does exist a function, say U', that represents $\succcurlyeq$ on X: Choose for each indifference class $\{x' \in X : x' \sim x\}$ a *probability equivalent*, i.e., a p such that $\{(x^0, x^1); (1-p, p)\} \sim x$. By Axiom 3 there exists at least one such p. Then define $U'(x') = p$ for all x' from the indifference class. By Axiom 2'a, $x'' \succ x'$ must imply $U'(x'') > U'(x')$. This, and constancy of U' on outcome $\sim$ indifference classes, shows that U' is representing. The existence of a representing U' excludes the existence of an uncountable number of disjoint preference intervals $\{x \in X : x'' \succcurlyeq x \succcurlyeq x'\}$ for $x'' \succ x'$, as the latter would lead to uncountably many distinct rational numbers, one from each interval $]U'(x'), U'(x'')[$. So it suffices, for contradiction, to derive an uncountable number of such preference intervals.

Either $\mu \neq 0$ or $\mu \neq 1$; say the latter. Take any $\mu < \nu < 1$. By Stage 5, there exists x_ν such that $U(x_\nu) = \nu$. Now, by (A5), $V\{(x'', x_\nu); (\frac{1}{2}, \frac{1}{2})\} = (\mu + \nu)/2 = V\{(x', x_\nu); (\frac{1}{2}, \frac{1}{2})\}$, whereas, by Axiom 2'b, $\{(x'', x_\nu); (\frac{1}{2}, \frac{1}{2})\} \succ \{(x', x_\nu); (\frac{1}{2}, \frac{1}{2})\}$. We take $x''_{(\mu+\nu)/2} \sim \{(x'', x_\nu); (\frac{1}{2}, \frac{1}{2})\}$ and $x'_{(\mu+\nu)/2} \sim \{(x', x_\nu); (\frac{1}{2}, \frac{1}{2})\}$. Then $x''_{(\mu+\nu)/2} \succ x'_{(\mu+\nu)/2}$, but $U(x''_{(\mu+\nu)/2}) = U(x'_{(\mu+\nu)/2})$. Such outcomes can be constructed for each ν between μ and 1, and $\{\{x \in X : x'_{(\mu+\nu)/2} \preccurlyeq x \preccurlyeq x''_{(\mu+\nu)/2}\}\}_{\nu \in]\mu, 1[}$ gives an uncountable number of mutually disjoint preference intervals.

Stage 7 (Jensen's Equation). Fix $\mathbf{p} = (p_1, ..., p_n)$ in this stage. Because U represents $\succcurlyeq$ over outcomes, and because of Lemma 3.1, we can write $V\{(x_1, ..., x_n); \mathbf{p}\} = W(U(x_1), ..., U(x_n))$ for a function W. For simplicity of notation, from now on we identify outcomes with their U values in this stage. The domain of W is the set $[0, 1]^n_\uparrow$ of all $\mathbf{x} = (x_1, ..., x_n) \in [0, 1]^n$ with $0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1$. We show that W satisfies Jensen's equation; i.e., for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n_\uparrow$,

$$W\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right) = \frac{W(\mathbf{x}) + W(\mathbf{y})}{2}.$$

Define $c_i \sim \{\{x_i, y_i\}; (\frac{1}{2}, \frac{1}{2})\}$ for all i. Then, by (A5) and the definitions, $c_i = (x_i + y_i)/2$ for all i. As $c_1 \leqslant \cdots \leqslant c_n$, $\mathbf{c} \in [0, 1]^n_\uparrow$. Let $x \sim \{\mathbf{x}; \mathbf{p}\}$, $y \sim \{\mathbf{y}; \mathbf{p}\}$. By Axiom 4, $\{\mathbf{c}; \mathbf{p}\} \sim \{\{x, y\}; (\frac{1}{2}, \frac{1}{2})\}$. So, substituting (A5), we get $W(\mathbf{c}) = (U(x) + U(y))/2$. The latter is equal to $(x+y)/2$ and to $(W(\mathbf{x}) + W(\mathbf{y}))/2$, and $W(\mathbf{c}) = W((\mathbf{x}+\mathbf{y})/2)$. Thus Jensen's equation follows.

Stage 8 (W is Linear and Gives the AU Form). By standard techniques it can be shown that W as obtained in Stage 7, must be linear. In general, solutions of Jensen's equation exist that are nonlinear, but these are very irregular. As will be elaborated below, the monotonicity of W excludes those irregular cases. From the definition of f, it follows that the weights employed in the linear W are exactly what they should be according to AU. The remainder of this stage gives a formal derivation of linearity of W for a fixed $(p_1, ..., p_n)$.

Define $e_1 := (1, ..., 1), ..., e_2 := (0, 1, ..., 1), ..., e_n := (0, ..., 0, 1)$. On a rank-ordered cone it is convenient to take $e_1, ..., e_n$ as basis, because then standard results of Aczél [1] can be applied literally. The details are as follows. Define $W': \mathbb{R}^n_+ \to \mathbb{R}$ in the following way. If $\sum_{i=1}^n y_i \leqslant 1$, $W'(\mathbf{y}) := W(\sum_{i=1}^n y_i e_i)$. On the domain covered so far, W' satisfies Jensen's equation; in particular, given $W'(0, ..., 0) = 0$, $W'(\mathbf{x}/2^m) = W'(\mathbf{x})/2^m$ for all m. For a general $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$, find any 2^m large enough to ensure that, for $y_i := x_i/2^m$, $\sum_{i=1}^n y_i \leqslant 1$. Next define $W'(\mathbf{x}) := 2^m W'(\mathbf{y})$. From Jensen's equation on the domain covered before, it follows that the definition of $W'(\mathbf{x})$ does not depend on the particular choice of m and $\mathbf{y}$, and that in fact W' satisfies Jensen's equation throughout its domain. For the fixed $\mathbf{p}$ we get, by the definition of f, $f(p_1 + \cdots + p_n) = W(e_1) = W'(1, 0, ..., 0)$, $f(p_2 + \cdots + p_n) = W(e_2) = W'(0, 1, 0, ..., 0), ..., f(p_n) = W(e_n) = W'(0, ..., 0, 1)$. The proof is complete if a contradiction is derived from nonlinearity of W'.

This follows from Aczél [1, Sect. 2.1 and 2.2.3, extended in Sect. 5.1.1]. Because $W'(0, ..., 0) = 0$, W' satisfies Cauchy's equation ($W'(\mathbf{x}+\mathbf{y}) = W'(\mathbf{x}) + W'(\mathbf{y})$). Functions satisfying Cauchy's equation, but nonlinear, are very "irregular," and for instance must be unbounded from both sides. The function W', however, is nonnegative, so bounded from below. It follows that W' is linear. This completes the proof of Stage 8.

A rereading of the proof, plus substitution of AU, shows that any choice of $U(0) = \sigma$, $U(1) = \tau$, for general $\tau > \sigma$ instead of $\tau = 1$, $\sigma = 0$, could be made and would uniquely determine a positive affine transform of the function U as in the proof above, and that the function f is uniquely determined.

Finally, we relax Assumption A1. If all outcomes are indifferent then, by R.2, all prospects are indifferent and the result is trivial. So suppose there are nonindifferent outcomes. We fix some $x^1 \succ x^0$. For each $y \succcurlyeq x^1 \succ x^0 \succcurlyeq z$, we can construct an AU representation for prospects with outcomes $\{x \in X : y \succcurlyeq x \succcurlyeq z\}$, similar to the construction under Assumption A1. By the uniqueness results for U and f as established above, this AU representation for outcomes $\{x \in X : y \succcurlyeq x \succcurlyeq z\}$ can be made to coincide with the AU representation established above, which uniquely determines the extended AU representation. As the outcomes involved in any prospect are finite, thus bounded, the AU representation is uniquely determined for all prospects. This completes the proof of Theorem 3.2.

Note that the set of $\sim$ indifference classes of the outcome set is isomorphic to an interval; the set X, when endowed with the order topology, is a connected topological space.

References

1. J. Aczél, "Lectures on Functional Equations and Their Applications," Academic Press, New York, 1966.
2. M. Allais, Fondements d'une théorie positive des choix comportant un risque et critique des postulats et axiomes de l'école américaine, *in* "Colloq. Internat. CNRS **40**, Econométrie," pp. 257–332, Centre National de la Recherche Scientifique, Paris, 1953. Translated into English, with additions, as The foundations of a positive theory of choice involving risk and a criticism of the postulates and axioms of the American school, *in* "Extended Utility Hypotheses and the Allais Paradox" (M. Allais and O. Hagen, Eds.), pp. 27–145, Reidel, Dordrecht, 1979.
3. M. Allais, The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function, *in* "Risk, Decision and Rationality" (B. R. Munier, Ed.), pp. 233–289, Reidel, Dordrecht, 1988.
4. S. H. Chew, The rank-dependent quasilinear mean, unpublished manuscript, Department of Economics, University of California, Irvine, 1989.
5. W. Edwards, Subjective probabilities inferred from decisions, *Psych. Review* **69** (1962), 109–135.
6. P. C. Fishburn, On Handa's 'New theory of cardinal utility' and the maximization of expected return, *J. Political Econ.* **86** (1978), 321–324.
7. D. Kahneman and A. Tversky, Prospect theory: An analysis of decision under risk, *Econometrica* **47** (1979), 263–291.
8. R. D. Luce, Rank-dependent, subjective expected-utility representations, *J. Risk Uncertainty* **1** (1988), 305–332.
9. Y. Nakamura, Multi-symmetric structures and non-expected utility, *J. Math. Psych.* **36** (1992), 375–395.
10. M. G. Preston and P. Baratta, An experimental study of the Auction value of an uncertain outcome, *Amer. J. Psych.* **61** (1948), 183–193.
11. J. Quiggin, Risk perception and risk aversion among Australian farmers, *Austral. J. Agricult. Econ.* **25** (1981), 160–169.
12. J. Quiggin, A theory of anticipated utility, *J. Econ. Behav. Organization* **3** (1982), 323–343.
13. J. Quiggin, "Generalized Expected Utility Theory—The Rank-Dependent Model," Kluwer Academic Press, Dordrecht, 1992.

14. D. SCHMEIDLER, Subjective probability and expected utility without additivity, *Econometrica* **57** (1989), 571–587.
15. A. TVERSKY AND D. KAHNEMAN, Advances in prospect theory: Cumulative representation of uncertainty, *J. Risk Uncertainty* **5** (1992), 297–323.
16. P. P. WAKKER, Separating marginal utility and probabilistic risk aversion, *Theory and Decision* **36** (1994), 1–44.
17. J. A. WEYMARK, Generalized Gini inequality indices, *Math. Social Sci.* **1** (1981), 409–430.
18. M. E. YAARI, The dual theory of choice under risk, *Econometrica* **55** (1987), 95–115.