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CONCAVE ADDITIVELY DECOMPOSABLE REPRESENTING FUNCTIONS AND RISK AVERSION

1. Introduction

We study preference relations on sets that are finite-fold cartesian products. Elements of these sets will be called <u>alternatives</u>. The preference relation represents the opinion of a decision maker.

Examples are consumer demand theory, where alternatives are "commodity bundles"; and decision making under uncertainty, where alternatives are "acts".

Our purpose is to <u>characterize</u> (i.e. give properties of the preference relation, necessary and sufficient to guarantee) the existence of special kinds of representing functions, mainly continuous concave functions that are additively decomposable.

An often used property of preference relations is known under various names such as (strong/strict) separability, (preferential) independence, the sure-thing principle. We shall use the term "coordinate independence" (CI) for it. The property was introduced in Sono (1945, 1961) and Leontief (1947a, 1947b) in terms of derivatives of a (presupposed) representing function. See also Samuelson (1947, pp. 174-180). In Debreu (1960) it was formulated in its present, more appealing, form, in terms of the preference relation, without differentiability assumptions. Before, Savage (1954) had introduced the "sure-thing principle" for DMUU. This in fact is identical to CI, as is well known nowadays. It can be seen to underly the "likelihood principle" in statistics, thus should be fruitful there too. See Birnbaum (1962, 1972), Savage (1962), Berger (1980, 1.6.2), or Barnard & Godambe (1982). For an extensive study of generalizations of CI, see Blackorby, Primont and Russell (1978).

In this paper we shall characterize, under a continuity and nontriviality assumption, the existence of <u>concave</u> additively decomposable representing functions. It is known that the combination of CI and convexity for

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the preference relation does not suffice for this, see section 4. We shall introduce a property for preference relations, the "concavity assumption", that achieves the desired characterization. It is an extension of "axiom Q" of Yaari (1978), which there is not studied for its own sake, but only as an intermediate between a stronger and weaker property, and concerns the case where every component equals R_+.

For continuity and concavity to have meaning, at least some topological and convexity structure is required. A reader, not interested in these notions in full generality, may simply assume that the coordinates refer to convex subsets of Euclidean spaces, and skip over the specific details of mixture spaces in section 2. Our notation is chosen such that this can be done without any problem. Note that we do not assume monotonicity, to achieve maximal generality.

In section 5 we consider the case where all components of the cartesian product are identical. The main application for this is DMUU, with a finite state space. First the characterization of continuous expected utility maximization, provided in Wakker (1984a) for a set of consequences that is a convex open subset of R, is extended to the case where the set of consequences is any topologically connected space. Next it is combined with the concavity assumption, thus characterizing continuous expected utility maximization with risk aversion. In this we do not need differentiability assumptions or methods. A result, using differentiability assumptions, is Stigum (1972).

The existence of special kinds of representing (="utility") functions can be verified/justified (falsified/criticized) if and only if the involved characterizing properties can be verified/justified (falsified/criticized). Thus characterization results are central for the discussion on the foundations of utility.

Proofs are given or referenced in the Appendix.

2. Product Topological Mixture Spaces

First we introduce "mixture spaces", i.e. sets with a "mixture operation" on them. These were already used by von Neumann and Morgenstern (1944) and Herstein and Milnor (1953). There the mixture operation mainly served as a generalization of lotteries. Extensive use of mixture spaces is made in Fishburn (1982). See also Luce & Suppes (1965). The applicability of mixture operations to fields such as quantum mechanics, and colour perception in psychology, is indicated in Gudder (1977) and Gudder and Schroeck (1980). Our main intended application is that a mixture operation is a generalization of the convex combination operation in linear spaces.

DEFINITION 2.1: A function $\Theta : C \times [0,1] \times C \Rightarrow C$, where C is a nonempty set, and where we write $p\alpha + (1-p)\beta$ for $\Theta(\alpha,p,\beta)$, is a <u>mixture operation</u> (<u>on</u> C) if the following i, ii, and iii are satisfied for all $\alpha,\beta \in C$; and $p,\mu, \in [0,1]$:
i : commutativity $p\alpha + (1-p)\beta = (1-p)\beta + p\alpha$.
ii : associativity $\mu(p\alpha+(1-p)\beta)+(1-\mu)\beta = (\mu p)\alpha+(1-(\mu p))\beta$.
iii: identity $1\alpha + 0\beta = \alpha$.
Here (C,Θ), or simply C, is called a <u>mixture space</u>.

It may be argued that ii could be called "distributivity" instead of "associativity". We write α/μ for $(1/\mu)\alpha$, and $p\alpha/\mu$ for $(p/\mu)\alpha$. We say "<u>y is between α and β</u>" if $p \in [0,1]$ exists such that $y = p\alpha + (1-p)\beta$. The definitions of convex sets, and affine/convex/concave/quasiconcave functions, are as in Euclidean spaces.

LEMMA 2.1: If C is a mixture space, then for all $\alpha,\beta \in C$; $p,\mu \in [0,1]$:
iv: $\mu\alpha + (1-\mu)\alpha = \alpha$.
v : $p(\mu\alpha + (1-\mu)\beta) + (1-p)(v\alpha + (1-v)\beta) =$
$= (p\mu + (1-p)v)\alpha + (p(1-\mu) + (1-p)(1-v)\beta)$.

DEFINITION 2.2: (C,T,Θ), also denoted as C, is a <u>topological mixture space</u> if T is a topology on C, Θ a mixture operation on C, and $\Theta: C \times [0,1] \times C$ (with the product topology) $\Rightarrow$ C is continuous.

LEMMA 2.2: A topological mixture space C is topologically connected.

LEMMA 2.3: Let V from a topological mixture space $(C,T,\Theta) \Rightarrow \mathbb{R}$ be continuous. The following propositions are equivalent:
i : There exists $\pi > 0$ such that $V[(\alpha/2) + (\beta/2)] \geq [V(\alpha) + V(\beta)]/2$ whenever $0 \leq V(\alpha)-V(\beta) \leq \pi$.
ii : $V(\alpha/2 + \beta/2) \geq [V(\alpha) + V(\beta)]/2$ for all α,β.

iii: V is concave.

DEFINITION 2.3: For a sequence of mixture spaces $(C_i,\Theta_i)_{i=1}^n$, the product mixture operation

$$\Theta : (X_{i=1}^n C_i) \times [0,1] \times (X_{i=1}^n C_i) \Rightarrow X_{i=1}^n C_i$$

assigns to every $(x,p,y) = [(x_1,\ldots,x_n),p,(y_1,\ldots,y_n)]$, the image element $(px_1 + (1-p)y_1,\ldots,px_n + (1-p)y_n)$, also denoted as $p(x_1,\ldots,x_n) + (1-p)(y_1,\ldots,y_n)$, or $px + (1-p)y$. We then call $(X_{i=1}^n C_i, \Theta)$, or simply $X_{i=1}^n C_i$, the product mixture space. If the C_i's are topological mixture spaces, then we endow $X_{i=1}^n C_i$ with the product topology and call it a product topological mixture space.

For the above definitions to be suited the following:

THEOREM 2.1: A product mixture operation is a mixture operation. A product topological mixture space is a topological mixture space.

3. The Concavity Assumption

Throughout this section $C = X_{i=1}^n C_i$ is a product topological mixture space. For instance any C_i may be (a convex subset of) R^{m_i}. Elements of C, called alternatives, are denoted by x,y,v, etc., with coordinates x_1, v_2, etc. Elements of $U_{i=1}^n C_i$ are also denoted by α,β,Γ, etc. Less standard is the following notation: $x_{-i}\alpha$ is the alternative with i-th coordinate α, other coordinates equal to those of x.

The decision maker is denoted by T, his preference relation on the set of alternatives by $\succeq$. We write $x \succeq y$ if T thinks x at least as good as y. We write $x \preceq y$ if $y \succeq x$, $x \prec y$ if not $x \succeq y$, $x \succ y$ if not $y \succeq x$, and $x \approx y$ if $x \succeq y$ and $y \succeq x$. A weak order $\succeq$ is complete and transitive, i.e. $[x \succeq y$ or $y \succeq x]$, and $[x \succeq y$ & $y \succeq z \Rightarrow x \succeq z]$ for all $x,y,z \in C$; thus it induces an "equivalence relation", $\approx$. A preference relation $\succeq$ is convex if $\{x|x \succeq y\}$ is convex for all alternatives y, and continuous if $\{x|x \succeq y\}$ and $\{x|x \preceq y\}$ are closed for all y. Coordinate i is essential (w.r.t. $\succeq$) if $x_{-i}\alpha \succ x$ for some $x \in C$, $\alpha \in C_i$. A function $V : C \Rightarrow R$ represents $\succeq$ if $x \succeq y \Leftrightarrow V(x) \geq V(y)$ for all $x,y \in C$.

LEMMA 3.1: If $\succeq$ is a weak order, then $x \approx y$ whenever $x_j = y_j$ for all essential j.

The above Lemma shows that, for a weak order $\succeq$, the inessential coordinates do not affect $\succeq$. Hence they can be left out, as will be used in Definition 3.2.

LEMMA 3.2: Let $\succeq$ be a weak order on a topological mixture space C. Then the following propositions are equivalent:
i : $x \succeq y \Rightarrow px + (1-p)y \succeq y$ for all x,y; $p \in [0,1]$.
ii: $\succeq$ is convex.
If $\succeq$ is continuous, then a further equivalent proposition is:
iii: $x \succeq y \Rightarrow x/2 + y/2 \succeq y$ for all x,y.

DEFINITION 3.1: We say $\succeq$ is coordinate independent (CI) if $(x_{-i}\alpha) \succeq (y_{-i}\alpha) \Rightarrow (x_{-i}\beta) \succeq (y_{-i}\beta)$ for all x,y,i,α,β.

For the case of exactly two essential coordinates we shall need one more property:

DEFINITION 3.2: Let exactly two coordinates be essential. We say a weak order $\succeq$ satisfies the Thomsen condition if, after removal of the inessential coordinates, $(\alpha,\mu) \approx (\Gamma,v)$ & $(\Gamma,\sigma) \approx (\beta,\mu)$ imply $(\alpha,\sigma) \approx (\beta,v)$ for all $\alpha,\ldots,v$.

The following property is a generalization of "Axiom Q" in Yaari (1978), which is formulated for the case where $C_i = R_+$ for all i, and for this case by some elementary calculus can be seen to be equivalent to our present definition.

DEFINITION 3.3: We say $\succeq$ satisfies the concavity assumption if $x_{-i}\Gamma \succeq y_{-i}\delta$ whenever $x_{-i}\alpha \succeq y_{-i}\beta$ and $\beta = p\alpha + (1-p)\delta$, $\Gamma = p\delta + (1-p)\alpha$ for some $p \in [0,1]$.

A way to see the meaning of this is by substitution in Theorem 3.1.i. The following Lemmas adapt to the present context the Remark at section 4, and the Lemma 2 of section 5 and by that the implication of axiom D through axiom Q, of Yaari (1978).

LEMMA 3.3: The concavity assumption implies CI.

LEMMA 3.4: If $\succeq$ is a continuous weak order that satisfies the concavity assumption, then $\succeq$ is convex.

If three or more coordinates are essential, the above lemma can also be obtained as a corollary of Theorem 3.1 below.

DEFINITION 3.4: $(V_j)_{j=1}^n$ is an <u>array of additive value functions for</u> $\succsim$ if $[x \succsim y \Leftrightarrow \Sigma_{j=1}^n V_j(x_j) \geq \Sigma_{j=1}^n V_j(y_j)]$ for all $x,y \in X_{j=1}^n C_j$. With then $V(x) := \Sigma_{j=1}^n V_j(x_j)$, V is called <u>additively decomposable</u>.

Now we are ready for our main result:

THEOREM 3.1: Let $\succsim$ be a binary relation on a product topological mixture space $X_{i=1}^n C_i$ (e.g. $X_{i=1}^n R^{m_i}$). Let at least two coordinates be essential. Then the following propositions are equivalent:

i : There exists an array of continuous concave additive value functions $(V_j)_{j=1}^n$ for $\succsim$.

ii: The binary relation $\succsim$ is a continuous weak order that satisfies the concavity assumption; if exactly two coordinates are essential, then furthermore $\succsim$ satisfies the Thomsen condition.

Furthermore, if i applies, then $(W_j)_{j=1}^n$ is an array of additive value functions for $\succsim$ if and only if $(\mu_j)_{j=1}^n$ exist, and positive k, such that $W_j = kV_j + \mu_j$ for all j.

The following Corollary applies the above result to a common context, where $C = R_{++}^n$ ($R_{++} = \{\alpha \in R | \alpha > 0\}$), $\succsim$ is <u>monotone</u> ($x_j \geq y_j$ for all $j \Rightarrow x \succsim y$). The property after "furthermore" in ii below, simply is a reformulation of the concavity assumption, so of Yaari's axiom Q. It reflects the idea of nonincreasing marginal utility.

COROLLARY 3.1: Let $\succsim$ be a binary relation on $(R_{++})^n$, $n \geq 3$. Then the following propositions are equivalent:

i : There exist concave (thus continuous) nondecreasing nonconstant functions $V_j : R_{++} \Rightarrow R$, $j = 1,\ldots,n$, such that $[x \succsim y \Leftrightarrow \Sigma_{j=1}^n V_j(x_j) \geq \Sigma_{j=1}^n V_j(y_j)$ for all x,y].

ii: $\succsim$ is a continuous weak order, it is monotone, every coordinate is essential, and furthermore $x_{-i}\alpha \succsim y_{-i}\beta \Rightarrow x_{-i}(\alpha - e) \succsim y_{-i}(\beta - e)$ whenever $(\alpha-\beta)e \geq 0$.

4. Completion of Logical Relations

Throughout this section $\succsim$ is a continuous weak order on a

product topological mixture space $X_{i=1}^m C_i$, and $n \leq m$ is the number of <u>essential</u> coordinates. In figure 1 we have indicated the logical relations between the propositions, numbered 4.1 - 4.4 there.

FIGURE I: (for continuous weak order $\succsim$)

<u>There exists an array of continuous concave additive value functions for</u> $\succsim$ (4.1)

↑: n=1: counterexample $V = f^5$; n=2: counterexample $V = f^2$; n≥3: correct by Theorem 3.1 — by Th. 3.1 :↓

<u>$\succsim$ satisfies the concavity assumption</u> (4.2)

↑: n=1: counterexample $V=f^1$; n=2: counterexample $V=f^3$; n≥3: counterexample $V=f^3$ — by Lemma 3.3 & 3.4 :↓

<u>$\succsim$ is convex and CI</u> (4.3)

↑: n=1: correct; n=2: counterexample $V=f^4$; n≥3: counterexample $V=f^2$ — direct :↓

<u>$\succsim$ is convex</u> (4.4)

n: number of essential coordinates;
V: function, representing $\succsim$;
$f^j : R_{++}^m \Rightarrow R$ for all $1 \leq j \leq 5$; with $f^1(x) = 1$ if $x_1 \leq 1$, $f^1(x) = x_1$ if $x_1 \geq 1$; $f^2(x) = \Sigma_{j=1}^m x_j + \min\{x_j : 1 \leq j \leq m\}$; $f^3(x) = (m-1)e^{x_1} + \Sigma_{j=2}^m \log x_j$; $f^4(x) = -(\Sigma_{j=1}^m (x_j - 2))^2$; $f^5(x) = x_1 - 1$ for $0 < x_1 < 1$, $f^5(x) = (x_1-1)^2$ for $1 \leq x_1 < 2$, $f^5(x) = 3 - x_1$ for $x_1 \geq 2$.

For n=1, proposition 4.2 in figure 1 does not imply 4.1, even if a representing function V exists. This follows from Kannai (1977, p. 17), or from f^5 in figure 1. This, and f^2, can be seen to represent a $\succsim$, that satisfies the concavity assumption. f^5 is a minor variation on the example of Artstein in Kannai (1981, p. 562), where it is shown not to be "concavifiable". For n=2, 4.2 does imply 4.1 iff $\succsim$ satisfies the Thomsen condition. That $\succsim$, represented by f^2, does not satisfy this for n=2, and has no additive value functions, can be seen from $(1,4,9,\ldots,9) \approx (2,2,9,\ldots,9)$, $(2,8,9,\ldots,9) \approx (4,4,9,\ldots,9)$, $(1,8,9,\ldots,9) \succ (4,2,9,\ldots,9)$.

That $\succsim$, represented by f^1, does not satisfy the concavity assumption, follows from $(1/2,1,\ldots,1) \succsim (1,\ldots,1) \prec (3/2,1,\ldots,1)$. That f^3 is quasiconcave, thus represents a convex $\succsim$, can be derived from 6.28 of Arrow and Enthoven (1961). For $\succsim$, represented by f^3, by the "furthermore"-statement of Theorem 3.1, no <u>concave</u> additive value functions exist, if $n \geq 2$: $\succsim$ must violate the concavity assumption. The observation that 4.3 does not imply 4.2, for $n \geq 2$, is closely related to the observation that quasiconcavity and additive decomposability of V do not imply 4.1, i.e. concavity of V. This latter observation has been made some times in the literature. The earliest reference to this, given in Debreu and Koopmans (1982), is Slutsky (1915).

That $\succsim$, represented by f^4, is not CI for $n \geq 2$, follows from $(2,\ldots,2) \succ (2,3,2,\ldots,2)$ and $(1,2,\ldots,2) \prec (1,3,2,\ldots,2)$. Finally, that for $n \geq 3$ $\succsim$, represented by f^2, is not CI, follows from $(1,6,1,\ldots,1) \succ (3,3,1,\ldots,1)$, $(1,6,3,\ldots,3) \prec (3,3,\ldots,3)$.

5. Expected Utility with Risk Aversion

Let $S = \{s_1,\ldots,s_n\}$ be a finite <u>state space</u>. Its elements are <u>(possible) states (of nature)</u>. Exactly one is the true state, the other states are not true. The decision maker T is uncertain about which of the states is true. C is the set of <u>consequences</u>, and $x = (x_1,\ldots,x_n) \in C^n$ is the <u>act</u> (= alternative) yielding consequence x_j if s_j is true.

DEFINITION 5.1: We say $\succsim$ <u>maximizes subjective expected utility (SEU)</u> w.r.t. $(P_j)_{j=1}^n$, U if $U : C \Rightarrow R$, $P_j \geq 0$, $\Sigma_{j=1}^n P_j = 1$, and $[x \succsim y \Leftrightarrow \Sigma_{j=1}^n P_j U(x_j) \geq \Sigma_{j=1}^n P_j U(y_j)]$ for all x,y. (Here U is the "<u>utility function</u>").

The results of Wakker (1984a), formulated for the special case that C is a convex open subset of R, are generalized in the sequel to the case where C is any connected topological space, e.g. a topological mixture space, such as a (convex subset of) R^m.

DEFINITION 5.2: We say $\succsim$ is <u>cardinally coordinate independent (CCI)</u> if $v_{-j}\Gamma \succsim w_{-j}\delta$ whenever $x_{-i}\alpha \precsim y_{-i}\beta$, $x_{-i}\Gamma \succsim y_{-i}\delta$, $v_{-j}\alpha \succsim w_{-j}\beta$, and i essential.

THEOREM 5.1: Let $\succsim$ be a binary relation on C^n where $n \in N$, and C a connected topological space (e.g. R_+^m or R). Let C be topologically separable if exactly one coordinate is essential. Then the following propositions are equivalent:

i : $\succsim$ maximizes SEU w.r.t. some $(P_j)_{j=1}^n$, U, where U is continuous.

ii: $\succsim$ is a continuous weak order that is CCI.

The concavity assumption and CCI can be combined as follows, to give in Theorem 5.2. iii a very concise characterization of SEU maximization with risk aversion, i.e. concavity of U.

DEFINITION 5.3: Let $\succsim$ be a binary relation on a product topological mixture space C^n. We say $\succsim$ is <u>concavely cardinally coordinate independent (CCCI)</u> if $v_{-j}\sigma \succsim w_{-j}\tau$ whenever $x_{-i}\alpha \precsim y_{-i}\beta$, $x_{-i}\Gamma \succsim y_{-i}\delta$, $v_{-j}\alpha \succsim w_{-j}\beta$, i is essential, and $p \in [0,1]$ exists such that $\sigma = p\tau + (1-p)\Gamma$, $\delta = p\Gamma + (1-p)\tau$.

THEOREM 5.2: Let $\succsim$ be a binary relation on a product topological mixture space C^n (e.g. $(R_+^m)^n$ or R^n). Let at least two coordinates be essential. Then the following propositions are equivalent:

i: $\succsim$ maximizes SEU w.r.t. some $(P_j)_{j=1}^n$, U, where U is concave and continuous.

ii: $\succsim$ is a continuous weak order that is CCI, and satisfies the concavity assumption or is convex.

iii: $\succsim$ is a continuous weak order that is CCCI.

APPENDIX: PROOFS

The proof of Lemma 2.1 is in Fishburn (1970, section 8.4). Lemmas 2.3 and 3.2 are as in Euclidean spaces, Theorem 2.1 is elementary topology. Proofs for these are given in Wakker (1984b).

PROOF OF LEMMA 2.2: Suppose $A \subset C$ is open and nonempty, say $x \in A$, and A^c is open and nonempty, say $y \in A^c$. Then by elementary topology (by Lemma A1 in Wakker, 1984b) the set $\{p \in [0,1] : px + (1-p)y \in A\}$ and its complement $\{p \in [0,1] : px + (1-p)y \in A^c\}$ are open. But one contains $p = 1$, the other $p = 0$, so both are nonempty, contradicting connectedness of $[0,1]$.

PROOF OF LEMMA 3.1: As an example, let 1,2,3 be not essential, $x_j = y_j$ for all $j \geq 4$. Then $x \approx x_{-1}y_1 \approx (x_{-1}y_1)_{-2}y_2 \approx ((x_{-1}y_1)_{-2}y_2)_{-3}y_3 = y$.

PROOF OF LEMMA 3.3: Let $\alpha = \beta, \Gamma = \delta, p = 1$ in Definition 3.3.

PROOF OF LEMMA 3.4: By Lemma 3.2 we only have to prove that $v \succsim w \Rightarrow v/2 + w/2 \succsim w$. It is sufficient to suppose $v/2 + w/2 \precsim w \precsim v$, and then to derive $v \leq v/2 + w/2$. We define, inductively, for all $0 \leq j \leq n$, v^j and w^j by $v^0 = v/2 + w/2$, $w^0 = w$, $v^j = v^{j-1}{}_{-j}v_j$, $w^j = w^{j-1}{}_{-j}(v_j/2 + w_j/2)$, thus getting $v^n = v$, $w^n = v/2 + w/2$. For $j = 0$ we have, by assumption, $v^0 \precsim w^0$. Suppose now for some $1 \leq j \leq n$, that $v^{j-1} \precsim w^{j-1}$. Then we apply the concavity assumption with $i = j$, $p = 1/2$, $x = w^{j-1}$, $y = v^{j-1}$, $\alpha = w_j$, $\beta = \Gamma = v_j/2 + w_j/2$, $\delta = v_j$ to obtain $v^j \precsim w^j$. Now by induction $v^n \precsim w^n$, i.e. $v \precsim v/2 + w/2$.

PROOF OF THEOREM 3.1: i => ii is straightforward. So we assume ii, and derive i. By Lemma 3.1 the inessential coordinates do not affect $\succsim$. Hence they can be left out. The additive value functions, to be constructed in the sequel, simply are to be taken constant for these coordinates. So we assume in the sequel that all coordinates are essential.
By Lemma 2.2 every C_i is topologically connected.
By Lemma 3.3 we get CI for $\succsim$. Hence, if $n \geq 3$, then everything of i, except concavity of the V_j's, follows from Theorem 14 of section 6.11.1 of Krantz et al (1971). This theorem is a strenghtening of Theorem 3 of Debreu (1960) because no topological separability of the C_i's is required. If $n = 2$ the same as above (so without the demand of topological separability for the components) can be derived from Theorem 2 of section 6.2.4 of Krantz et al. (1971), the same way as their Theorem 14 of section 6.11.1 is derived from their Theorem 13 there. Their reasoning of section 6.12.3 applies literally for $n = 2$. (See also their exercise 34 of Chapter 6.) These theorems

of Krantz et al. also guarantee the assertion "Furthermore...". So all that remains is concavity of the V_j's. We show concavity of V_1.
Since coordinate 2 is essential, x_2 and y_2 in C_2 exist with $V_2(x_2) - V_2(y_2) = \pi > 0$. V_2 is continuous, and C_2 connected, so $V_2(C_2)$ is connected too. Thus for any $0 \leq \Phi \leq \pi$ there exists z_2 in C_2 such that $V_2(z_2) - V_2(y_2) = \Phi$. By Lemma 2.3, concavity of V_1 is guaranteed if we show that $V_1(\beta) \geq [V_1(\alpha) + V_1(\Gamma)]/2$ for any $\alpha, \beta, \Gamma \in C_1$ such that $\beta = \alpha/2 + \Gamma/2$, and $0 \leq V_1(\alpha) - V_1(\Gamma) \leq \pi$. To this end let $z_2 \in C_2$ be such that $V_2(z_2) - V_2(y_2) = [V_1(\alpha) - V_1(\Gamma)]/2$. We apply, for arbitrary v, the concavity assumption, to obtain $(v_{-1}\alpha)_{-2}y_2 \succsim (v_{-1}\beta)_{-2}z_2 \Rightarrow (v_{-1}\beta)_{-2}y_2 \succsim (v_{-1}\Gamma)_{-2}z_2$, i.e.: $V_1(\alpha) - V_1(\beta) \geq V_2(z_2) - V_2(y_2) \Rightarrow V_1(\beta) - V_1(\Gamma) \geq V_2(z_2) - V_2(y_2)$.
This shows:
$V_1(\alpha) - V_1(\beta) \geq [V_1(\alpha) - V_1(\Gamma)]/2 \Rightarrow V_1(\beta) - V_1(\Gamma) \geq [V_1(\alpha) - V_1(\Gamma)]/2$.
This can only be if $V_1(\beta) \geq [V_1(\alpha) + v_1(\Gamma)]/2$.

PROOF OF THEOREM 5.1: i => ii is straightforward. So we assume ii, and derive i. The cases of one or no essential coordinates are treated as in Wakker (1984a, proof of Theorem 2.1). Taking $\alpha = \beta$, $\Gamma = \delta$, $x = y$ in Definition 5.2 shows that CCI implies CI. Thus, if three or more coordinates are essential, then existence of an array of continuous additive value functions $(V_j)_{j=1}^n$ follows as in the proof of Theorem 3.1. If exactly two coordinates are essential, then we first observe that CCI implies "triple cancellation", i.e. $v_{-j}\Gamma \succsim w_{-j}\delta$ whenever $x_{-j}\alpha \precsim y_{-j}\beta$, $x_{-j}\Gamma \succsim y_{-j}\delta$, $v_{-j}\alpha \succsim w_{-j}\beta$. If j is essential this follows from CCI by taking $i = j$, if j is inessential it is direct. An array of continuous additive value functions$(V_j)_{j=1}^n$ for $\succsim$ is constructed, but now with triple cancellation instead of the Thomsen condition, the same way as was indicated in the proof of Theorem 3.1. See the end of section 6.2.4 of Krantz et al. (1971).
The demonstration that $V_j = p_jU$, for some nonnegative $(p_j)_{j=1}^n$, and continuous $U: C \Rightarrow R$, is performed the same way as in Wakker (1984 a, proof of Theorem 2.1). If no coordinate is essential, we now take $P_j = 1/n$ for all j, and U constant; if one coordinate i is essential we take $P_i = 1$, $P_j = 0$ for all $j \neq i$. If two or more

coordinates are essential, then with p_j as above, at least two p_j's are nonzero, so we can take $P_i = p_i/\Sigma_{j=1}^{n} p_j$ for all i. Thus $\Sigma_{i=1}^{n} P_i = 1$ is always satisfied.

PROOF OF THEOREM 5.2: i => ii and i => iii are straightforward, so we only prove ii => i and iii => ii. First we assume ii, and derive i. By Theorem 5.1 we see $\succcurlyeq$ maximizes SEU w.r.t. some $(P_j)_{j=1}^{n}$, and continuous U. Thus $(P_jU)_{j=1}^{n}$ is an array of additive value functions for $\succcurlyeq$. This implies that $\succcurlyeq$ satisfies the TB-condition if exactly two coordinates are essential. If now $\succcurlyeq$ satisfies the concavity assumption, then concavity of U, thus i, follows from Theorem 3.1 and the substitution $W_j = P_jU$. That concavity of U is also implied by convexity of $\succcurlyeq$, is well-known.

Now finally we assume iii, and derive ii. CCI for $\succcurlyeq$ is straightforward, set p = 0 in Definition 5.3. So only the concavity assumption remains to be derived. Let $x_{-j}\Gamma \succcurlyeq y_{-j}\delta$, and $\sigma = p\tau + (1-p)\Gamma$, $\delta = p\Gamma + (1-p)\tau$, for some $p \in [0,1]$, $\sigma,\tau \in C$. To prove is that $x_{-j}\sigma \succcurlyeq y_{-j}\tau$. If j is not essential this is direct. So let j be essential. If no $\pi,\Theta \in C$ exist such that $x_{-j}\pi \prec y_{-j}\Theta$, then $x_{-j}\sigma \succcurlyeq y_{-j}\tau$, as desired. So suppose $x_{-j}\pi \prec y_{-j}\Theta$ for some π, Θ. We now first construct α, β such that $x_{-j}\alpha \approx y_{-j}\beta$. To this end, first suppose $y_{-j}\delta \succcurlyeq x_{-j}\pi$. Then we have $x_{-j}\Gamma \succcurlyeq y_{-j}\delta \succcurlyeq x_{-j}\pi$. Thus $\{w \in C \mid x_{-j}w \succcurlyeq y_{-j}\delta\}$ and $\{w \in C \mid x_{-j}w \preccurlyeq y_{-j}\delta\}$ are both nonempty. Also they are closed, by continuity of $\succcurlyeq$, and Wakker (1984b, Lemma A1 applied to complements). By connectedness of C, the intersection of the above two subsets of C is nonempty: it contains some α with $x_{-j}\alpha \approx y_{-j}\delta$. Finally, take $\beta = \delta$.

Remains the construction of α, β as above for the case where $y_{-j}\delta \prec x_{-j}\pi$. Then we have $y_{-j}\delta \prec x_{-j}\pi \prec y_{-j}\Theta$. The same way as above now β is found such that $x_{-j}\pi \approx y_{-j}\beta$. Now finally take $\alpha = \pi$.

We thus always have α,β such that $x_{-j}\alpha \approx y_{-j}\beta$. We can now apply CCCI as in Definition 5.3, with i = j, v = x, w = y, to obtain $x_{-j}\sigma \succcurlyeq y_{-j}\tau$.

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