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Reconciling Savage's and Luce's modeling of uncertainty: The best of both worlds

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HIGHLIGHTS

- Mosaics of events are more suited for modeling uncertainty than (σ-)algebras.
- We connect Luce's and Savage's ways of modeling uncertainty.
- Luce's modeling of uncertainty can be applied to modern decision theories.
- Most models of uncertainty can be embedded in Savage's model.

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ABSTRACT

This paper recommends using mosaics, rather than (σ-)algebras, as collections of events in decision under uncertainty. We show how mosaics solve the main problem of Savage's (1954) uncertainty model, a problem pointed out by Duncan Luce. Using mosaics, we can connect Luce's modeling of uncertainty with Savage's. Thus, the results and techniques developed by Luce and his co-authors become available to currently popular theories of decision making under uncertainty and ambiguity.

1. Introduction

Savage (1954) introduced the best-known and most-used model for decision under uncertainty, with gambles[1] mapping states to consequences. A decision maker chooses a gamble, nature independently chooses a state, and the corresponding consequence results. Duncan Luce pointed out some serious drawbacks to Savage's model. Throughout his career, Luce used the following example to illustrate these drawbacks. We use it as the lead example in our paper:

> If one is considering a trip from New York to Boston, there are a number of ways that one might go. Probably the primary ones that most of us would consider are, in alphabetical order, airplane,[2] bus, car, and train.

When you consider each transportation alternative, you can focus on the uncertainties relevant to that alternative only. However, Savage's model requires you to consider not only the separate uncertainties regarding each alternative, but also all joint uncertainties. Thus, when choosing between airplane and car, you have to consider your degree of belief that both the airplane and the car (had it been taken) would be delayed jointly. This joint event is, however, irrelevant to the decision to be made. Savage's requirement may lead to large and intractable event and gamble spaces. Further, the resolution of joint uncertainties often is not even observable. For instance, if you had chosen to travel by airplane, then you could never fully learn about the delays of the car trip, which did not even take place.

Luce developed various conditional decision models to avoid the aforementioned drawbacks. In the lead example, one then only considers the uncertainties relevant to (conditioned on) each

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[1] This is Luce's term. Savage (1954) used the term act. We use Luce's (2000) terminology as much as possible.

[2] The exact quote is from Luce (2000, Section 1.1.6.1). During his childhood, Luce was much interested in airplanes (besides painting), and he majored in aeronautical engineering. His parents advised against an art career, and astigmatism ruled out military flying, so that he turned to academic research. This history may have contributed to the adoption of this example. Luce used the example also in Krantz, Luce, Suppes, and Tversky (1971, Section 8.2.1) and Luce and Krantz (1971, Section 2).

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transportation alternative separately, with no need to inspect the irrelevant joint uncertainties. As pointed out by Luce and others, these models, while avoiding some problems, create other problems. As we will argue in further detail later, one drawback of Luce's models is that they do not have Savage's clean separation between chance (in nature) and the human will of the decision maker. In Luce's models, both the decision maker and nature may choose (conditioning) events to happen. Another drawback is that part of the mathematical elegance of Savage's model is lost (pointed out by Luce, 2000 p. 7 and discussed below).

This paper reconciles Savage's and Luce's models, with the aforementioned problems solved and the best of both worlds preserved. For this purpose, we propose a generalization of Savage's model, based on Kopylov's (2007)[3] mosaics. Mosaics relax the intersection-closedness requirement of algebras, which is the cause of the aforementioned problems in Savage's model. Using mosaics we can model Luce's lead example without considering irrelevant and inconceivable combinations of uncertainties. At the same time, we maintain Savage's mathematical elegance and his clear separation of nature's influence and the decision maker's influence. We will show that for every Luce (2000) model there exists an isomorphic Savage model, which implies that this isomorphic model can capture all structures and phenomena that Luce's model can, and it can do so in the same tractable manner. In addition, our model satisfies all principles of Savage's model: One state space captures all uncertainties, and the moves of nature and the decision maker are completely separated. In this sense, our model has the best of both worlds.

Our result shows the usefulness of mosaics. The main conclusion of this paper, entailing a blend of Savage's and Luce's ideas, extends beyond the reconciliation obtained. We recommend the use and study of mosaics rather than (σ-)algebras as the event spaces for decision under uncertainty in general. This raises a research question: To what extent can the appealing and useful mathematical results obtained for algebras in the literature be generalized to mosaics? Abdellaoui and Wakker (2005) and Kopylov (2007) provided several positive results.[4]

This paper is organized as follows. Section 2 discusses Savage's (1954) model and Section 3 discusses Luce's (2000) model, the most comprehensive account of his views. Our reconciliation of these two models is in Section 4. Section 5 overviews some other deviations from Savage's model, including Luce and Krantz (1971), which contributed to and preceded Luce (2000). Unlike Luce (2000), the models considered there are not isomorphic to (a generalization of) Savage's (1954) models, but we show that they can still be embedded (i.e., are isomorphic to substructures). Thus we show for all models considered how they can be related to the revealed preference paradigm of economics. Bradley (2007) provides a general logical model that can embed all models considered in this paper as substructures. Section 6 presents a discussion and Section 7 concludes. The Appendix discusses some other generalizations of Savage's model that Luce considered, being compounding, coalescing, and joint receipts, which are tangential to our main topic: connecting Luce's uncertainty model with other uncertainty models popular in the literature today. Our connection allows the introduction of Luce's techniques, including those in the Appendix and follow-up papers,[5] into modern decision theories.

2. Savage (1954)

This section reviews Savage's (1954) model. Savage models uncertainty through a state space S. One state $s \in S$ is true and the other states are not true, but it is uncertain which state is the true one. S is endowed with an algebra $\mathcal{E}$ of subsets called *events*.[6] An *algebra* contains S and is closed under union and complementation. It follows from elementary manipulations that an algebra also contains $\emptyset$ and is closed under finite unions and intersections. An event is *true* if it contains the true state of nature. $\mathcal{C}$ is a set of *consequences*; it can be finite or infinite. A decision maker has to choose between *gambles* (generic symbol G), which are mappings from S to $\mathcal{C}$ with finite image[7] that are measurable with respect to $\mathcal{E}$. *Measurability* of G means that for each consequence x its inverse under G, $G^{-1}(x)$, is an event. It implies that $G^{-1}(D)$ is an event for every subset $D \subset \mathcal{C}$: $G^{-1}(D)$ is a finite union of events $G^{-1}(x)$ of the elements $x \in D$, where only finitely many of these events $G^{-1}(x)$ are nonempty.

The decision maker's comparisons between gambles constitute a preference relation $\succcurlyeq$. Some approaches do not take states and consequences as primitives, with gambles derived, but take gambles and consequences, or gambles and states, as primitives (Fishburn, 1981 Section 8.4; Karni, 2006, 2013). Yet these approaches can be recast in terms of the original Savage model for the purposes of this paper (Schmeidler & Wakker, 1987).

Savage gave a preference foundation for expected utility theory:

$$G \to \int_S U(G(s))\, dP(s)\,. \tag{2.1}$$

Here $U : \mathcal{C} \to \mathbb{R}$ is a *utility function*, and P is a *probability measure* defined on the events. This paper does not discuss which particular decision theory (such as expected utility theory, prospect theory, multiple priors, and so on) is to be used. Its topic concerns the general modeling of uncertainty.

As regards Savage's drawback of involving a complicated event space, we not only have to specify all joint uncertainties but also have to posit axioms sufficiently wide-ranging to generate all likelihoods. Then, further, all gambles whose consequences are contingent on the complicated event space have to be considered. This drawback was elaborated by Luce (2000, p. 6):

> It is certainly not unreasonable to suppose that each mode of travel entails, as a bare minimum, at least 10 distinct [uncertain events].[8] To place this simple decision situation in the Savage

[3] In 2007, Kopylov worked at the economics department of the University of California at Irvine, within a mile of Luce's office who was at the psychology department there. Yet Kopylov developed his idea independently of Luce's work.

[4] Kopylov (2007), while working from a different motivation (see below), in fact already gave a positive answer for Savage's (1954) foundation of expected utility, by extending it to mosaics. Abdellaoui and Wakker (2005, written after and building on Kopylov's paper) provided further generalizations and preference foundations for a number of popular nonexpected utility theories for risk and uncertainty: the Gilboa (1987)–Schmeidler (1989)–Quiggin (1982) rank-dependent utility (including Choquet expected utility), Tversky & Kahneman's (1992) prospect theory (which applies not only to risk but also to ambiguity), and Machina & Schmeidler's (1992) probabilistic sophistication. Generalizations of other models of uncertainty to mosaics is a topic for future research, as is the extension of measure-theoretic concepts to mosaics.

[5] References include (Liu (2003), Luce (2010), Luce and Marley (2005), Marley and Luce (2005), Marley, Luce, and Kocsis (2008)).

[6] In his main analysis, Savage (1954) assumed that $\mathcal{E}$ is the power set, but he pointed out that it suffices that it is a σ-algebra (Section 3.4, pp. 42–43). His preference conditions, especially his P6, imply that S is infinite. Technical aspects such as the difference between σ-algebras and algebras are not important in this paper and we keep these aspects as simple as possible.

[7] We throughout make this assumption, common in decision theory and made throughout Luce (2000, see his p. 3), to simplify the mathematics.

[8] Luce instead used the term outcome. This term commonly refers to uncertain events (states of nature) in probability theory, a convention followed by Luce. In decision theory, however, the term outcome commonly refers to consequences rather than events. To avoid confusion, we do not use this term.

Table 2.1
Decisions impacting states of nature.

	s_1: lung cancer	s_2: no lung cancer
G_1: smoke	Pleasant life, then disease	Pleasant life, healthy
G_2: don't smoke	Unpleasant life, then disease	Unpleasant life, healthy

> framework we must set $S = A \times B \times C \times T$,[9] and so there are at least 10,000 states of nature. Make the problem a bit more complex and it is easy to see that millions or billions of states must be contemplated. I think very few of us are able or willing to structure decisions in this fashion. Rather, we contemplate each of the alternatives as something unitary.

The 10,000 states of nature calculated by Luce involve 9999 subjective probabilities under subjective expected utility theory, and $2^{10,000} - 2$ nonadditive weights under Luce's rank- and sign-dependent theory.[10] Hence Savage's model quickly becomes intractable for empirical applications.

A crucial assumption in Savage's model is that the decision maker does not have any influence on which state is true (Fishburn, 1981, Section 2.2). A classical example showing what goes wrong if this assumption is violated is as follows. For two states $S = \{s_1, s_2\}$ and two gambles G_1, G_2, Table 2.1 displays the consequences.

By Savage's expected utility, and even just by dominance, the only rational choice seems to be G_1: "smoke", giving the better consequence and more utility in every state. This analysis obviously misses an essential point: The choice of the decision maker impacts the probability of the state of nature, making G_2: "don't smoke" a good choice. Hence Savage's model, and the dominance principle, cannot be used in Table 2.1.

In applications, it may be hard to fully satisfy Savage's requirement of a strict separation of the decision maker's and nature's moves. A proper definition of states of nature, specifying the relevant uncertainties beyond control of the decision maker is not easy to achieve in the example of Table 2.1. It is difficult in many applications, e.g. when there is uncertainty about whether you pass an exam next week, whether your next working paper will be accepted by the first outlet tried, whether your new company will survive the first year, whether you will be healthy in five years from now, and so on. Studies of moral hazard in economics (Mas-Colell, Whinston, & Green, 1995; Zeckhauser, 1970), and of causal decision theory in philosophy (Joyce, 1999[11]), do allow for influence on states as an add-on to Savage's model, generalizing it. We will discuss (Sections 3 and 5) more fundamental deviations from Savage's model. They seek to incorporate influences of the decision maker on states and conditioning events so as to simplify Savage' model rather than to generalize it. We start with Luce (2000).

3. Luce (2000)

This section presents Luce's (2000) model of uncertainty, the most comprehensive and most up-to-date version of Luce's views. For a book review of Luce (2000), see Bleichrodt (2001). An historical development of Luce's ideas and some important historical influences are presented in Section 5. Luce (2000) introduced a new primitive: (chance) experiments in a two-stage decision process. First, there is a set $(E^i)_{i\in I}$ of *chance experiments*, with I an index set. Following Luce (2000), we write the indexes as superscripts. $\mathcal{C}$ denotes a set of consequences. The decision maker first chooses an experiment from some available experiments, and then chooses one of the gambles related to the chosen experiment. One experiment may be traveling by airplane, and another experiment may be traveling by car. Each chance experiment E has its own *universal set* Ω_E which, given that chance experiment, plays the same role as Savage's state space, specifying all uncertain events from there on. It is endowed with an algebra $\mathcal{E}_E$ of *events*. The chance experiment E is often identified with its universal set Ω_E. Given one experiment, Luce's model is like an unconditional Savage model. Once the experiment is chosen, there remain several gambles available conditional on the experiment. For example, if one chooses to travel by car, then one may still have to choose which commitments to take at one's destiny, with different pros and cons resulting that depend on travel delays of the car ride.

A gamble conditional on E is of the form $(E_1 : g_1, \ldots, E_n : g_n)$ where $(E_1, \ldots, E_n)$ are events partitioning Ω_E and consequence g_j is assigned to every element of E_j. Under expected utility theory, the gamble is evaluated by

$$\sum_{j=1}^{n} P(E_j)U(g_j) \tag{3.1}$$

with P a probability measure conditional on E and U utility as usual. These expected utility evaluations are also used for comparisons between different chance experiments. The central decision theory in Luce (2000) is a generalization of expected utility: rank- and sign-dependent utility, introduced by Luce and Fishburn (1991). It is essentially equivalent to Tversky & Kahneman's (1992) prospect theory. We focus on the general modeling of uncertainty, without commitment to any specific decision theory. Therefore, an explanation of details of decision theories is not needed for this paper.

Luce's model is more tractable for empirical analyses than Savage's. Consider the lead example where each mode of travel involves 10 distinct further uncertain events. Under Luce's model, the number of states of nature to be contemplated is $4 \times 10 = 40$ instead of 10,000 as under Savage's model. Under subjective expected utility theory, for each mode of travel 9 probabilities are involved, totaling $4\times 9 = 36$, considerably fewer than Savage's 9999. Under rank- and sign dependent theory, $4 \times (2^{10} - 2)$ nonadditive weights are involved, considerably fewer than Savage's $2^{10,000} - 2$.

One drawback of Luce's model is that it has a set of chance experiments without further structure or relations between them. This way of modeling is not mathematically elegant. To illustrate a concrete drawback, assume that a richness requirement is needed that the event space is a continuum, as in Savage (1954).[12] In Luce's model this then needs to be imposed on every chance experiment separately instead of only on one unifying state space.

Also valuing mathematical elegance, Luce (2000, p. 7) considered a step halfway in the direction of Savage's (1954) model. For this purpose, different chance experiments are interpreted as mutually exclusive events. Their union $\bigcup_{i\in I} \Omega_{E^i}$ (called *master experiment*) then is a state space similar to Savage's. However, Luce cautioned against this step conceptually. It leads to a conditional decision model, under which the decision maker first chooses a chance experiment, and then chooses a decision conditional on the experiment chosen. This entails choosing an event (subset of the state space) to come true. Luce (2000, p. 7) wrote critical comments

9 A: airplane; B: bus; C: car; T: train. In his book, Luce used Ω instead of S to denote the universal set.

10 For simplicity we focus on gains. If there are losses and sign-dependence is allowed, then the number of required calculations doubles for all cases considered in this paper.

11 Fishburn (1964, Chs. 2 and 3) similarly considered dependence of probabilities on gambles.

12 His probability measure is atomless with full range [0, 1] (Savage, 1954, Theorem 3.3.3, item 7), which implies a continuum of events.

about the danger of confusing nature's moves with the decision maker's moves:

> A choice of when and how to travel differs deeply from the statistical risks entailed by that choice. The potential for confusion in trying to treat them in a unitary fashion is so great that I eschew this perhaps mathematically more elegant approach.

Luce's (2000) approach avoids the most serious problem of the earlier approach in Luce and Krantz (1971), discussed further in Section 5.2. However, it still does not achieve the clear separation of influences that Savage did. By choosing a chance experiment the decision maker does impact what the true state of nature will be.

The next section presents an alternative to, and a complete reconciliation of Luce's and Savage's models. Both the mathematical elegance of Savage's model and the empirical tractability of Luce's model are obtained, as is Savage's clean separation between nature and human influence. A special case appeared in Abdellaoui and Wakker (2005, Example 5.4.v).

4. Reconciling Luce (2000) and Savage (1954)

In our model, as in Savage's (1954), S denotes the state space, $\mathcal{C}$ the consequence space, and $\mathcal{E}$ the collection of subsets of S called events. To resolve the problems of Savage's model, we relax intersection-closedness. The interest of relaxing intersection-closedness was pointed out before by Luce and his co-authors in Krantz et al. (1971, Section 5.4.1) and was based on findings of quantum mechanics. There a particle's speed and location can be known separately, but they cannot both be known exactly. Hence, Krantz et al. (1971, Section 5.4.1) recommended using Dynkin systems[13] to model such uncertainties. Dynkin systems generalize algebras by requiring closedness only for disjoint unions. Then intersection-closedness is no more implied.

An interest in relaxing intersection-closedness also arose recently in the theory of decision under ambiguity (uncertainty without objective or subjective probabilities), where the intersection of two unambiguous events need not be unambiguous. For example, consider an urn with 100 balls where each ball has a number 0 or 1 and a color red or green. Even if the proportion of the numbers and the proportion of the colors are known and unambiguous, their intersection may not be, if the correlation between number and color is unknown. Zhang (2002) therefore recommended taking the set of unambiguous events to be a Dynkin-system.

Kopylov (2007) proposed yet a further generalization and required the set of unambiguous events to be a mosaic (defined later). He still required that the general domain of events $\mathcal{E}$ be an algebra. He used the mosaic structure only to analyze the subcollection of unambiguous events. We use mosaics for a different purpose. We require that $\mathcal{E}$ itself, the whole domain of events, is a mosaic rather than an algebra. We do so to solve the problems in Savage's model pointed out by Luce, while avoiding the problems of Luce's model.

Definition 4.1. The collection $\mathcal{E}$ of subsets of the state space S is a *mosaic* if it satisfies the following conditions:

(i) It is complementation-closed;
(ii) It contains S;
(iii) For every finite partition $(E_1, \ldots, E_n)$ of S consisting of events (elements of $\mathcal{E}$), $\mathcal{E}$ contains all unions of E_j's. □

To see that mosaics are more general than Dynkin systems, consider two different partitions $(E_1, \ldots, E_n)$ and $(F_1, \ldots, F_n)$. A Dynkin system would still consider the union $E_i \bigcup F_j$ if $E_i \bigcap F_j = \emptyset$, but mosaics do not and, therefore, are more flexible for applications involving multiple chance experiments. *Gambles* G are again mappings from S to $\mathcal{C}$ with finite image that are measurable (for each consequence x, $G^{-1}(x)$ is an event). It still implies that $G^{-1}(D)$ is an event for every subset $D \subset \mathcal{C}$, as $G^{-1}(D)$ is a union of events $G^{-1}(x) : x \in D \cap G(S)$ from a finite partition $\{G^{-1}(x) : x \in G(S)\}$ of S.

We next show how the flexibility of mosaics enables us to construct, for every Luce model, an isomorphic Savage model. We assume a Luce model with $\left\{E^i\right\}_{i \in I}$, Ω_{E^i}, $\mathcal{E}_{E^i}$, and $\mathcal{C}$ as in Section 3. We define the isomorphic Savage model as follows. We keep $\mathcal{C}$ as it is. We define the state space S of Savage not as the union of the E^is (as in Luce's 2000 master experiment), but instead as their product set $\prod_{i \in I} \Omega_{E^i}$. In this product there is a dimension for each chance experiment. A Savage-state $s \in S = \prod_{i \in I} \Omega_{E^i}$ specifies a Luce-state $s^i \in E^i$ for each chance experiment E^i. It may seem at this stage that we still consider joint resolutions of uncertainty as Savage did, because our states refer to such joint resolutions. However, states are not very relevant because they need not be evaluated. Events are relevant for this purpose, and they will not reflect joint uncertainties, as explained next.

The domain $\mathcal{E}$ of *events* in our isomorphic Savage model is defined as $\cup_{j \in I} \left(\mathcal{E}_{E^j} \times \prod_{i \neq j} \{\Omega_{E^i}\}\right) = \left\{A : A = A^j \times \prod_{i \neq j} \Omega_{E^i},\ A^j \in \mathcal{E}_{E^j} \text{ for some } j\right\}$. Thus an event specifies what happens for one chance experiment E^j, leaving the events for all other chance experiments unspecified. In measure theory, such sets are called *cylinder sets*.

It is well-known that $\mathcal{E}$ must be extended, with many sets added, to turn it into an algebra or a σ-algebra. This extension brings many artificial extra gambles, requiring many extra considerations that are practically irrelevant, and this lies at the heart of Savage's problem. For mosaics, however, no extension is needed, because the collection of cylinders itself is a mosaic. This follows mainly because for a partition of S consisting of events, all those events $A_j \times \prod_{i \neq j} \Omega_{E^i}$ with $A_j \in \mathcal{E}_{E^j}$, so as to be disjoint, must concern the same chance experiment E^j. Their unions are again cylinder sets concerning that same chance experiment and, hence, are again contained in $\mathcal{E}$.

Measurability of a gamble G implies, informally speaking, that G depends on only one chance experiment E^j. To see this point, assume that $\{x_1, \ldots, x_n\}$ is the image of G. Then the sets $G^{-1}(x_j)$ constitute a partition of S. Hence, as we saw before, these sets all concern the same chance experiment E^j.[14] We call E^j the chance experiment *relevant* to gamble G. We now obtain a one-to-one correspondence between gambles G in Savage's model and gambles G_L in Luce's model. The chance experiment E^j relevant for the Savage gamble G is the domain of the Luce gamble G_L, and we have

$$G^{-1}(x) = A^j \times \prod_{i \neq j} \Omega_{E^i} \Longleftrightarrow G_L^{-1}(x) = A^j. \tag{4.1}$$

Although our state space is big, this does not affect tractability because we only consider measurable acts and the event space is tractable. Single states, indeed, never need to be considered or

13 Other terms are d-systems, QM-algebras, or λ-systems. Such collections play a role in the mathematics of probability theory when constructing probability measures on σ-algebras (Billingsley, 1968).

14 To see this point, assume that $G^{-1}(x_1)$ refers to chance experiment E^i, imposing restrictions on the ith coordinate of the true state s. Assume, for contradiction, that $G^{-1}(x_2)$ refers to another chance experiment E^j ($j \neq i$), imposing restrictions on the jth coordinate of the true state s. Then $G^{-1}(x_1) \cap G^{-1}(x_2) \neq \emptyset$, consisting of the states that satisfy the requirements for both coordinate i and coordinate j. This contradicts the disjointness of $G^{-1}(x_1)$ and $G^{-1}(x_2)$.

evaluated. The isomorphism of our model with Luce's model shows that we have the same tractability. Consider again the lead example where each mode of travel entails 10 distinct further uncertain events. Our model involves 10,000 states of nature, which is as many as Savage (1954) and way more than Luce's (2000) 36 states. Yet this does not entail a greater complexity because our mosaic does not involve many events. The number of probabilities or weights involved is the same as in Luce's model: 36 probabilities for 36 cylinder sets under subjective expected utility theory, and $4 \times (2^{10} - 2)$ nonadditive weights of cylinder sets under Luce's rank- and sign dependent theory. Hence, we have maintained the tractability of Luce's model.

We have re-established Savage's principle that nature's moves and the decision maker's moves are completely separated. The decision maker cannot in any way influence which state in S is true. She can influence which partition (chance experiment) $\{G^{-1}(x) : x \in \mathcal{C}\}$ is relevant to her, but this is always the case with every choice of gamble in Savage's model. For instance, choosing to fly by plane in economy class rather than (a) flying in business class or (b) traveling by train 1st class or (c) traveling by train 2nd class, consists of two stages in Luce's model: First, the airplane chance experiment is chosen. Second, economy class is chosen. In our Savage model it is only the one-stage choice of the act of flying economy class. The choice of this act automatically implies that the relevant chance experiment/partition describes the uncertainties about the airplane. In the same way, every choice of any act G in Savage's model always implies the "choice" of the relevant events $\{G^{-1}(x) : x \in \mathcal{C}\}$. Thus we have incorporated Luce's choice of chance events, not as influence on which subset of the universal event happens in a separate stage, but as a standard choice of act that fully maintains Savage's separation of choices of acts from choices of states of nature. Influencing the relevant partition in no way influences the true state of nature.

Using mosaics, we have solved a problem in Savage's model that Luce discussed throughout his career. We fully meet the flexibility required by Luce but at the same time maintain all principles of Savage's model and its elegance with one unifying state space. Using the described isomorphism, all results in Luce (2000) and many papers building on it can now be transposed to the currently popular Savage model.

5. Different ways to model uncertainty that preceded and influenced Luce (2000)

Luce was not alone in his search of an alternative to Savage's model. Different models had been proposed before, making different assumptions about where uncertainty comes from. Luce's (2000) model was the culmination of this development. This section discusses this history. We show that all models can be obtained as submodels of Savage (1954) in a mathematical sense, even if at first the concepts of those models seem to be very different.

Savage (1954) is at one extreme of the spectrum: The separation of the decision maker's control over the gamble chosen and nature's control over the state chosen is complete. At the other extreme is Jeffrey's (1965) model: He treats both gambles and events as propositions and uncertainty concerns the truth or falsity of propositions. Thus he does not differentiate at all between moves by the decision maker and moves by nature. Luce throughout took a middle ground between Savage and Jeffrey. In particular, he never recognized Savage's strict separation between the decision maker's and nature's moves.

We next reference some other discussions of models for decision under uncertainty. Fishburn (1981) is an extensive and impressive review of models up to 1981, providing many insights that are still relevant to the literature today. Bradley (2007) presented a general logical model that comprises the models of Jeffrey (1965), Savage (1954), and their intermediate Luce and Krantz (1971) (when interpreted as logical models) as submodels. Spohn (1977) also discussed the models developed up to that point, arguing for the desirability of maintaining Savage's separation.

5.1. Jeffrey (1965): the other extreme

Jeffrey (1965) took an approach fundamentally different from Savage. Bolker (1966, 1967) provided generalizations of Jeffrey's model. Whereas Savage distinguishes states and gambles completely, Jeffrey's model makes no such distinction at all, and even equates them. Both probabilities $P(A)$ and utilities $u(A)$ (called desirability by Jeffrey) are carried by propositions A, which we model as subsets of a space S.[15] To Jeffrey, beliefs and desires are just two sorts of attitude toward the same proposition, say A. We can thus assign a probability, but also a degree of desirability, to a proposition such as passing an exam next week. The interpretation of propositions can be very broad. In terms of Savage's model, propositions can refer to events, gambles, consequences, or their mix. A decision problem can thus be modeled using a single set of propositions. The characteristic condition in Jeffrey's decision theory, making it a relative to Savage's expected utility, is that, whenever $A \cap B = \emptyset$:

$$u(A \cup B) = \frac{P(A)\,u(A) + P(B)u(B)}{P(A) + P(B)} \tag{5.1}$$

where P denotes a probability measure on S. This formula suggests that $u(A)$ may play a role as a conditional expected utility. This suggestion will be formalized next.

Though Jeffrey's and Savage's models are two extremes of the philosophical spectrum, one can formally obtain every Jeffrey model as a substructure of a Savage model. To see this point, we first start from a Savage model. Take his state space S and consequence space $\mathcal{C}$. We assume expected utility theory with probabilities P and utilities U. Suppose that we want to measure the desirability of a gamble $G : \mathcal{S} \to \mathcal{C}$ conditional on an event $A \subset \mathcal{S}$, i.e., $\frac{\int_A U(G(s))dP}{P(A)}$. This can be inferred from the *conditional certainty equivalent* $c_{G_A} \in \mathcal{C}$ of G given A, defined by

$$\begin{aligned} &c_{G_A} \quad \text{is such that } G' \sim G \text{ if} : G'(s) = G(s) \text{ for } s \in \mathcal{S} \setminus A \text{ and} \\ &G'(s) \equiv c_{G_A} \text{ for } s \in A. \end{aligned} \tag{5.2}$$

Based on expected utility theory,

$$\begin{aligned} EU(G) &= \int_{\mathcal{S}} U(G(s))dP \\ &= \int_{\mathcal{S}\setminus A} U(G(s))dP + \int_A U(G(s))dP \\ EU(G') &= \int_{\mathcal{S}\setminus A} U(G'(s))dP + \int_A U(G'(s))dP \\ &= \int_{\mathcal{S}\setminus A} U(G(s))\,dP + U\left(c_{G_A}\right) \cdot P(A). \end{aligned}$$

Since $G \sim G'$, we have $EU(G) = EU(G')$, resulting in $U(c_{G_A}) = \frac{\int_A U(G(s))dP}{P(A)}$, which can be defined as the desirability of G conditional on event A. Using the same method and keeping G fixed, the desirability of other events ("propositions") can be calculated, which makes the comparison between propositions possible. For

15 Jeffrey considered a Boolean algebra of propositions, endowed with logical operations. Such an algebra is isomorphic to an algebra of subsets of a set S (Stone, 1936). For the purposes of this paper, the latter is more convenient.

example, we can now compare a decision maker's desirability of "walking outside in the rain tomorrow" versus "walking outside in strong wind tomorrow".

Formally, for each fixed gamble G, ordering events A by their conditional certainty equivalents $u(A) = U(c_{G_A})$ yields a Jeffrey structure. Conversely, under some richness, every Jeffrey structure can be identified with such a Savage substructure.[16] This way, Jeffrey preferences between events A become observable from Savagean preferences. In particular, Jeffrey's desirability can now be related to the decision-based revealed preference paradigm prevailing in economics. And hence, at least in a mathematical sense, a Jeffrey model can be interpreted as a conditional certainty equivalent submodel of a Savage model with one gamble fixed. Under revealed preference interpretations of desirability, Jeffrey's model is almost dual to Savage's in the sense that the fixed gamble G has been decided outside the control of the decision maker but now the decision maker seems to be choosing between events.

Jeffrey's model can be reinterpreted as an extreme form of state-dependent expected utility.[17] Now utility not only depends on the event considered, but is entirely determined by it. There have been many debates about the identifiability of probabilities under state-dependent utility (Karni, 2003). Drèze (1987, Ch. 2) showed that probabilities then can become observable if the decision maker has some influence on the states of nature. Jeffrey's model is in this spirit.

Ramsey's (1931) famous analysis may be between Jeffrey's and Savage's. It never clearly specifies whether events and consequences can be the same or should be separated. Bradley (2004) provides a detailed analysis of Ramsey (1931). We now turn to another model in between Jeffrey's and Savage's.

5.2. *Luce & Krantz's (1971) conditional decision model: the middle ground*

Inspired by the lead example (see our introduction), Luce and Krantz (1971) developed a conditional decision model.[18] It consists of a set $\mathcal{C}$ of consequences and a state space S with an algebra $\mathcal{E}$ of events $E \subset S$,[19] as in Savage's model. Different from gambles in the unconditional Savage setting, which map the entire space S to $\mathcal{C}$, here *conditional gambles* G_A are considered, which map subsets A of S to $\mathcal{C}$. That is, they are restrictions of gambles G to events A. For example, A can designate the event, decided upon by you, of you traveling by car and G_A then specifies the further uncertainties and results of the travel, laying down a consequence $G(s)$ for each state $s \in A$. If B designates traveling by airplane, then the preference $G_A \succcurlyeq G'_B$ indicates that traveling by car with the consequences specified by G is preferred to traveling by airplane with the consequences specified by G'.

The conditional decision structure synthesizes Savage and Jeffrey. It allows an event, such as A (traveling by car) to be under the control of the decision maker. A choice between G_A and G'_B determines not only the contingencies with which various consequences arise, but also which conditioning event (A or B) will occur. Hence, while Savage's S can be interpreted as one universe, here events can constitute parallel universes, the realization of which depends on the choice of the decision maker. For an event A different conditional decisions G_A, G'_A are conceivable, and this is one way in which the model is richer than Jeffrey's.

For disjoint events A, B and conditional decisions G_A and G'_B, their union

$$G_A \cup G'_B \tag{5.3}$$

denotes the function restricted to $A \cup B$ and equal to G on A and G' on B. We postpone discussing the problematic interpretation of such unions until the end of this section, and now proceed with the characteristic condition of Luce & Krantz's (1971) decision theory:

$$u\left(G_A \cup G'_B\right) = \frac{P(A)\,u(G_A) + P(B)u(G'_B)}{P(A) + P(B)}. \tag{5.4}$$

Again, P denotes a probability measure on S. As in Jeffrey's model, the condition suggests that $u(G_A)$ plays a role as conditional expected utility. This indeed follows because, under some natural conditions (Luce & Krantz, 1971, Section 4):

$$u(G_A) = \frac{\int_A U(G(s))\,dP}{P(A)}. \tag{5.5}$$

Fishburn (1981, Section 8.4) reviews related follow-up studies. For example, Balch and Fishburn (1974) considers a combination of conditional gambles with gamble-dependent probabilities.

We next show that for every Savage model, we can consider a Luce & Krantz substructure: For each G_A we construct the conditional certainty equivalent c_{G_A} as in Eq. (5.2). Through conditional certainty equivalents, Luce & Krantz preferences between conditional gambles G_A become observable from Savagean preferences.

Conversely, for every Luce & Krantz model we can obtain a Savage substructure, simply because all primitives of Savage's model have been provided, with unconditional Savagean preferences equated with Luce & Krantz preferences conditioned on the universal event. This "unconditional" subpart of the Luce & Krantz model indeed completely determines U and P, and thus not only the whole Savage model but also the whole Luce & Krantz model with all its conditional preferences. The Luce & Krantz model can be interpreted as a conditional certainty equivalent submodel of a Savage model but now not with one fixed gamble G considered, as in Jeffrey's model, but with various gambles $G, G', \ldots$ considered.

We finally discuss interpretations of a union $G_A \cup G'_B$ of conditional acts for disjoint events A and B. Whether A or B happens is determined by chance here, with A happening with its probability determined through P, in $\frac{P(A)}{P(A)+P(B)}$. In a choice between G_A and G'_B, however, it is under the control of the decision maker whether A or B happens. We agree with Spohn (1977) that this double interpretation of events is hard to conceive of.

5.3. *From Luce and Krantz (1971) to Luce (2000)*

Possibly Luce (2000) came to agree with Spohn (1977) and later works that the double role of events in Luce & Krantz's (1971) approach (Eq. (5.3)), with both nature and the decision maker choosing between them, is hard to conceive of. Although not explicitly referring to such unions, Luce's citation at the end of our Section 3, criticizing confusions of nature's and the decision maker's moves, suggests so. His model (Luce, 2000), both with and without a master experiment, does not have events that are both under control of nature and the decision maker, which we consider to be an improvement. But Luce (2000) does not yet achieve the clear separation of influences that Savage did and does involve conditional decisions.

There was yet another reason that probably led Luce to abandon unions of conditional gambles as in Eq. (5.3). Since the 1980s an

16 See Fishburn (1981, p. 186). Assume countable additivity. Then the measure $P(A)\,U(A)$ is absolutely continuous with respect to P. With u its Radon–Nikodym derivative, we take U and G in Savage's model such that $u(s) = U(G(s))$ for all s.

17 This relation was suggested by a referee.

18 Krantz et al. (1971, Ch. 8) present it with some modifications and extensions.

19 Null events cannot play the role of conditioning events in what follows. To avoid technicalities, we do not consider null events.

interest has arisen in theories deviating from expected utility, the most popular one being prospect theory (Tversky & Kahneman, 1992). It has not been very widely known that Luce and Fishburn (1991) essentially introduced the same theory called rank- and sign-dependent utility. This theory is central in Luce (2000). Once the realm of expected utility is left, it is no longer clear how to update or weigh unions. There then is no undisputed analogue of Eq. (5.4) (Denneberg, 1994; Dominiak, Duersch, & Lefort, 2012; Miranda & Montes, 2015). This complication will have added to Luce's decision to abandon unions of conditional gambles.

6. Discussion

Although our Savage model in Section 4 is isomorphic to Luce's model and can accommodate the same empirical phenomena, we note some formal differences. In Luce's model, different chance experiments are simply different unrelated entities and there is no formal unifying structure. In the step halfway to Savage, using the master experiment as described in Section 3, different chance events would be disjoint subevents of one encompassing master experiment, but Luce does not commit to such a formalization. His claim "the events of one experiment never appear in the formulation of a different experiment" (p. 6) has no clear formal meaning. It does suggest that the universal sets Ω_{E^j} of all possibilities in a chance experiment are different, and may be mutually exclusive, for different chance experiments. In our model they are all the same, being the state space S (the universal statement that is always true). For us, different chance experiments refer to different partitions of the same state space, with that state space reflecting one universal truth.

The term Savage model as used in this paper is defined in Section 2, with states, acts, consequences, events, and a complete preference relation. A technical difference with Savage's (1954) decision theory is that we did not impose his richness conditions, mainly his P6. They imply that S is infinite for instance, whereas we allow finite S. Another difference is that his set of events is an algebra and even a σ-algebra, which is essentially used in his analyses. Hence his preference foundation cannot be directly used, for instance. Here Kopylov's generalization is useful.

In our Savage models that are isomorphic to Luce's models, two events $A^i \in \Omega_{E^i}$ and $B^j \in \Omega_{E^j}$ of different chance experiments have a nonempty intersection $A^i \times B^j \times \prod_{k \neq i,j} \Omega_{E^k}$. The true state of nature may be such that both events are true. However, this intersection is no event: it is not contained in the mosaic. It then is not relevant to any decision to be made. We are not interested in it, and need never evaluate, analyze, or even think about it.

In natural language, different choices of gambles, with different partitions of S relevant, can be called different events. In the formal language of decision theory and probability theory, as in Savage's model, such terminology is not possible. Events only refer to subsets of the state space. That is, they only describe moves by nature. Luce never followed this strict separation of nature's moves from decision maker's moves in Savage's model.

In our Savage models isomorphic to Luce's models, any event (except $\emptyset$ and S) appearing in one chance experiment does not appear in any other, in agreement with Luce's assumptions. But this need not hold for general mosaics. In general it may well happen that an event A occurs in two partitions $\{A, B_2, \ldots, B_n\}$ and $\{A, C_2, \ldots, C_m\}$ that have nothing other in common. Here, intersections $B_i \cap C_j$ of other events of the chance experiments may not be events, i.e., they may not be contained in the mosaic. In the lead example, event A could entail that an earthquake destroys the route to the destination, preventing all modes of transportation.[20]

We prefer mosaics to Dynkin systems. That is, in the preceding paragraph we prefer not to speculate on whether $B_i \cap C_j = \emptyset$ or on committing to then having $B_i \cup C_j$ as an event (that should be evaluated according to the decision theory considered). In this regard mosaics better capture a desirable feature of Luce's chance experiments than Dynkin systems do. We consider this to be a desirable extra flexibility of mosaics.

There is a renewed interest in the foundations of uncertainty in the modern literature on uncertainty. Ellsberg (1961) showed that, surely from a descriptive perspective, we often cannot assign traditional probabilities to uncertain events, calling for generalizations ("ambiguity") of Savage's expected utility formula. After initiating work by Gilboa (1987), Gilboa and Schmeidler (1989), and Schmeidler (1989), many ambiguity theories have been developed, including Luce & Fishburn's (1991) rank- and sign-dependent utility. This may have also contributed to Luce's (2000) decision to modify Luce & Krantz's (1971) uncertainty model.

Although many modern nonexpected utility theories stayed within Savage's general uncertainty model[21], there also have been more fundamental deviations. Those involved uncertainties about the state space, as in unforeseen contingencies (Ahn & Ergin, 2010; Dekel, Lipman, & Rustichini, 1998; Karni & Vierø, 2013) and case-based decisions (Gilboa & Schmeidler, 2001). Ahn (2008) proposed an ambiguity model that used Jeffrey's (1965) uncertainty model, also deviating from Savage's general model. Ahn and Ergin (2010) incorporated Luce's partition dependence (see our Appendix), unaware of Luce's precedence. Our analysis has shown how Luce's models and theories can be invoked in uncertainty models commonly used today. The many results and tools provided in Luce's book can now be used in modern studies, and new researchers can better become aware of Luce's precedences.

7. Conclusion

We have shown how various models of uncertainty can be embedded into the most well-known model today: Savage's (1954) state space model. Thus we have shown how all these models can be related to the decision-based revealed preference paradigm of economics. In particular, we showed that Luce's most comprehensive model, in Luce (2000), not only can be embedded, but even can be related isomorphically to Savage's model. To this effect, we needed a version of Savage's model more general than used by Savage (1954), taking the collection of events as a mosaic rather than as an algebra. This way we avoided the overly restrictive intersection-closedness requirement of algebras. We can now handle parallel uncertainties without having to consider their joint resolutions. The results, techniques, and empirical flexibility of Luce (2000) and the follow-up works by him and his colleagues now become available to modern decision theories.

The main recommendation of our paper is that the literature on decision under uncertainty use and study mosaics rather than the common algebras. This way, more tractable models that are better suited for applications result. It raises a research question for future studies: How can existing results for algebras be generalized to mosaics?

20 Such an event is plausible in Luce's domicile, Irvine in California, both in 1971 and in 2000.

21 For the purposes of this paper, the Anscombe and Aumann (1963) model, popular in modern decision theory, belongs to Savage's general uncertainty model, with a state space and a clear separation between moves by the decision maker and moves by nature.

Appendix. Luce's generalizations of Savage (1954) that are tangential to our analysis

We next turn to three other generalizations of Savage's model that Luce incorporated. They are tangential to the proper definition of a state space and thus to the main topic of this paper. They are add-ons that can be added or removed from every model considered in the main text. An important motivation for the results provided in the main text is to make the following generalizations available to modern Savage-based analyses.

First, in some parts of Luce's analysis, he allowed for compound gambles. Let $G_0 = \mathcal{C}$ and let G_1 denote the set of all gambles defined before; i.e., they are mappings from a chance experiment into G_0. Hence, $G_1 \supset G_0$ and G_1 is defined as the set of first-order gambles. One can continue this process recursively, defining gambles in G_k as kth-order gambles, being mappings from a chance experiment into G_{k-1}. Compounding implies that all uncertain events considered should be repeatable, which of course is a restrictive assumption.

Luce assumed backward induction. That is, each first-order subgamble in a compound gamble can be replaced by its certainty equivalent. This certainty equivalent is assumed to be the same as if the subgamble had been an unconditional first-order gamble that was not a subgamble of another gamble. In this way we can use backward induction to evaluate a gamble. Although Luce (2000, Definition 2.3.1) called this backward induction assumption monotonicity, it is controversial outside of expected utility (Luce, 2000, Sections 2.3.1–2.3.2). Monotonicity with respect to subjective underlying orderings amounts to separability, and restrictive dynamic decision principles are then implied (Machina, 1989).

Another problem for backward induction arises for uncertainty, when probabilities of events are unknown. Luce's model precludes learning when the same chance event is used for compounding. The subjective beliefs remain the same for that chance event, ignoring the preceding sampling evidence. Assume that we have randomly selected a ball from an Ellsberg urn of unknown composition 100 times with replacement, and all 100 times the color of the ball was red. Then for the 101th sample our belief of red showing up has increased, and our certainty equivalent for the gamble (red: \$100, not-red: 0) is higher than it was in the beginning. In Luce's approach, the 100 preceding events should provide no information about the 101th replication and its certainty equivalent should be the same as in the beginning, which is hard to imagine. Luce (2000, p. 10) referred to personal communication with one of us about this problem, but left it as an open problem.

If we only have Savage's model without replications or compounding, then we can nevertheless make the entire compounding structure observable. For any compound gamble, we only need Savage's first-order preferences to carry out backward induction: we replace final-stage gambles by certainty equivalents step by step, ending up with the certainty equivalent of the compound gamble. In this way any preference between compound gambles in Luce's setup can be observed using merely Savage-type preferences. In this procedure, learning effects are excluded. It therefore provides a solution to the open problem of the preceding paragraph, suggested to us by Anthony Marley.

Luce considered another generalization of Savage's setup. Here $(E_1 : g_1, \ldots, E_n : g_n)$ is not interpreted as a function from Ω_E to $\mathcal{C}$ with the E_js partitioning Ω_E. Instead, it is taken as a general $2n$-tuple. It is then allowed to treat $(E_1 : \alpha, E_2 : \alpha, E_3 : g_3, \ldots, E_n : g_n)$ differently than $(E_1 \cup E_2 : \alpha, E_3 : g_3, \ldots, E_n : g_n)$ even though they concern the same function from Ω_E to $\mathcal{C}$ and, hence, must be indifferent and even identical in Savage's approach. In this manner, violations of event coalescing (Luce, 2000, Section 1.1.6.3) and other elementary rationality conditions can be accommodated. A recent model incorporating this idea, and unaware of Luce's precedence, is Ahn and Ergin (2010).

Luce also considered joint receipts $x \oplus y$ of consequences, meaning that one receives both x and y. If consequences are monetary, then decision makers may perceive $x \oplus y$ differently than $x + y$. For example, a gain of \$200 followed by two losses of \$50 may be perceived differently than a gain of \$100. Thaler's (1985) mental accounting is an empirical counterpart to Luce's joint receipts. Luce introduced joint receipts so as to obtain (in our terminology) cardinal evaluations of consequences (Luce, 2000, 4th approach in Section 1.3). An alternative way is to use multiattribute techniques (Luce's 1st approach in his Section 1.3), but not in combination with probability mixtures as in Keeney and Raiffa (1976) and indicated by Luce, but instead with the measurement techniques of Krantz et al. (1971). Then standard sequences of Krantz et al. (1971) give cardinal evaluations of consequences. This underlies the tradeoff technique of Abdellaoui (2000), Bleichrodt and Pinto (2000), and Wakker (1984, 2010). This technique has also been used in many papers by Prelec (e.g., Prelec, 1998, Definition 1) and Tversky (e.g., Tversky, 1977, p. 351 axiom 5: invariance), and occasionally by Luce (Luce & Krantz, 1971, axiom 5). Modern papers in uncertainty commonly use yet another way to obtain cardinal evaluations of consequences: through Anscombe & Aumann's (1963) model (Etner, Jeleva, & Tallon, 2012; Gilboa & Marinacci, 2013). There are however drawbacks to this model (Machina, 2014 3rd example and p. 3835, 3rd bullet point; Wakker, 2010 Section 10.7.3), and Luce's joint receipts can serve as an alternative tool.

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