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CONTINUITY OF PREFERENCE RELATIONS FOR SEPARABLE TOPOLOGIES*

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1. INTRODUCTION

In Debreu (1954, 1959) some classical results were provided for consumer theory. Necessary and sufficient conditions were given for the existence of (continuous) utility functions to represent preference relations of consumers. Further results are given in Bowen (1968), Jaffray (1975), Richter (1980), and Chateauneuf (1985).

A basic procedure is to first derive the utility function on a countable subset. Next, some denseness property of this subset is used to extend the domain of the utility function. In view of this, a good starting point for the study of derivations of utility functions may be the study of preference relations which are continuous w.r.t. a *separable* topology (i.e., a topology with a countable dense subset). This will be the topic of the present paper. It is well-known (see Richter 1980, Remark 1; or Fleischer 1961, p. 50) that for such preference relations no utility function has to exist.

Let X be a *set of alternatives*, and $\geq$ a binary (preference) relation on X. We write $x \leq y$ if $y \geq x$; $x > y$ or $y < x$ if $x \geq y$ and not $y \geq x$; and $x \approx y$ if $x \geq y$ and $y \geq x$. The binary relation $\geq$ is a *weak order* if it is complete (i.e. $x \geq y$ or $y \geq x$ for all $x, y \in X$) and transitive. Its *natural topology* $\Upsilon_n(\geq)$ is the smallest topology for which $\geq$ is *continuous*; i.e., $\Upsilon_n(\geq)$ is the smallest topology which contains all sets of the form $\{x \in X: x > y\}$ and $\{x \in X: x < y\}$.

The *lexicographic ordering* $\geq_L$ on $\mathbb{R} \times \{0, 1\}$ is defined by

$$(x_1, x_2) \geq_L (y_1, y_2)$$

if

$$x_1 > y_1 \quad \text{or} \quad [x_1 = y_1 \;\&\; x_2 \geq y_2].$$

For a set V with a binary relation $\geq'$ on it, $(X, \geq)$ is *embeddable in* $(V, \geq')$ if there exists an *embedding* $f: X \to V$; i.e., $x \geq y \Leftrightarrow f(x) \geq' f(y)$ for all $x, y \in X$. A *utility function* is an embedding in $(\mathbb{R}, \geq)$.

(A topology on X) is *second countable* if it has a countable basis. Then X is separable and every subset of X is second countable. A topology on X is *con-*

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nected if there is no non-trivial subset of X which is simultaneously open and closed.

2. THE MAIN THEOREM

In this section, we provide a characterization of weak orders which are continuous w.r.t. a separable topology. Note for the following preparatory lemma that $\mathbb{R} \times \{0, 1\}$ with the natural topology induced by the lexicographic ordering is not second countable, so that there is no direct way to conclude that any subset of $\mathbb{R} \times \{0, 1\}$ is separable.

LEMMA 2.1. *Every subset A of $\mathbb{R} \times \{0, 1\}$ is separable w.r.t. the restriction of $\Upsilon_n(\geq_L)$ to A.*

PROOF. Note that the restriction of $\Upsilon_n(\geq_L)$ to $\mathbb{R} \times \{1\}$ consists of unions of sets of the form $]\alpha, \beta[\times \{1\}$ and $[\alpha, \beta[\times \{1\}$.

Now let $A \subset \mathbb{R} \times \{0, 1\}$. We construct a countable dense subset C in A, where C_0 will be $C \cap \mathbb{R} \times \{0\}$, and C_1 will be $C \cap \mathbb{R} \times \{1\}$. For $k = 0, 1$, C_k will firstly contain an element in every non-empty $\{(\alpha, k): r^1 < \alpha < r^2\} \cap A$ with r^1, r^2 rational; further C_k will contain all endpoints in A of gaps of A in $\mathbb{R} \times \{k\}$. Here a *gap* of A in $\mathbb{R} \times \{k\}$ is an element, maximal w.r.t. inclusion, of the collection

$$\{A^* \times \{k\}: \quad A^* \times \{k\} \text{ is contained in } (\mathbb{R} \times \{0, 1\})\backslash A, \text{ and } A^* \subset \mathbb{R} \text{ is convex and has non-empty interior}\}.$$

Different gaps must be disjoint, so there can be at most countably many. Hence, indeed C_k can be taken countable. Let $C := C_0 \cup C_1$.

There remains to be shown that C is dense. So we consider an open non-empty subset of A. Such a set can be written as $D \cap A$ for some open D in $\mathbb{R} \times \{0, 1\}$. To show is that $C \cap D \cap A$ is non-empty. Let $(\alpha, j) \in D \cap A$. Say $j = 0$. (The proof for $j = 1$ is similar.) There must exist a, b with $(\alpha, 0) \in \{z: a <_L z <_L b\} \subset D$. We can take $b = (\alpha, 1)$. From $a = (a_1, a_2) <_L (\alpha, 0)$, it follows that $a_1 < \alpha$. Let $\tilde{a}_1$ be a rational number such that $a_1 < \tilde{a}_1 < \alpha$. If no b'_1 exists s.t. $(b'_1, 0) \in A$ and $\tilde{a}_1 < b'_1 < \alpha$, then $(\alpha, 0)$ is the right endpoint of a gap as described above, and $(\alpha, 0)$ itself is in C. If a b'_1 exists s.t. $(b'_1, 0) \in A$ and $\tilde{a}_1 < b'_1 < \alpha$, then take $\tilde{b}_1$ rational s.t. $b'_1 < \tilde{b}_1 < \alpha$. Then $(b'_1, 0) \in (]\tilde{a}_1, \tilde{b}_1[\times \{0\}) \cap A \neq \emptyset$. So C intersects the latter set.

Always C has an element in $D \cap A$. □

The following is our main theorem.

THEOREM 2.2. *The following three statements are equivalent for the binary relation $\geq$ on the set X:*

(i) *$(X, \geq)$ has an embedding f in $(\mathbb{R} \times \{0, 1\}, \geq_L)$.*
(ii) *$\geq$ is a weak order with a separable natural topology $\Upsilon_n(\geq)$.*

(iii) $\geq$ *is a weak order that is continuous w.r.t. a separable topology* Υ.

PROOF. First we derive the implication (i) $\Rightarrow$ (ii). Suppose (*i*) holds. By Lemma 2.1, the set $f(X) \subset \mathbb{R} \times \{0, 1\}$ is separable w.r.t. $\Upsilon_n(\geq_L)$, therefore has a countable dense subset $\tilde{C}$. Let $C \subset X$ be countable s.t. $f(C) = \tilde{C}$. We show that C is dense. Let S be a non-empty basis element of $\Upsilon_n(\geq_L)$; so S has one of the following forms: $\{y \in X: y > w\}$, $\{y \in X: y < v\}$, or $\{y \in X: v > y > w\}$. Then $f(S)$ is open in $f(X)$, and is non-empty since S is empty. Therefore $f(S)$ must contain an element of $\tilde{C}$. The entire original under f of that element of $\tilde{C}$ must be contained in S. Hence, S contains an element of C. Indeed C is dense, and (*ii*) follows.

The implication (*ii*) $\Rightarrow$ (*iii*) follows by taking $\Upsilon = \Upsilon_n$ $(\geq)$. Finally, we assume (*iii*) and derive (*i*). Let C be a countable dense subset of X. If X has "maximal" elements x (i.e., $x \geq y$ for all y), we let C contain at least one such maximal element. Also let C contain at least one minimal element, if one exists. There exists a function $\tilde{\varphi}$, embedding $(C, \geq)$ in $(Q[0, 1], \geq)$, with $Q[0, 1]$ the set of rational numbers in [0, 1] (see Jaffray 1975, the Lemma). We extend $\tilde{\varphi}$ to all of X by defining $\varphi(x) := \inf \{\tilde{\varphi}(y): y \in C, y \geq x\}$. This infimum is between 0 and 1. And of course, $x \geq y \Rightarrow \varphi(x) \geq \varphi(y)$. Still $[x > y$ and $\varphi(x) = \varphi(y)]$ can occur, so φ is not necessarily an embedding.

We consider, for every $\alpha \in \varphi(X)$, $\varphi^{-1}(\alpha)$. Suppose $v > w > x$, $\varphi(v) = \varphi(w) = \varphi(x) = \alpha$, and suppose $w \approx w'$ for no $w' \in C$. A contradiction will follow. Since $\{z: v > z > x\}$ is open and non-empty, it contains an element w' of C. Either $w' > w$ or $w > w'$, say the latter. Then $v > w > w'$. Again, C must contain an element w'' with $v > w'' > w'$. This implies $\varphi(w'') > \varphi(w')$. Then $\varphi(x) = \varphi(v) \geq \varphi(w'') > \varphi(w') \geq \varphi(x)$; i.e., a contradiction has been obtained. Apparently, if $\varphi^{-1}(\alpha)$ contains three equivalence classes, then the middle one must contain an element of C. This also reveals that $\varphi^{-1}(\alpha)$ cannot contain four (or more) equivalence classes; then the middle two would have to contain elements of C, and hence be assigned different φ-values. This is impossible since both φ-values should be equal to α. So $\varphi^{-1}(\alpha)$ can contain at most three equivalence classes, of which at most one can contain an element of C; this one is the middle equivalence class if $\varphi^{-1}(\alpha)$contains three equivalence classes.

Now, in preparation for the definition of f, we first define $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$: $X \to \mathbb{R} \times \{0, 1\}$. We set $\tilde{f}_1 = \varphi(x)$ for all x. Furthermore, $\tilde{f}_2(x) = 0$ if $\varphi^{-1}(\varphi(x))$ contains only the equivalence class of x. If $\varphi^{-1}(\varphi(x))$ contains two equivalence classes, then $\tilde{f}_2$ assigns 0 to every element of the worst (in terms of $\geq$) equivalence class, and 1 to every element of the best equivalence class. If $\varphi^{-1}(\varphi(x))$ contains three equivalence classes, then $\tilde{f}_2$ assigns 0 to every element of the worst and middle, and 1 to every element of the best equivalence class. Thus, with only one exception, we have $[x \geq y \Rightarrow \tilde{f}(x) \geq_L \tilde{f}(y)]$ and $[x > y \Rightarrow \tilde{f}(x) >_L \tilde{f}(y)]$ for all x, y. The exception concerns the case where x is from a middle class, and y from a worst class of a $\varphi^{-1}(\alpha)$. Then $x \approx x^j$ for some $x^j \in C$. For these exceptions a final rearrangement is performed. C is countable; hence, we can also enumerate as x^1, x^2, ... the x^j's in C occurring in a middle equivalence class as above. Finally we define f by

$$f: x \mapsto \left(\left[\tilde{f}_1(x) + \sum_{j:x^j \leqslant x} 2^{-j}\right], \tilde{f}_2(x)\right).$$

This gives:

$$[x \geq y] \Rightarrow [\tilde{f}(x) \geq_L \tilde{f}(y) \text{ and } f(x) \geq_L f(y)];$$

$$[x > y \text{ and } \tilde{f}(x) >_L \tilde{f}(y)] \Rightarrow [f(x) >_L f(y)];$$

$$[x > y \text{ and } \tilde{f}(x) = \tilde{f}(y)] \Rightarrow [x \approx x^j \text{ for some } j, \text{ so } f(x) >_L f(y)].$$

Conclusion: f embeds $(X, \geq)$ in $(\mathbb{R} \times \{0, 1\}, \geq_L)$. Note that f is constructed such that, for all $\alpha \in \mathbb{R}$:

(2.1) $$[(\alpha, 1) \in f(X)] \Rightarrow [(\alpha, 0) \in f(X)].$$ □

It is known (see for instance Chipman 1960, Theorem 3.1; or Chipman 1971, Appendix) that any $(X, \geq)$, with $\geq$ a weak order, can be embedded in $(\mathbb{R}^v, \geq_L)$ where v is an ordinal number. Chipman addressed the question concerning the smallest ordinal number v as above, for several cases of $(X, \geq)$. In particular, he showed that in some "natural cases" the minimal v may still be uncountable.

Our theorems consider embeddability in $(\mathbb{R} \times S, \geq_L)$ for sets S. The above theorem has shown that the minimal cardinality v of S is not larger than 2 if and only if the order topology is separable. Since separability of the order topology does not guarantee the existence of a utility function, indeed $v = 2$ may occur as minimal value. Let us further note that $v = 3$ is too high in the sense that $(\mathbb{R} \times \{0, 1, 2\}, \geq_L)$ is not embeddable in $(\mathbb{R} \times \{0, 1\}, \geq_L)$, as it is not separable w.r.t. the natural topology: $\{\{z: (\alpha, 2) >_L z >_L (\alpha, 0)\}: \alpha \in \mathbb{R}\}$ is an uncountable family of open non-empty mutually disjoint sets.

3. EXISTENCE OF UTILITY FUNCTIONS

The above theorem and proof indicate what problems remain for the derivation of a utility function, once a countable dense subset has been obtained.

LEMMA 3.1. *Let X, $\geq$, and f be as in the proof of Theorem 2.2. The following two statements are equivalent*:

(*i*) *There exists a utility function u for $\geq$.*
(*ii*) *$R := \{\alpha \in \mathbb{R}: (\alpha, 1) \in f(X)\}$ is countable.*

PROOF. Let (*i*) hold. For every rational r, there can exist at most one $\alpha \in \mathbb{R}$ such that $r \in \,]u(x), u(y)[$, where $f(x) = (\alpha, 0)$ and $f(y) = (\alpha, 1)$. By (2.1), for every $\alpha \in R$, there exists such a rational r. So (*ii*) follows.

Next suppose that (*ii*) holds. Let $R = (\alpha^j)_{j=1}^{\infty}$. Define $u : X \to \mathbb{R}$ by

$$u : x \mapsto f_1(x) + \sum_{j:(\alpha^j, 0) <_L f(x)} 2^{-j}.$$ □

The above result is related to the theorem in Fleischer (1961). Fleischer presupposes antisymmetry of $\geq$ and then shows that a utility function exists for $\geq$ if and only if the natural topology $\Upsilon_n(\geq)$ is separable and has countably many so-called "jumps." These jumps are related to the set R in (*ii*) above.

Lemma 3.1, in itself not attractive since the "characterization" in (*ii*) is not formulated directly in terms of elementary properties of $\geq$, does show how a countable dense subset leads to the derivation of a utility function. As an illustration, we show how the above lemma implies some well-known results. (For the case where $\geq$ is antisymmetric, Corollaries 3.2 and 3.3 below are like the remark after the theorem in Fleischer 1961.)

COROLLARY 3.2. *If $\geq$ is continuous w.r.t. a connected separable topology, then a utility function u exists.*

PROOF. Connectedness implies that no two open disjoint non-empty sets V, W, with union X, can exist. Any $(\alpha, 1)$ in the set R of Lemma 3.1 (*ii*) induces a violation of this because, with w such that $f(w) = (\alpha, 1)$ and v (existing by (2.1)) such that $f(v) = (\alpha, 0)$, we can define $V := \{x : x > v\}$ and $W := \{x : x < w\}$. □

COROLLARY 3.3. *If $\geq$ is continuous w.r.t. a second countable topology Υ, then a utility function u exists.*

PROOF. Υ being second countable means that an array $(A_j)_{j=1}^{\infty}$ of open sets exists, s.t. every open set is a union of some A_j's. Let α be in the set R of Lemma 3.1 (*ii*); say $f(x_\alpha^1) = (\alpha, 1)$. By (2.1), and x_α^0 must exist with $f(x_\alpha^0) = (\alpha, 0)$. By openness of $\{x: x < x_\alpha^1\}$, there must exist an A_{j_α} containing x_α^0 and containing no $y > x_\alpha^0$. So every $\beta \neq \alpha \in R$ has $A_{j_\beta} \neq A_{j_\alpha}$. Therefore R is countable. □

As a further illustration:

COROLLARY 3.4. *The following two statements are equivalent*:

(*i*) *There exists a utility function u for* $\geq$.
(*ii*) $\geq$ *is a weak order for which a countable set* $V \subset X$ *exists s.t. for all* $x > y$ *there is a* $v \in V$ *with* $x \geq v \geq y$.

PROOF. Suppose (*i*) holds. Then $(X, \geq)$ is embeddable in $\mathbb{R}$, so in $\mathbb{R} \times \{0, 1\}$. Hence Theorem 2.2 applies. For V we take $C \cup R'$, where (f and) C are as in the proof of Theorem 2.2, and R' is a countable subset of X such that $f(R') = \{(\alpha, 1): (\alpha, 1) \in f(X)\}$, i.e., $f(R') = R \times \{1\}$ with R as in Lemma 3.1. Obviously V is countable. If $x > y$ and $Z := \{z: x > z > y\}$ is non-empty, then $v \in C$ is taken such that $f(x) >_L f(v) >_L f(y)$. If Z is empty, and not $x \approx v$ or $y \approx v$ for some $v \in C$, then $f_1(x) = f_1(y), f_2(x) = 1, f_2(y) = 0$, and $x \approx v$ for some $v \in R'$. So (*ii*) has been established.

Next, suppose that (*ii*) holds. We can let V contain a maximal element, if one exists, and a minimal element, if one exists. Let $W \subset X$ be a countable set s.t., for every $(x, y) \in V$ with $\{z: x > z > y\}$ non-empty, W contains an element in $\{z: x > z > y\}$. We first show that $V \cup W$ is dense w.r.t. $\Upsilon_n(\geq)$. It suffices to show that, for any $x > y \in X$ with $\{z: x > z > y\}$ non-empty (say $z^0 \in \{z: x > z > y\}$), there exists a $b \in V \cup W$ s.t. $b \in \{z: x > z > y\}$. Let $v_1, v_2 \in V$ be such that $x \geq v_1 \geq z^0 \geq v_2 \geq y$. If not $x \approx v_1$, then we take $b = v_1$. If not $y \approx v_2$, then we can take $b = v_2$. If $x \approx v_1 > z^0 > v_2 \approx y$, we can take b from W. This way separability of $\Upsilon_n(\geq)$ follows. And since for every two different $\alpha, \beta \in R$, there must exist different $v^\alpha, v^\beta \in V$ s.t. $(\alpha, 0) \leq_L f(v^\alpha) \leq_L (\alpha, 1)$ and $(\beta, 0) \leq_L f(v^\beta) \leq_L (\beta, 1)$, R in Lemma 3.1 must be countable. So (i) follows. □

The condition in *(ii)* above is called "perfect separability" in Chateauneuf (1985). We have not considered continuity properties of utility functions. For these, the reader can consult the following references: Debreu (1964), Bowen (1968), Jaffray (1975), and Richter (1980).

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