METHODS OF OPERATIONS RESEARCH

54

X. SYMPOSIUM ON OPERATIONS RESEARCH

Organized by
Gesellschaft für Mathematik, Ökonomie und Operations Research e. V.

Universität München, August 26-28, 1985

Proceedings

Part II: SECTIONS 6-10

Edited by: Martin J. Beckmann, München
             Karl-Walter Gaede, München
             Klaus Ritter, München
             Hans Schneeweiss, München

Wakker, P.P, H.J.M. Peters, & T.B.P.G. van Riel (1986), 'Comparisons of Risk aversion, with an Application to Bargaining'.

COMPARISONS OF RISK AVERSION, WITH AN APPLICATION TO BARGAINING

Peter WAKKER, Nijmegen

Hans PETERS, Maastricht

Tom van RIEL, Nijmegen

ABSTRACT: Key results of Pratt (1964), Arrow (1971), Yaari (1969), and Kihlstrom and Mirman (1974) on the comparability of the risk aversion of different decision makers, are extended to completely general consequence sets, without any restriction on the utility functions, for decision making under risk, and to topologically connected consequence spaces with continuous utility functions, for decision making under uncertainty. An application to bargaining game theory is given.

1. INTRODUCTION. Around 1965 Pratt and Arrow, independently, found a key tool to compare the risk aversion of different decision makers who maximize expected utility. Yaari (1969) gave further results for the context of decision making under uncertainty. All this was done for real numbers (amounts of money) as consequences. Kihlstrom and Mirman (1974) gave results for multidimensional consequences. In all these papers differentiability assumptions for the utility functions were made, so the Euclidean space structure on the consequence set was used. In Kihlstrom, Roth and Schmeidler (1981) these results on risk aversion were introduced into bargaining game theory, to define and study "risk sensitivity" of bargaining solutions. In Peters and Tijs (1981) a slightly different definition of "more risk averse than" has been introduced to define and study risk sensitivity.

The present paper extends results for decision making under risk (DMUR) to completely general consequence sets and utility functions. For decision making under uncertainty (DMUU) matters are more complicated. Here different decision makers may, in principle, have different subjective probabilities. Hence we shall assume here that the utility functions are continuous w.r.t. a connected topology on the consequence set. This still is less restrictive than the assumptions, usually made in literature. Remark 4.2, under expected utility maximization an ex-

tension of Remark 1 in Yaari (1969), now shows that it is only possible to compare the risk aversion of decision makers who (can be thought to) have the same subjective probabilities.

Our work enables the extension of comparisons of risk aversion to cases where consequences (contrary to money or commodity bundles) have no (known) physical quantification. Although for such cases risk aversion as such is undefined, "more risk averse than" can be defined; as we shall see in subsection 2.4; and use in section 6.

2. ELEMENTARY DEFINITIONS

2.1. GENERAL DEFINITIONS.

Let X be a nonempty set. Elements of X are called alternatives, and denoted by x,y,v,w, etc. By $\succ$, a binary relation on X, we denote the preference relation of a person T on X. We write $x \prec y$ if $y \succ x$, $x \succ y$ or $y \prec x$ if $x \succ y$ and not $y \succ x$, and $x \approx y$ if $x \succ y$ and $y \succ x$. $\succ$ is a weak order if it is complete ($x \succ y$ or $y \succ x$ for all $x,y \in X$) and transitive. A weak order induces an equivalence relation $\approx$. $\succ$ is trivial if $x \succ y$ for all x,y.

C is another nonempty set. Its elements are consequences, and denoted by $\alpha, \beta, \gamma, \delta$. Intuitively, an alternative x will yield a consequence; but T has not sufficient information to know for sure which one. It is custom to distinguish between two ways to model the insufficiency of information: decision making under risk (DMUR) and decision making under uncertainty (DMUU).

2.2. DEFINITIONS FOR DMUR.

Here X is $L^S(C)$, the set of simple lotteries on C, i.e. probability measures on $(C, 2^C)$ that assign probability 1 to a finite subset of C. Simple lotteries (= alternatives) are also denoted as $(p_j; x_j)_{j=1}^n$, where n can be any natural number, $x_j \in C$ and $p_j \geq 0$ for all j, $\sum_{j=1}^n p_j = 1$. It assigns probability p_j to every consequence x_j. For any $\alpha \in C$, we write $\bar\alpha$ for $(1; \alpha)$.

DEFINITION 2.1. A function $U: C \to \mathbb{R}$ is a von Neumann-Morgenstern (vNM) utility function (for $\succ$) if:

$$(p_j; x_j)_{j=1}^n \succ (q_i; y_i)_{i=1}^m \Leftrightarrow \sum_{j=1}^n p_j U(x_j) \geq \sum_{i=1}^m q_i U(y_i)$$

for all $(p_j; x_j)_{j=1}^n$ and $(q_i; y_i)_{i=1}^m$ in X. Here $\sum_{j=1}^n p_j U(x_j)$, notation $EU((p_j; x_j)_{j=1}^n)$, is the expected utility of $(p_j; x_j)_{j=1}^n$.

2.3. DEFINITIONS FOR DMUU.

Here we deal with a finite state space, i.e. a set $\{s_1, \ldots, s_n\}$ of (possible) states (of nature). Exactly one state is the true state, the others are untrue. T does not know for sure which state is true, and cannot influence the truth of the s_j's. And now X is the cartesian product C^n. An act (= alternative) $(x_1, \ldots, x_n)$ is interpreted to yield consequence x_j, with j such that s_j is true. For $\alpha \in C$, we write $\bar\alpha$ for $(\alpha, \ldots, \alpha)$. As in DMUR, $\bar\alpha$ yields consequence α with certainty. For DMUU, we shall throughout assume:

ASSUMPTION 2.2. C is a connected topological space; C^n is endowed with the product topology.

For instance C may be $\mathbb{R}$, or $\mathbb{R}_+^m$.

DEFINITION 2.3. $((p_j)_{j=1}^n, U)$ is a subjective expected utility (SEU) model (for $\succ$) if $U: C \to \mathbb{R}$ (U is the (subjective) utility function), $p_j \geq 0$ for all j, $\Sigma p_j = 1$ (p_j is the subjective probability of T for state s_j), and:

$$x = (x_1, \ldots, x_n) \succ (y_1, \ldots, y_n) = y \Leftrightarrow \sum_{j=1}^n p_j U(x_j) \succ \sum_{j=1}^n p_j U(y_j)$$

for all $x,y \in X$. Here $\sum_{j=1}^n p_j U(x_j)$, notation SEU(x), is the subjective expected utility of x.

2.4. FURTHER GENERAL DEFINITIONS.

In this subsection we introduce two ways to compare the risk aversion of two decision makers T^1 and T^2, with preference relations $\succ^1$, $\succ^2$. The first definition has been introduced by Yaari (1969, p. 316) (his formulation in terms of "acceptance sets" is logically equivalent to ours), as an alternative for the formulations used in Pratt (1964) and Arrow (1971). The second, slightly different property (that we have renamed) has been introduced in Peters

and Tijs (1981).

DEFINITION 2.4. $\succ^1$ (or T^1) is <u>more risk averse than</u> (MRA) $\succ^2$
(or T^2) if $x\succ^1\bar{\alpha} \Rightarrow x\succ^2\bar{\alpha}$ for all $x \in X$, $\alpha \in C$.

DEFINITION 2.5. $\succ^1$ (or T^1) is <u>less risk prone than</u> (LRP) $\succ^2$
(or T^2) if $x\succ^1\bar{\alpha} \Rightarrow x\succ^2\bar{\alpha}$ for all $x \in X$, $\alpha \in C$.

Note that the trivial $\succ$ is both the least risk averse, and the
least risk prone, preference relation.

3. THE MAIN RESULT FOR DMUR

THEOREM 3.1. Let, for DMUR, U^k be a vNM utility function for
$\succ^k$, $k = 1,2$. Then:
(3.1.a) $\succ^1$ MRA $\succ^2 \Leftrightarrow U^2 = \psi \circ U^1$ with $\psi: U^1(C) \rightarrow U^2(C)$
       nondecreasing and convex.
(3.1.b) $\succ^1$ LRP $\succ^2 \Leftrightarrow U^1 = \phi \circ U^2$ with $\phi: U^2(C) \rightarrow U^1(C)$
       nondecreasing and concave.

PROOF: To derive "$\Rightarrow$" in (3.1.a), suppose $\succ^1$MRA$\succ^2$. Then in par-
ticular $\bar{\beta}\succ^1\bar{\alpha} \Rightarrow \bar{\beta}\succ^2\bar{\alpha}$, i.e. $U^1(\beta) \geq U^1(\alpha) \Rightarrow U^2(\beta) \geq U^2(\alpha)$, for
all α,β. By Lemma A7.7 $U^2 = \psi \circ U^1$ for a nondecreasing ψ:
$U^1(C) \rightarrow U^2(C)$. To show convexity of ψ, suppose $U^1(\mu)$ is a
weighted mean $\sum_{j=1}^n p_j U^1(\mu_j)$. Then $(p_j;\mu_j)_{j=1}^n \succ^1\bar{\mu}$, by
MRA $(p_j;\mu_j)_{j=1}^n\succ^2\bar{\mu}$. Hence $\sum_{j=1}^n p_j U^2(\mu_j) \geq U^2(\mu)$, i.e.
$\sum_{j=1}^n p_j\psi(U^1(\mu_j)) \geq \psi(U^1(\mu)) = \psi(\sum_{j=1}^n p_j U^1(\mu_j))$: ψ is convex.

   To derive "$\Leftarrow$" in (3.1.a), suppose $U^2 = \psi \circ U^1$ for a convex
nondecreasing ψ. And suppose $(p_j;x_j)_{j=1}^n\succ^1\bar{\alpha}$, i.e.
$\sum_{j=1}^n p_j U^1(x_j) \geq U^1(\alpha)$. To derive is $(p_j;x_j)\succ^2\bar{\alpha}$, i.e.
$\sum_{j=1}^n p_j U^2(x_j) \geq U^2(\alpha)$, or $\sum_{j=1}^n p_j\psi(U^1(x_j)) \geq \psi(U^1(\alpha))$. This is
direct if $U^1(x_j) \geq U^1(\alpha)$ for all j, so suppose $U^1(x_j) < U^1(\alpha)$.

Let $0 < p \leq 1$ be s.t. $(1-p)U^1(x_1) + p\sum_{j=1}^n p_j U^1(x_j) = U^1(\alpha)$.
Since $(1-p) + p\sum_{j=1}^n p_j = 1$, convexity of ψ implies
$(1-p)\psi(U^1(x_1)) + p\sum_{j=1}^n p_j\psi(U^1(x_j)) \geq \psi(U^1(\alpha))$. Since
$\psi(U^1(x_1)) \leq \psi(U^1(\alpha))$ and $p > 0$, we have $\sum_{j=1}^n p_j\psi(U^1(x_j)) \geq \psi(U^1(\alpha))$,
or $\sum_{j=1}^n p_j U^2(x_j) \geq U^2(\alpha)$.

   For (3.1.b), note that $\succ^1$LRP$\succ^2$ if and only if
$[x\prec^2\bar{\alpha} \Rightarrow x\prec^1\bar{\alpha}$ for all $x,\alpha]$. Now the proof of (3.1.b) is completely
analogous to that of (3.1.a), mainly by interchanging $\prec$ and $\succ$,
1 and 2, MRA and LRP, $\geq$ and $\leq$, ψ and ϕ, concave and convex,
$>$ and $<$. $\square$

4. THE MAIN RESULT FOR DMUU.
Recall that we assume C to be a
connected topological space.

THEOREM 4.1. Let $((p_j^k)_{j=1}^n, U^k)$ be a SEU-model for $\succ^k$, with U^k
continuous, $k = 1,2$. Equivalent are:
(4.1.i) $\succ^1$MRA$\succ^2$ [respectively $\succ^1$LRP$\succ^2$].
(4.1.ii) (a),(b) or (c) below applies:
(a) (The nondegenerate case) $U^2 = \psi \circ U^1$ for a convex nondecreasing
    continuous ψ [respectively $U^1 = \phi \circ U^2$ for a concave nondecrea-
    sing continuous ϕ] and $(p_j^1)_{j=1}^n = (p_j^2)_{j=1}^n$.
(b) (The degenerate case of certainty) $U^2 = \psi \circ U^1$ for a nondecrea-
    sing continuous, possibly nonconvex ψ [respectively $U^1 = \phi \circ U^2$
    for a nondecreasing continuous ϕ, possibly nonconcave] and
    $p_j^1 = 1 = p_j^2$ for some j.
(c) (The degenerate case of triviality) $\succ^2$ is trivial [respec-
    tively $\succ^1$ is trivial].

PROOF: (iic) $\Rightarrow$ (i) and (iib) $\Rightarrow$ (i) are straightforward. (iia) $\Rightarrow$
(i) is exactly as the "$\Leftarrow$" 's in Theorem 3.1.

   So now we assume (i), and derive (ii). First the case where
$\succ^1$MRA$\succ^2$, so the parts <u>not</u> between brackets $[...]$. If $\succ^2$ is tri-
vial, i.e. U^2 is constant, nothing remains to be proved. So let
U^2 not be constant. As in the proof of Theorem 3.1., $U^2 = \psi \circ U^1$

with ψ nondecreasing. U^1 and ψ are nonconstant too. By Lemma A.7.7 ψ is continuous.

For all $(\lambda=)U^1(\alpha), (\nu=)U^1(\gamma)$ in the connected $U^1(C)$ we can find $U^1(\beta)$ with $U^1(\beta) = p_1^1 U^1(\alpha) + (1-p_1^1)U^1(\gamma)$. So $(\alpha,\gamma,\ldots,\gamma)\approx^1\bar{\beta}$. By (i) $(\alpha,\gamma,\ldots,\gamma)\succ^2\bar{\beta}$, i.e. $p_1^2 U^2(\alpha) + (1-p_1^2)U^2(\gamma) \geq U^2(\beta)$. So:

(4.2) (Key inequality) $p_1^2\psi(\lambda) + (1-p_1^2)\psi(\nu) \geq \psi(p_1^1\lambda + (1-p_1^1)\nu)$

for all $\lambda,\nu \in U^1(C)$.

Since ψ is a continuous nondecreasing nonconstant function on a connected domain, $p_1^1 = 1 \Leftrightarrow p_1^2 = 1$ and $p_1^1 = 0 \Leftrightarrow p_1^2 = 0$ straightforwardly follow from (4.2). Analogously $p_j^1 = 1 \Leftrightarrow p_j^2 = 1$ and $p_j^1 = 0 \Leftrightarrow p_j^2 = 0$ follow for all $j \neq 1$. For the case where some $p_j^k = 1$, everything has been proved, we are in case (b) then.

Remains the case where not only U^2 is nonconstant, $U^2= \psi\circ U^1$ with ψ nondecreasing and continuous, ψ and U^1 nonconstant, but where also $0 < p_j^1 < 1$ for some j, say j=1. Then also $0 < p_j^2 < 1$. First convexity of ψ is derived. Let $\sigma > \tau \in U^1(C)$. If $p_1^1 \geq p_1^2$, then $p_1^1\psi(\sigma) + (1-p_1^1)\psi(\tau) \geq p_1^2\psi(\sigma) + (1-p_1^2)\psi(\tau) \geq$ (4.2) $\psi(p_1^1\sigma + (1-p_1^1)\tau)$. If $p_1^1 \leq p_1^2$, then $(1-p_1^1)\psi(\sigma) + p_1^1\psi(\tau) \geq (1-p_1^2)\psi(\sigma) + p_1^2\psi(\tau) \geq$ (4.2) $\psi((1-p_1^1)\sigma + p_1^1\tau)$. By Theorem A 7.6 ψ is convex.

Finally in the above mentioned remaining case we must show that $p_j^1 = p_j^2$ for all j. We do it only for $0 < p_1^1 < 1$ and $0 < p_1^2 < 1$, other cases are analogous or have been handled above. Since ψ is convex and nonconstant on a connected domain, there must exist $\mu \in \text{int } U^1(C)$ where ψ is differentiable with $\psi'(\mu) > 0$. Let $\varepsilon \neq 0$ with $|\varepsilon|$ so small that $\lambda := \mu + (1-p_1^1)\varepsilon$ and $\nu := \mu - p_1^1\varepsilon$ in $U^1(C)$. Now $\mu = p_1^1\lambda + (1-p_1^1)\nu$. By (4.2) $p_1^2\psi(\mu + (1-p_1^1)\varepsilon) + (1-p_1^2)\psi(\mu -p_1^1\varepsilon) \geq \psi(\mu)$. Letting ε approach zero, this can only be if $(p_1^2(1-p_1^1)\varepsilon)\psi'(\mu) - (p_1^1(1-p_1^2)\varepsilon)\psi'(\mu) \geq 0$

for all ε close to zero, both positive and negative. $\psi'(\mu)$ being > 0, we conclude that $p_1^2 = p_1^1$.

For the parts not between brackets, (i) $\Rightarrow$ (ii) has been derived. So we turn to the parts between brackets.

[Let $\succ^1\text{LRP}\succ^2$. Then $x\prec^2\bar{\alpha} \Rightarrow x\prec^1\bar{\alpha}$. As in Theorem 3.1, the proof is completely analogous to that where $\succ^1\text{MRA}\succ^2$.] □

REMARK 4.2. Note that, also in (4.1.ii.c) equality of subjective probabilities can be arranged, because the probabilities for the trivial preference relation can be chosen completely arbitrary.

5. SOME GENERAL OBSERVATIONS. In literature usually risk aversion of preference relations $\succ^1$ and $\succ^2$ is compared under the presupposition that $\succ^1$ and $\succ^2$ induce the same ordering on the certain alternatives, i.e. have $U^1= \phi\circ U^2$ for strictly increasing ϕ. Our theorems, in the present general context, show as an implication of comparability of risk aversion, that $U^1= \phi\circ U^2$ or $U^2= \psi\circ U^1$, with ϕ, respectively ψ, only nondecreasing. Note that a nondecreasing concave (convex) $\phi(\psi)$ can be not-strictly-increasing only where it is maximal (minimal). In the case of strict increasingness, MRA and LRP by Lemma A 7.5 are equivalent; MRA and LRP hold simultaneously, only in the case of strict increasingness.

By Theorem A 7.4 and Lemma A 7.7 a characterization of comparability of risk aversion can be obtained by means of quotients of differences of utilities, in the spirit of (e) in Theorem 1 in Pratt (1964), also in our present general context.

Finally we refer to observation 85 in section 3.4 of Hardy, Littlewood and Polya (1959). This mathematical result on comparability of "means" (= "certainty equivalents") in fact is very close to the results of Pratt (1964), and to our Theorem 3.1 for $C \subset \mathbb{R}$.

6. RISK SENSITIVITY OF BARGAINING SOLUTIONS FOR GENERAL

CONSEQUENCE SETS. A (two-person) bargaining situation is a
triple $<C,U^1,U^2>$ where C is a nonempty set of consequences
and U^1, $U^2:C \to \mathbb{R}_+$ are bounded functions, with
$(U^1(\alpha),U^2(\alpha)) > 0$ for some $\alpha \in C$. Such a bargaining situation
$<C,U^1,U^2>$ is interpreted to involve two bargainers who either
agree on some $x \in L^s(C)$ giving $EU^i(x)$ to bargainer i, or dis-
agree, in which case each bargainer ends up with 0 utility.
A (two-person) bargaining game is a compact convex subset S
of $\mathbb{R}^2_+$, with $x > 0$ for some $x \in S$, and with $y \in S$ if $y \in \mathbb{R}^2_+$
with $y \leq x$ for some $x \in S$. B denotes the family of all bar-
gaining games. A bargaining solution is a map $f: B \to \mathbb{R}^2$ with
$f(S) \in S$ for every $S \in B$. Note that, if $\Gamma = <C,U^1,U^2>$ is a
bargaining situation, then

$$S_\Gamma := \{y \in \mathbb{R}^2_+: y \leq x \text{ for some } x \in \text{conv cl } \{(U^1(\alpha),U^2(\alpha)): \alpha \in C\}\}$$

(where "cl" means "closure") is a bargaining game since S
is compact (cf. Rockafellar (1970, Th. 17.2)). The following
property for bargaining solutions was introduced in Kihl-
strom, Roth, and Schmeidler (1981). For its interpretation,
see (3.1.a), (3.1.b); and also section 5, first paragraph.

DEFINITION 6.1. The bargaining solution f is risk sensitive
on a family A of bargaining situations if, for all $i,j \in \{1,2\}$
with $i \neq j$, we have $f_i(S_{\Gamma'}) \geq f_i(S_\Gamma)$ for every
$\Gamma = <C,U^1,U^2> \in A$ where Γ' is the bargaining situation ob-
tained from Γ by replacing U^j by $\phi \circ U^j$ with $\phi: U^j(C) \to \mathbb{R}_+$
strictly increasing, concave, and continuous.

(Strict increasingness and continuity are assumed for con-
venience.) As far as the authors know, all results in the
literature on risk sensitivity are derived for bargaining si-
tuations $<C,U^1,U^2>$ with C a compact subset of a Euclidean
space and with the U^i continuous. Here, in particular in view
of Theorem 3.1, we omit these restrictions. Also, most results
in the literature are derived for bargaining situations
$\Gamma = <C,U^1,U^2>$ satisfying:

$$(6.1) \quad P(S_\Gamma) \subset \text{cl}\{(U^1(\alpha),U^2(\alpha)): \alpha \in C\}.$$

$(P(T) := \{x \in T: y=x \text{ if } y \in T \text{ and } y \geq x\}$ is the Pareto set
of $T \subset \mathbb{R}^2$.) Assumption (6.1) would, in Definition 6.1, imply
that ϕ is defined on the second coordinate of every point of
$P(S_\Gamma)$; which simplifies the analysis. Here, we relax assump-
tion (6.1) and, instead, denote by BS a family of bargaining
situations such that:

(6.2) For every $S \in B$, there is a $\Gamma = <C,U^1,U^2> \in BS$ with
$S = S_\Gamma$ and $P(S_\Gamma) = \text{cl}\{(U^1(\alpha),U^2(\alpha)): \alpha \in C\}$.

Further, by BSC ("certain") we denote the subfamily of bar-
gaining situations in BS satisfying (6.1). For such families
BS with BS $\neq$ BSC, only few results on risk sensitivity of
bargaining solutions have been derived (see Roth and Roth-
blum (1982), and Peters and Tijs (1985)). The following
theorem enables us to extend risk sensitivity results, ob-
tained on BSC, to BS.

THEOREM 6.2. Let the bargaining solution f be risk sensitive
on BSC. Then f is risk sensitive on BS.

In order to prove this theorem, we give some notations and a
lemma. For a nonempty compact convex subset Y of $\mathbb{R}^2$, we de-
note by $\pi^Y:[\bar{p}^Y_1,p^Y_1] \to \mathbb{R}$ the function whose graph is $P(Y)$ (here,
$\underline{p}^Y$ and p^Y are the left and right endpoints of $P(Y)$, respecti-
vely).

LEMMA 6.3. Let $X \subset \mathbb{R}^2$ be nonempty and compact, with $X = P(X)$,
and $Y := \text{conv}(X)$. Let $\phi: \{x_2:(x_1,x_2) \in X \text{ for some } x_1\} \to \mathbb{R}$ be
a continuous strictly increasing concave function such that
$X' \subset P(Y')$ where $X':= \{(x_1,\phi(x_2)): x \in X\}$ and $Y':= \text{conv}(X')$.
Then $\pi^{Y'} = \psi \circ \pi^Y$ with $\psi: [\underline{p}^Y_2,\bar{p}^Y_2] \to \mathbb{R}$ continuous, strictly in-
creasing, and concave.

PROOF: The proof is in a few steps, some of which will be
only outlined, for briefness' sake. For ever $x \in X$, denote
by $\bar{x}$ the point in $P(Y)$ with first coordinate x_1. Define the

function $\bar{\phi}$: $W:= \{\bar{x}_2: x \in X\} \rightarrow \mathbb{R}$ by $\bar{\phi}(\bar{x}_2) := \phi(x_2)$ for every $\bar{x}_2 \in W$. Since $X = P(X)$, $\bar{\phi}$ is well-defined, and since, more-over, ϕ is strictly increasing, so is $\bar{\phi}$. We proceed with showing that $\bar{\phi}$ is concave. In view of Lemma A7.4, it is sufficient to show, for $\bar{x}_2, \bar{y}_2, \bar{z}_2 \in W$ with $\bar{x}_2 > \bar{z}_2$ and $\bar{y}_2 = \alpha \bar{x}_2 + (1-\alpha)\bar{z}_2$ for some $0 < \alpha < 1$, that $\bar{\phi}(\bar{y}_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$. There are eight cases, depending on whether $x,y,z \in P(Y)$ or not, or equivalently, depending on whether $x=\bar{x}$, $y=\bar{y}$, $z=\bar{z}$, or not. The case $x=\bar{x}$, $y=\bar{y}$, $z=\bar{z}$, follows from the concavity of ϕ. The other cases are dealt with in (i) - (iii) below.

(i) Suppose $x=\bar{x}$, $y=\bar{y}$, $z\neq\bar{z}$. We write $\bar{y}_2 = \beta\bar{x}_2 + (1-\beta)z_2$ with $1 > \beta > \alpha$. Then $\bar{\phi}(\bar{y}_2) =$
$\phi(y_2) \geq \beta\phi(x_2) + (1-\beta)\phi(z_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$.

(ii) Suppose $x=\bar{x}$, $z=\bar{z}$, $y\neq\bar{y}$. There are $s,t \in X \cap P(Y)$ with $x_2 \geq s_2 > \bar{y}_2 > t_2 \geq z_2$ such that $\bar{y} = \delta s + (1-\delta)t$ for some $0 < \delta < 1$. Concavity of ϕ implies $\delta\bar{\phi}(\bar{s}_2) + (1-\delta)\bar{\phi}(\bar{t}_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$. Hence $\bar{\phi}(\bar{y}_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$ since otherwise $(\bar{y}_1, \bar{\phi}(\bar{y}_2)) = (y_1, \phi(y_2)) \in X' \subset P(Y')$ would be contra-dicted.

(iii) The case $x\neq\bar{x}$, $y=\bar{y}$, $z=\bar{z}$, is analogous to (i). In the remaining cases at least two out of $\{\bar{x},\bar{y},\bar{z}\}$ are not in X. Suppose, e.g., $x\neq\bar{x}$, $y\neq\bar{y}$, $z=\bar{z}$. First prove the lemma for the case $\{x\} = X\backslash P(Y)$; then replace x by $\bar{x}$ and ϕ by the so obtained $\bar{\phi}$. This construction brings us back in case (ii). Similarly for the other cases.

(iv) We now describe ψ. Let $\lambda \in [\underline{p}_2^Y, \bar{p}_2^Y]$. If $\lambda = \bar{x}_2$ for some $\bar{x}_2 \in W$, then $\psi(\lambda) = \bar{\phi}(\bar{x}_2)$. Otherwise, $\lambda = \alpha\bar{x}_2 + (1-\alpha)\bar{z}_2$ where $0 < \alpha < 1$, and $\bar{x}_2 := \min\{\bar{y}_2 \in W: \bar{y}_2 > \lambda\}$ and $\bar{z}_2 := \max\{\bar{y}_2 \in W: \bar{y}_2 < \lambda\}$ exist in view of the compact-ness of X: then $\psi(\lambda) = \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$. ψ can be seen to be strictly increasing and concave since $\bar{\phi}$ is; and hence continuous (also for $\lambda = \underline{p}_2^Y$). $\square$

PROOF OF THEOREM 6.2. Let $\Gamma = <C,U^1,U^2> \in BS$, and let $\Gamma' = <C,U^1,\phi\circ U^2>$ where $\phi: U^2(C) \rightarrow \mathbb{R}_+$ is strictly increasing, continuous, and concave. We show:

(6.3) $f_1(S_{\Gamma'}) \geq f_1(S_\Gamma)$.

On account of ϕ's properties, we can continuously extend it to a function on $cl(U^2(C))$; we call this extension also ϕ. Next, we apply Lemma 6.3 with $X := \{z \in Z: (z_1, \phi(z_2)) \in P(S_{\Gamma'})\}$ where $Z := cl \{(U^1(\alpha), U^2(\alpha)): \alpha \in C\}$. Then, with notations as in Lemma 6.3, $S_\Gamma = \{x \in \mathbb{R}_+^2: x \leq y$ for some $y \in P(Y)\}$ and $S_{\Gamma'} = \{x \in \mathbb{R}_+^2: x \leq y$ for some $y \in P(Y')\}$. In view of (6.2), $\Delta = <\mathcal{D},V^1,V^2> \in BSC$ exists with $S_\Gamma = S_\Delta$. Then $S_{\Gamma'} = S_{\Delta'}$, where $\Delta' = <\mathcal{D},V^1,\psi\circ V^2>$, with ψ as in Lemma 6.3, is well-defined in view of (6.2). Since f is risk sensitive on BSC, we have $f_1(S_{\Delta'}) \geq f_1(S_\Delta)$, from which (6.3) follows. $\square$

7. (APPENDIX) CONVEX FUNCTIONS ON NON-CONVEX DOMAINS.

In this appendix we adapt some more or less elementary results on con-vex functions to the case of nonconvex domains. No literature on this case is known to us, apart from the following defini-tion, which is given in Peters and Tijs (1981, (2.2)).

DEFINITION A 7.1. Let V be a subset of a real vector space, $\phi: V \rightarrow \mathbb{R}$. Then ϕ is <u>convex</u> [respectively <u>concave</u>] if:
$$\phi(\sum_{j=1}^n p_j\mu_j) \leq [\text{respectively} \geq] \sum_{j=1}^n p_j\phi(\mu_j) \text{ whenever } 0 \leq p_j \leq 1$$
for all j, $\sum_{j=1}^n p_j = 1$, and $\mu_1,\ldots,\mu_n$ <u>and</u> $\sum_{j=1}^n p_j\mu_j$ are in V.

REMARK A 7.2. ϕ is concave if and only if $-\phi$ is convex. All results below for convex ϕ's can be reformulated for concave ϕ's by the substitution $\phi \mapsto -\phi$.

We shall only consider domains in $\mathbb{R}$. Our main tool:

DEFINITION A 7.3. Let $V \subset \mathbb{R}$, $\phi: V \rightarrow \mathbb{R}$, $\mu, \nu \in V$, $\mu < \nu$. Then $[\phi(\nu) - \phi(\mu)]/(\nu - \mu)$ is the <u>average increase of ϕ over</u> $[\mu,\nu]$, denoted $AI_\phi[\mu,\nu]$.

THEOREM A 7.4. Let $V \subset \mathbb{R}$, $\phi:V \rightarrow \mathbb{R}$. Equivalent are:
(A 7.4.i) ϕ is convex.

(A 7.4.ii) Let $0 \le p \le 1, \mu, \tau$ <u>and</u> $p\mu + (1-p)\tau \in V$. Then

$$\phi(p\mu + (1-p)\tau) \le p\phi(\mu) + (1-p)\phi(\tau).$$

(A 7.4.iii) If $\mu < \nu < \tau$ for $\mu, \nu, \tau \in V$, then

$$AI_\phi(\mu,\nu) \le AI_\phi(\nu,\tau).$$

(A 7.4.iv) If $\mu < \nu, \sigma < \tau$, $\mu \le \sigma$, $\nu \le \tau$, for $\mu, \nu, \sigma, \tau \in V$,

then $AI_\phi(\mu,\nu) \le AI_\phi(\sigma,\tau)$.

<u>PROOF</u>: We write AI instead of AI_ϕ. (i) $\Rightarrow$ (ii) is immediate.
For (ii) $\Rightarrow$ (iii), let $p = (\tau-\nu)/(\tau-\mu)$, so $\nu = p\mu + (1-p)\tau$.
Now $\phi(\nu) \le p\phi(\mu) + (1-p)\phi(\tau) \Leftrightarrow AI(\mu,\nu) \le AI(\nu,\tau)$.

For (iii) $\Rightarrow$ (iv), suppose (iii). If $\nu = 0$, (iv) equals (iii).
If $\nu < \sigma$, then by (iii): $AI(\mu,\nu) \le AI(\nu,\sigma) \le AI(\sigma,\tau)$. Finally,
if $\sigma < \nu$, then, with the convention $AI(\mu,\sigma) := AI(\sigma,\nu)$ in case
$\mu = \sigma$, and $AI(\nu,\tau) := AI(\sigma,\nu)$ in case $\nu = \tau$, (iii) gives:
$AI(\mu,\sigma) \le AI(\sigma,\nu) \le AI(\nu,\tau)$. Now $AI(\mu,\nu) \le AI(\sigma,\tau)$ follows,
since $AI(\mu,\nu)$ is a weighted mean of $AI(\mu,\sigma)$ and $AI(\sigma,\nu)$, and
$AI(\sigma,\tau)$ is a weighted mean of $AI(\sigma,\nu)$ and $AI(\nu,\tau)$.
For (iv) $\Rightarrow$ (i), assume (iv). Let $\mu \in V$ be a weighted mean
$\sum_{k=1}^{n} p_k \mu_k$ of elements of V. Let $I = \{i : \mu_i < \mu\}$, $J = \{j : \mu_j > \mu\}$.
Then $\sum_I p_i[\mu-\mu_i] = \sum_J p_j[\mu_j-\mu]$. To prove is $\phi(\mu) \le \sum_{k=1}^{n} p_k\phi(\mu_k)$,

i.e. $\sum_I p_i[\phi(\mu)-\phi(\mu_i)] \le \sum_J p_j[\phi(\mu_j)-\phi(\mu)]$. Let $I \ne \emptyset \ne J$ (other

cases are trivial). Let $\mu_k = \max\{\mu_i : i \in I\}$. Now

$$\sum_I p_i[\phi(\mu)-\phi(\mu_i)] = \sum_I p_i[\mu-\mu_i]AI(\mu_i,\mu) \le \text{(by (iv))} \sum_I p_i[\mu-\mu_i]AI(\mu_k,\mu)$$

$$= \sum_J p_j[\mu_j-\mu]AI(\mu_k,\mu) \le \text{(by (iv))} \sum_J p_j[\mu_j-\mu]AI(\mu,\mu_j) =$$

$$\sum_J p_j[\phi(\mu_j)-\phi(\mu)]. \qquad \square$$

<u>LEMMA A 7.5.</u> Let $V \subset \mathbb{R}$, $\phi: V \to \mathbb{R}$ strictly increasing,
$\psi: \phi(V) \to V$ the inverse of ϕ. Then ϕ is concave if and only
if ψ is convex.

<u>PROOF</u>: ϕ is concave $\Leftrightarrow [\phi(\nu) - \phi(\mu)]/(\nu-\mu) \ge [\phi(\tau)-\phi(\nu)]/(\tau-\nu)$
for all $\mu < \nu < \tau$ in $V \Leftrightarrow (\nu-\mu)/[\phi(\nu)-\phi(\mu)] \le (\tau-\nu)/[\phi(\tau)-\phi(\nu)]$
for all $\mu < \nu < \tau$ in $V \Leftrightarrow [\psi(\tilde\nu)-\psi(\tilde\mu)]/(\tilde\nu-\tilde\mu) \le [\psi(\tilde\tau)-\psi(\tilde\nu)]/(\tilde\tau-\tilde\nu)$
for all $\tilde\mu < \tilde\nu < \tilde\tau$ in $\phi(V) \Leftrightarrow \psi$ is convex. We applied Remark
A 7.2, and tow times (A 7.4.iii) $\Leftrightarrow$ (A 7.4.i). $\square$

<u>THEOREM A7.6.</u> Let $V \subset \mathbb{R}$ convex. Let $\phi: V \to \mathbb{R}$ continuous. Let
for every $\mu > \nu \in V$ there exists $0 < p < 1$ such that
$\phi(p\mu + (1-p)\nu) \le p\phi(\mu) + (1-p)\phi(\nu)$. Then ϕ is convex.
<u>PROOF</u>: See observation 88 in section 3.7 of Hardy, Littlewood
and Polya (1959). $\square$

<u>LEMMA A7.7.</u> Let C be a nonempty set. Let $U^1: C \to \mathbb{R}$ and
$U^2: C \to \mathbb{R}$. Equivalent are:
(A 7.7.i): $U^2 = \phi \circ U^1$ for a nondecreasing $\phi: U^1(C) \to \mathbb{R}$.
(A 7.7.ii): $U^1(\alpha) \ge U^1(\beta) \Rightarrow U^2(\alpha) \ge U^2(\beta)$ for all $\alpha, \beta \in C$.
Furthermore, if C is a connected topological space, and U^1
and U^2 are continuous, then so is ϕ in (i).
<u>PROOF</u>: (i) $\Rightarrow$ (ii) is direct. For (ii) $\Rightarrow$ (i), note that by
(ii): $U^1(\alpha) = U^1(\beta) \Leftrightarrow [U^1(\alpha) \ge U^1(\beta)$ and $U^1(\beta) \ge U^1(\alpha)] \Rightarrow$
$U^2(\alpha) = U^2(\beta)$. So $U^2 = \phi \circ U^1$ for some ϕ. Nondecreasingness is
straightforward. For the furthermore statement note that a
nondecreasing function from a connected $U^1(C)$ onto a connec-
ted $U^2(C)$ must be continuous (it cannot make "jumps"). $\square$

REFERENCES

(1) K.J. ARROW (1971): <u>Essays in the Theory of Risk-Bearing</u>.
     North-Holland, Amsterdam.
(2) G.H. HARDY, J.E. LITTLEWOOD, G. POLYA (1959): <u>Inequalities</u>.
     University Press, Cambridge.
(3) R.E. KIHLSTROM, L.J. MIRMAN (1974): Risk Aversion with
     Many Commodities. Journal of Economic Theory 8, 361-388.
(4) R.E. KIHLSTROM, A.E. ROTH, D. SCHMEIDLER (1981): Risk
     Aversion and Nash's Solution to the Bargaining Problem.
     In: <u>Game Theory and Mathematical Economics</u> (O. Moeschlin,
     D. Pallaschke, eds.), North-Holland, Amsterdam, 65-71.
(5) H. PETERS, S. TIJS (1981): Risk Sensitivity of Bargaining
     Solutions. Methods of Operations Research 44, 409-420.
(6) H. PETERS, S. TIJS (1985): Risk Aversion in n-Person Bar-
     gaining. Theory and Decision 18, 47-72.
(7) J. PRATT (1964): Risk Aversion in the Small and in the
     Large. Econometrica 32, 122-136.
(8) R.T. ROCKAFELLAR (1970): <u>Convex Analysis</u>. Princeton
     University Press, Princeton, New Yersey.
(9) A.E. ROTH, U.G. ROTHBLUM (1982): Risk Aversion and Nash's
     Solution for Bargaining Games with Risky Outcomes. Econome-
     trica 50, 639-647.

(10) M.E. YAARI (1969): Some Remarks on Measures of Risk
     Aversion and on their Uses. Journal of Economic
     Theory 1, 315-329.

Peter Wakker, Tom van Riel   Hans Peters
Department of Mathematics   Faculty of Economics
University of Nijmegen   University of Limburg
Toernooiveld   P.O. Box 616
6525 ED NIJMEGEN   6200 MD MAASTRICHT
The Netherlands   The Netherlands