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Continuity of Transformations

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1. INTRODUCTION

Let $u: \Re \to \Re$ be a continuous function. Let $\varphi: u(\Re) \to \Re$ be a "transformation" (which in our terminology does not have to be bijective). We set $v := \varphi \circ u$. It is well known that continuity of φ implies continuity of v. We shall consider the reversed question: Does continuity of v imply continuity of φ? Elementary as this question may be, we did not find a place in literature where the answer is given. In fact it is our experience that the probability that a mathematician at first sight will gamble on the wrong answer, is a strictly increasing function of his familiarity with elementary analysis, and is always above 1/2. This paper will answer the reversed question above, in a somewhat more general setting, and give applications.

2. THE MAIN THEOREM

Throughout this section Γ denotes a connected topological space, $u: \Gamma \to \Re$ and $v: \Gamma \to \Re$ are continuous functions, and $\varphi: u(\Gamma) \to \Re$ is a transformation such that $v = \varphi \circ u$. This section studies the question whether φ is continuous.

LEMMA 2.1. *If φ is nondecreasing or nonincreasing, then it is continuous.*

Proof. Since φ is a nondecreasing, or nonincreasing, function from a (connected) interval $u(\Gamma)$ *onto* the (connected) interval $v(\Gamma)$, φ cannot make "jumps," and must be continuous. ∎

LEMMA 2.2. *The transformation φ has the intermediate value property.*

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Proof. Let $\lambda, \nu \in \Gamma$, with $\lambda < \nu$. Let $\varphi(\lambda) < \varphi(\nu)$. [The case $\varphi(\lambda) > \varphi(\nu)$ is analogous.] Let $\varphi(\lambda) < \beta < \varphi(\nu)$ for some β. We must find a μ between λ and ν such that $\varphi(\mu) = \beta$. Let $G := \{(u(\sigma), v(\sigma)) \in \Re^2: \sigma \in \Gamma\}$. G is the graph of φ. Since v and u, thus $\sigma \mapsto (u(\sigma), v(\sigma))$, are continuous, G is connected. Let $V = \{(\mu, \alpha) \in G: \mu \leqslant \lambda$ or $[\lambda \leqslant \mu \leqslant \nu$ and $\alpha \leqslant \beta]\}$, and $W = \{(\mu, \gamma) \in G: \mu \geqslant \nu$ or $[\lambda \leqslant \mu \leqslant \nu$ and $\gamma \geqslant \beta]\}$. V and W are closed subsets of G with $V \cup W = G$, $(\lambda, \varphi(\lambda)) \in V$ so $V \neq \emptyset$, and $(\nu, \varphi(\nu)) \in W$ so $W \neq \emptyset$. By connectedness of G, $V \cap W \neq \emptyset$. Say $(\mu, \beta') \in V \cap W$. It follows that $\beta' = \beta$ and that $\lambda < \mu < \nu$. So $\varphi(\mu) = \beta$, and the intermediate value property has been established. ∎

EXAMPLE 2.3. The transformation φ is not necessarily continuous: Let $\varphi: \Re \to \Re$ assign 0 to 0, and $\sin(1/\mu)$ to every $\mu \neq 0$. Let G be the graph of φ, and $\Gamma = G$. Let v be the projection on the second coordinate, u the projection on the first. Then indeed Γ is connected, v and u are continuous, and $v = \varphi \circ u$. Still φ is not continuous in 0. ∎

LEMMA 2.4. *If φ is bijective then it is either strictly increasing or strictly decreasing.*

Proof. It is sufficient to show, for any $\lambda < \mu < \nu$ in the domain of φ, that either $\varphi(\lambda) < \varphi(\mu) < \varphi(\nu)$, or $\varphi(\lambda) > \varphi(\mu) > \varphi(\nu)$. Say, for $\lambda < \mu < \nu$, that $\varphi(\lambda) < \varphi(\nu)$. We show that $\varphi(\lambda) < \varphi(\mu) < \varphi(\nu)$. If we had $\varphi(\mu) < \varphi(\lambda)$, then by Lemma 2.2 any value between $\varphi(\lambda)$ and $\varphi(\mu)$ would be assigned by φ to at least two arguments, one between λ and μ, and one between μ and ν. By bijectivity of φ this cannot hold. An analogous violation of bijectivity occurs if $\varphi(\mu) > \varphi(\nu)$. Also the equalities $\varphi(\mu) = \varphi(\lambda)$ and $\varphi(\mu) = \varphi(\nu)$ obviously violate bijectivity of φ. The only possibility left is $\varphi(\lambda) < \varphi(\mu) < \varphi(\nu)$. ∎

The following lemma shows that in the main case of interest for us, where Γ is a convex subset of a Euclidean space, the transformation φ is continuous.

LEMMA 2.5. *The transformation φ is continuous if Γ is arcwise connected.*

Proof. It is sufficient to show that any sequence $(u(\mu_k))_{k=1}^{\infty}$ in $u(\Gamma)$, converging to a $u(\mu)$ in $u(\Gamma)$, has a subsequence $(u(\mu_{k_i}))_{i=1}^{\infty}$ such that $\lim_{i \to \infty} \varphi(u(\mu_{k_i})) = \varphi(u(\mu))$. So let $(u(\mu_k))$ converge to $u(\mu)$. We may assume $u(\mu_k) \neq u(\mu)$ for all k. There must exist a subsequence $(u(\nu_j))_{j=1}^{\infty}$ of $(u(\mu_k))_{k=1}^{\infty}$ which either strictly increases or strictly decreases. We use the arcwise connectedness of Γ by taking an arc λ from ν_1 to μ, i.e., $\lambda: [0, 1] \to \Gamma$ is continuous, with $\lambda(0) = \nu_1$, $\lambda(1) = \mu$. Now $u \circ \lambda$ is continuous, $(u \circ \lambda)(0) = u(\nu_1)$, $(u \circ \lambda)(1) = u(\mu)$. By the intermediate value property, there exist $(\sigma_j)_{j=1}^{\infty}$ in $[0, 1]$ such that $(u \circ \lambda)(\sigma_j) = u(v_j)$ for all j. We define $\tau_j := \lambda(\sigma_j)$ for all j. Then $u(\tau_j) = u(v_j)$ for all j. Since $\lambda([0, 1])$ is compact, $(\tau_j)_{j=1}^{\infty}$ has a convergent subsequence $(\tau_{j_i})_{i=1}^{\infty}$, with limit say τ. Also $(u(\tau_{j_i}))_{i=1}^{\infty}$ and $(v(\tau_{j_i}))_{i=1}^{\infty}$ must converge to $u(\tau)$, respectively $v(\tau)$. This can hold only if $u(\tau) = u(\mu)$, and $\lim_{i \to \infty} \varphi(u(v_{j_i})) = \lim_{i \to \infty} \varphi(u(\tau_{j_i})) = \lim_{i \to \infty} v(\tau_{j_i}) = v(\tau) = \varphi(u(\tau)) = \varphi(u(\mu))$. So $(\mu_{k_i})_{i=1}^{\infty} = (v_{j_i})_{i=1}^{\infty}$ is taken. ∎

THEOREM 2.6. *The transformation φ is continuous if φ satisfies one of the following conditions*:

(1) *φ is nondecreasing*;

(2) *φ is nonincreasing*;

(3) *φ is bijective*;

(4) *The domain Γ of φ is arcwise connected.*

Proof. This follows from Lemmas 2.1, 2.2, 2.4, and 2.5. ∎

3. APPLICATIONS

The first application is well known in mathematical economics. Suppose $\succcurlyeq$ is a *preference relation* on $\mathfrak{R}^n_+$, i.e., $\succcurlyeq$ is a binary relation on $\mathfrak{R}^n_+$ which is *transitive* (for all $x, y, z \in \mathfrak{R}^n_+$, if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$) and *complete* (for all $x, y \in \mathfrak{R}^n_+$, $x \succcurlyeq y$ or $y \succcurlyeq x$). The elements of $\mathfrak{R}^n_+$ are *commodity bundles*, and $\succcurlyeq$ reflects the opinion of a consumer, $x \succcurlyeq y$ meaning that the consumer thinks x is at least as good as y. A function $u: \mathfrak{R}^n_+ \to \mathfrak{R}$ is a *utility function* if, for all commodity bundles x, y, $[x \succcurlyeq y \Leftrightarrow u(x) \geqslant u(y)]$. It is straightforwardly verified that a utility function is unique up to a strictly increasing transformation, i.e., if u is a utility function for $\succcurlyeq$ then the class of all utility functions for $\succcurlyeq$ is a class of the form $\{v: \mathfrak{R}^n_+ \to \mathfrak{R}: v = \varphi \circ u$ for a strictly increasing transformation $\varphi\}$. This indeterminateness of a utility function was the starting point of a controversy between economists in the beginning of this century; for a recent account, and references, see Cooter and Rappoport [2].

In literature one is usually interested only in continuous utility functions. It is well known that a continuous utility function exists for a preference relation if and only if the preference relation is *continuous*, i.e., for all commodity bundles x the sets $\{y \in \mathfrak{R}^n_+ : y \succcurlyeq x\}$ and $\{y \in \mathfrak{R}^n_+ : x \succcurlyeq y\}$ are closed (see for instance Debreu [3–5]). Obviously, if u is a continuous utility function, then for any continuous strictly increasing transformation φ also $\varphi \circ u$ is a continuous utility function. Lemma 2.1 shows that no other continuous utility functions exist. So a continuous utility function is unique up to a continuous strictly increasing transformation. We found that many textbooks on mathematical economics are not explicit w.r.t. this point.

Next we shall briefly sketch the application of Theorem 2.6 as used in Wakker [7, Sect. IV.4] and Wakker [8]. Again Γ denotes a nonempty connected topological space. Let $n \in \mathbb{N}$, $n \geqslant 2$. The Cartesian product Γ^n is endowed with the product topology. Its elements are called *alternatives*. An alternative $x \in \Gamma^n$ has ith coordinate x_i. There is given a preference relation $\succcurlyeq$ on Γ^n. We write $x \simeq y$ for $[x \succcurlyeq y$ and $y \succcurlyeq x]$. Obviously $\simeq$ is an equivalence relation.

Notation 3.1. For $1 \leqslant i \leqslant n, \alpha \in \Gamma, x \in \Gamma^n$,

$$x_{-i}\alpha := (x \text{ with } x_i \text{ replaced by } \alpha).$$

We shall assume that every coordinate has influence on the preference relation, i.e., *for all coordinates i there exists an alternative x, and an* $\alpha \in \Gamma$, *such that not* $x_{-i}\alpha \simeq x$.

We say that $\succcurlyeq$ is *weakly separable* if, for all i, x, y, α, β, we have

$$[x_{-i}\alpha \succcurlyeq x_{-i}\beta] \Leftrightarrow [y_{-i}\alpha \succcurlyeq y_{-i}\beta].$$

This is the analogue of monotonicity in $\mathfrak{R}^n_+$. Further we shall need:

DEFINITION 3.2. The binary relation $\succcurlyeq$ satisfies *equivalence cardinal coordinate independence* (ECCI) if for all $i, j, x, ..., \delta$ we have

$$\begin{aligned} &x_{-i}\alpha \simeq y_{-i}\beta \qquad \text{and} \qquad v_{-j}\alpha \simeq w_{-j}\beta \qquad \text{and} \\ &x_{-i}\gamma \simeq y_{-i}\delta \\ &\qquad \Rightarrow [v_{-j}\gamma \simeq w_{-j}\delta]. \end{aligned}$$

Note that the above condition cannot be immediately expressed in the derived tradeoffs of Wakker [9].

THEOREM 3.3. *The following two statements are equivalent for the binary relation* $\succcurlyeq$ *on* Γ^n:

(i) *There exist real* $(\lambda_j)_{j=1}^n$ *and a continuous function* $U: \Gamma \to \mathfrak{R}$, *such that* $x \mapsto \sum_{j=1}^n \lambda_j U(x_j)$ *is a utility function for* $\succcurlyeq$.

(ii) *The binary relation* $\succcurlyeq$ *is a continuous weakly separable ECCI preference relation.*

Sketch of Proof. The implication (i) $\Rightarrow$ (ii) is straightforward. So we assume (ii) and derive (i). As demonstrated in Wakker [7, Proof of Theorem IV.4.3] (see also Remark III.7.3 in Wakker [9]) there exists a utility function V of the form $V\colon x \mapsto \sum_{j=1}^{n} V_j(x_j)$ for $\succcurlyeq$, with all V_j's continuous and nonconstant. ECCI gives

$$V_i(\alpha) - V_i(\beta) = \sum_{k \neq i} [V_k(y_k) - V_k(x_k)] = V_i(\gamma) - V_i(\delta)$$

and

$$V_j(\alpha) - V_j(\beta) = \sum_{k \neq j} [V_k(w_k) - V_k(v_k)] \Rightarrow [V_j(\alpha) - V_j(\beta) = V_j(\gamma) - V_j(\delta)]. \tag{3.1}$$

Setting $\alpha = \beta, x = y, v = w$, we have $[V_i(\gamma) = V_i(\delta)] \Rightarrow [V_j(\gamma) = V_j(\delta)]$. Hence there exists a transformation φ such that $V_j = \varphi \circ V_i$. Analogously there exists a transformation ψ such that $V_i = \psi \circ V_j$. Thus, $\varphi\colon V_i(\Gamma) \to V_j(\Gamma)$ is bijective. Since V_i and V_j are continuous, φ is continuous by Theorem 2.6.

Now let $V_i(\zeta)$ be an arbitrary element of $V_i(\Gamma)$, the domain of φ. No V_j is constant, and all $V_j(\Gamma)$ are nondegenerate intervals. Since $n \geqslant 2$ and $V_j = \varphi \circ V_i$ with φ continuous, there exists an open interval S around $V_i(\zeta)$ so small that for all $V_i(\alpha)$, $V_i(\beta)$ in S, there are x and y for which $V_i(\alpha) - V_i(\beta) = \sum_{k \neq i} [V_k(y_k) - V_k(x_k)]$ and there are v and w for which $[\varphi(V_i(\alpha)) - \varphi(V_i(\beta))] = V_j(\alpha) - V_j(\beta) = \sum_{k \neq j} [V_k(w_k) - V_k(v_k)]$. Setting $\beta = \gamma$, and finding appropriate x, y, v, w, by (3.1) we get, for all $V_i(\alpha)$, $V_i(\beta)$, $V_i(\delta) \in S$: $[V_i(\alpha) - V_i(\beta) = V_i(\beta) - V_i(\delta)] \Rightarrow [V_j(\alpha) - V_j(\beta) = V_j(\beta) - V_j(\delta)]$. Thus, on S, φ satisfies Jensen's equality $\varphi((\sigma + \tau)/2) = [\varphi(\sigma) + \varphi(\tau)]/2$. By Theorem 1 of Section 2.1.4 of Aczél [1], or by (88) of Section 3.7 of Hardy, Littlewood, and Pólya [6], φ must be affine on S. Hence it has second derivative zero in all $V_i(\zeta)$. Consequently φ must be affine on $V_i(\Gamma)$.

We have shown that each V_j is an affine transform of V_1. The proof is completed by specifying U and the λ_j's as follows: For arbitrary fixed α and β in Γ with $V_1(\alpha) \neq V_1(\beta)$, set $U(\cdot) := V_1(\cdot) - V_1(\alpha)$ and set $\lambda_j := [V_j(\beta) - V_j(\alpha)]/[V_1(\beta) - V_1(\alpha)]$, $j = 1, ..., n$. ∎

Applications of the above theorem to the theory of economic indexes, and to production theory, have been indicated in Wakker [8].

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