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PREFERENCE FOUNDATIONS FOR NONEXPECTED UTILITY: A GENERALIZED AND SIMPLIFIED TECHNIQUE

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This paper examines a tradeoff-consistency technique for testing and axiomatically founding decision models. The technique modifies earlier tradeoff-consistency techniques by only considering indifferences, not strict preferences. The resulting foundations are both more general and more accessible than earlier results, and they are so regarding both the technical and the intuitive axioms. Axiomatizations based on other, bisymmetry-like, conditions also follow as corollaries from the results of this paper. The tradeoff technique is applied to three popular theories of individual decision under uncertainty and risk, i.e., expected utility, Choquet expected utility, and prospect theory. The conditions used are better suited for empirical measurements of utility than earlier tradeoff conditions and, accordingly, are easier to test.

1. Introduction. This paper presents preference foundations for expected utility, Choquet expected utility, and prospect theory. The technique that we use can be applied to any model based on weighted averages, such as intertemporal choices with discounted utility, utilitarian welfare, and their rank- and sign-dependent generalizations. We will formulate our results for decision under risk and uncertainty.

Many preference foundations have been proposed before. Such foundations typically use intuitive axioms such as transitivity, which refer to directly meaningful empirical properties of preferences, and technical axioms such as continuity, which describe the structural richness of the model and serve to simplify the mathematical analysis. Our new intuitive axioms are based on a tradeoff technique that improves upon earlier ones by only using indifferences, not preferences. This improvement is inspired by axioms used by Tversky (see Appendix A). The resulting tradeoff technique is more closely related to empirical measurements of utility, and, therefore, our intuitive axioms are easier to test. In all our results, except those for prospect theory, our technical axioms are of an algebraic nature as in Krantz et al. (1971), and do not require prior knowledge of topology. Still, they give results that are more general than the existing topological results, which follow as corollaries. Like our intuitive axioms, our algebraic axioms are directly related to the empirical measurement of utility, which makes them more accessible.

Section 2 presents the notation, and describes a simple but essential aspect of our technique for studying various nonexpected utility models, namely the restriction of preference conditions to subsets of the domain. By means of the notation developed, many results for subjective expected utility, Choquet expected utility, and prospect theory, can all be obtained efficiently at one blow. Throughout this paper, we use the term prospect theory in its modern sense, referring to what is also called new or cumulative prospect theory (Tversky and Kahneman 1992). This theory modifies original prospect theory (Kahneman and Tversky 1979) by incorporating Quiggin's (1982) and Schmeidler's (1989) ideas of rank-dependence.

Section 3 describes the indifference-tradeoff technique. Section 4 is the central section of this paper, with preference foundations for subjective expected utility in §4.1, Choquet

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expected utility in §4.2, and prospect theory in §4.3, all for finite state spaces. Section 5 describes extensions to infinite state spaces, binary (=two-outcome) acts, and decision under risk. Section 6 presents a comparison with the existing literature, and, in §7, comparisons with bisymmetry-based axiomatizations are given. All those axiomatizations follow as corollaries from the results of this paper. Section 8 concludes. Appendix A elaborates on the way in which our tradeoff techniques are more general than earlier tradeoff techniques, with details about earlier tradeoff-consistency axioms deferred to Appendix B. Proofs are in Appendixes C and D.

2. Basic definitions of decision under uncertainty and their restrictions to subsets of the domain. We use Savage's (1954) model for decision under uncertainty, with $S = \{1, \ldots, n\}$ the *state* space and X the *outcome* space. Subsets of S are called *events*. For simplicity, the state space is taken to be finite. Subsection 5.1 describes extensions to general state spaces. In the derivations of expected utility and Choquet expected utility (§§4.1 and 4.2), richness is imposed on X through a solvability axiom. This axiom can be satisfied if X is an interval in the reals, designating money, or $\mathbb{R}^m_+$, designating commodity bundles. It can also be satisfied for particular finite ("equally spaced") sets of, say, discrete and qualitative health states. In the derivation of prospect theory (§4.3), a more restrictive topological condition, namely, connectedness, is imposed on X, which is satisfied for intervals in the reals and for $\mathbb{R}^m_+$, but not for finite sets.

X^n is the set of *acts*. An act $f = (f_1, \ldots, f_n)$ is identified with the function assigning f_j to each state j. Outcomes are identified with constant acts. A *preference relation* $\succcurlyeq$ is a binary relation on X^n. *Strict preference* $\succ$, *indifference* or *equivalence* $\sim$, and *reverse preferences* $\preccurlyeq$ and $\prec$ are defined as usual. Given that outcomes are identified with constant acts, preferences also apply to outcomes.

A function V *represents* $\succcurlyeq$ on a set $F \subset X^n$ if

(1) $$\text{for all acts } f, g \in F,\ f \succcurlyeq g \text{ if and only if } V(f) \geq V(g).$$

In this definition, we implicitly assume that F is contained in the domain of V, and that V maps acts to the reals. Restrictions of representations and preference conditions to subsets of acts, such as F above, play a central role in this paper. In the above definition, as in other definitions throughout this paper, we omit "on F" if $F = X^n$. If a representing function exists (i.e. on X^n), then $\succcurlyeq$ is a *weak order*, meaning that it is *complete* ($f \succcurlyeq g$ or $f \preccurlyeq g$ for all acts f, g) and transitive. In all models in this paper, *weak monotonicity* will hold, i.e., if $f_j \succcurlyeq g_j$ for all j, then $f \succcurlyeq g$.

Subjective expected utility (*SEU*) holds on a set $F \subset X^n$ if there exist nonnegative probabilities $p_1, \ldots, p_n$ summing to one and a *utility function* U: $X \to \mathbb{R}$ such that

(2) $$\text{SEU: } f \to \sum_{j=1}^{n} p_j U(f_j)$$

represents preferences on F. We then call F an *SEU-set*.

For an outcome α, an act f, and a state i, $\alpha_i f$ denotes the act f with f_i replaced by α. A state i is *null* on $F \subset X^n$ if $\alpha_i f \sim f$ for all $\alpha_i f, f$ in F. Otherwise, the state is *nonnull* on F. If a state is nonnull on an SEU-set F, then its probability must be positive. The converse implication holds under minimal richness of F (if F contains $\alpha_i f, f$ with $U(\alpha) > U(f_i)$). Finally, $\succcurlyeq$ is *trivial* if all acts are indifferent.

3. A general tool for axiomatizing models. The following $\sim^*$ relation is central to this paper. It can be interpreted as a strength of preference relation over outcomes revealed from ordinal preferences over acts. For $F \subset X^n$ and outcomes α, β, γ, δ, we write

(3) $$[\alpha; \beta] \sim^*_F [\gamma; \delta] \quad \text{or} \quad \alpha\beta \sim^*_F \gamma\delta,$$

if there exist acts f, g and a state j that is nonnull on F, such that

$$\alpha_j f \sim \beta_j g \quad \text{and} \quad \gamma_j f \sim \delta_j g, \tag{4}$$

with all four displayed acts contained in F. The interpretation is that receiving α instead of β apparently does the same as receiving γ instead of δ; i.e., it exactly offsets the receipt of the g_is instead of the f_is contingent on the other states $i \neq j$. Köbberling and Wakker (2001), in a predecessor and informal counterpart to this paper, discussed the interpretation and normative appeal of this relation in detail for the special case of expected utility. We use the simpler notation $\alpha\beta$ instead of $[\alpha; \beta]$ whenever no confusion is likely to arise. If $F = X^n$, then we drop the subscript F in Equation (3) and write $\sim^*$. A similar relation, where intervals on one dimension are matched whenever they match the same interval on another dimension, was defined by Tversky (1977, Appendix: Invariance Axiom). Substitution shows that, for SEU-sets F,

$$\alpha\beta \sim_F^* \gamma\delta \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta). \tag{5}$$

We next list some symmetries for the relation $\sim_F^*$ that are desirable for the interpretation of derived strength of preference.

Observation 1. *If $\succcurlyeq$ is a weak order, then the following four relationships are equivalent*:
(i) $\alpha\beta \sim_F^* \gamma\delta$;
(ii) $\gamma\delta \sim_F^* \alpha\beta$;
(iii) $\beta\alpha \sim_F^* \delta\gamma$;
(iv) $\delta\gamma \sim_F^* \beta\alpha$. □

The second symmetry follows from the first by reversing the right-hand and left-hand indifferences in Equation (4), the third from reversing the right-hand and left-hand acts in both indifferences in Equation (4), and the fourth by combining these two operations. The $\sim_F^*$ relation need not be transitive and, consequently, need not be an equivalence relation. We will see in Observation 34 that transitivity is a strong restriction that comes close to implying SEU on the relevant set F.

Equation (5) shows that the $\sim_F^*$ relation can be used to elicit equalities of utility differences, which, for cardinal utility, suffices to determine the entire utility function. Obviously, we would want such measurements to be consistent, and not lead to contradictions. Therefore, there should not exist one pair of acts f, g and a nonnull state j that reveal $\alpha\beta \sim_F^* \gamma\delta$ as in Equation (4), and another pair of acts f', g' and a nonnull state j' that imply $\alpha'\beta \sim_F^* \gamma\delta$ for an $\alpha' \succ \alpha$. Such a finding would not combine well with the intuition that, for example, $\alpha'\beta$ should exceed $\alpha\beta$ as soon as $\alpha' \succ \alpha$, and that it is, therefore, not plausible that both $\alpha'\beta$ and $\alpha\beta$ match the same $\gamma\delta$. Substitution of SEU shows that the two $\sim_F^*$ relations do indeed require the equality $U(\alpha) = U(\alpha')$. The following definition is equivalent to the condition for α just stated because of symmetry in the four outcomes (Observation 1).

Definition 2. *Tradeoff consistency* holds on F if improving an outcome in any $\sim_F^*$ relationship breaks that relationship. □

The condition implies, for instance, that $\alpha\beta \sim_F^* \gamma\delta$ and $\beta' \succ \beta$ preclude $\alpha\beta' \sim_F^* \gamma\delta$, etc. Because of symmetry of β and β', the $\sim_F^*$ relationship is also precluded if $\beta' \prec \beta$. That is, worsening an outcome in any $\sim_F^*$ relationship breaks that relationship also.

Tradeoff consistency is the central idea of this paper, and several variations will be considered later on. The tradeoff-consistency condition is formulated in terms of the $\sim_F^*$-relation, which is not an empirical primitive but is derived from preferences. The condition can easily be reformulated directly in terms of preferences, through the requirement that the three indifferences $\alpha_i f \sim \beta_i g$, $\gamma_i f \sim \delta_i g$, and $\alpha_j x \sim \beta_j y$ should always imply the fourth indifference $\gamma_j x \sim \delta_j y$, under domain restrictions corresponding to the theory under consideration. This condition is equivalent to tradeoff consistency and is analyzed in Appendix B. We use the $\sim_F^*$-relations in the main text because evaluations of tradeoffs play a natural role in decision processes.

4. Applications of our technique. This section shows how our tradeoff technique can be applied to three popular models of decision under uncertainty: subjective expected utility, Choquet expected utility, and prospect theory. Gilboa, Schmeidler, and Wakker (2002) applied a similar technique to case-based decision theory.

4.1. The algebraic approach for subjective expected utility. We use the notation $\alpha_i\beta_j f$ to denote the act f with f_i replaced by α and f_j replaced by β, for $i \neq j$. Empirical measurements of utility often elicit *standard sequences* on sets $F \subset X^n$, i.e., sequences of outcomes $\alpha^0, \alpha^1, \ldots,$ for which there exist an act f, states $i \neq j$ that are nonnull on F, and ("gauge") outcomes $g \not\sim G$, such that $(\alpha^{k+1})_i g_j f \sim (\alpha^k)_i G_j f$ for all k, with all acts contained in F (Abdellaoui 2000, Abdellaoui and Munier 2001, Bleichrodt and Pinto 2000, 2001, Bleichrodt, Pinto, and Wakker 2001, de Blaeij and van Vuuren 2003, Fennema and van Assen 1998, Wakker and Deneffe 1996). A standard sequence entails a special way of observing that $\alpha^{k+1}\alpha^k \sim_F^* \alpha^k\alpha^{k-1}$ for all k, and, thus, of obtaining a sequence of outcomes that are equally spaced in utility units on an SEU-set. The nonequivalence of the gauge outcomes g and G serves to avoid trivial standard sequences of equivalent outcomes. The standard sequence is *increasing* if $G \succ g$, and *decreasing* if $G \prec g$. Under monotonicity requirements, $\alpha^{k+1} \succ \alpha^k$ for all k if the standard sequence is increasing, and $\alpha^{k+1} \prec \alpha^k$ for all k if the standard sequence is decreasing (Lemma 32). Formally, standard sequences can be finite or infinite. The two technical axioms of the "algebraic approach" defined next play a role in the method of measuring utility just described.

DEFINITION 3. *Solvability* holds if for all acts f, g, outcomes α, γ, and states i with $\alpha_i f \succ g \succ \gamma_i f$, there exists an "intermediate" outcome β such that $\beta_i f \sim g$. □

Solvability ensures, for instance, that a further outcome ($\beta=$) α^{k+1} can be defined to prolong an increasing standard sequence as just considered as long as a sufficiently extreme γ exists to have $\gamma_i g_j f \succ (\alpha^k)_i G_j f \succ (\alpha^k)_i g_j f$. In earlier papers, this condition was usually called restricted solvability. Given its importance, we prefer to use a more efficient term.

For the second technical axiom, we make two preparatory observations. First, a standard sequence is *bounded* if there exist an outcome weakly more preferred, and an outcome weakly less preferred, than all elements of the standard sequence. On SEU-sets, a bounded standard sequence must be finite. Otherwise, the standard sequence being equally spaced in utility units, either the utility of the more-preferred outcome would have to be infinite if the standard sequence were increasing, or the utility of the less-preferred outcome would have to be minus infinite if the standard sequence were decreasing. In each case, SEU, which does not allow for infinite utility, would be violated.

The second preparatory observation concerns the *degenerate case*, where exactly one state is nonnull. Then preferences do not reveal uncertainty. There do not exist nontrivial standard sequences, and finiteness of bounded standard sequences is vacuously satisfied. To avoid non-Archimedean representations, we then use a condition from representation theory for single-dimensional objects. It is well known that there exists a real-valued representation for a weak order on the set of outcomes if and only if there exists a countable *order-dense* set of outcomes, i.e., a set of outcomes such that for all outcomes α, γ with $\alpha \succ \gamma$, there exists an element β of the order-dense set with $\alpha \succcurlyeq \beta \succcurlyeq \gamma$ (Krantz et al. 1971, Theorem 2.2).

DEFINITION 4. The *Archimedean axiom* holds on F if every bounded standard sequence is finite, and, in the degenerate case of exactly one nonnull state on F, there exists a countable order-dense set of outcomes. □

The main axiom characterizing SEU is *tradeoff consistency*, which means that tradeoff consistency holds on X^n, the whole set of acts. At this point, we are ready to state the first representation theorem, which characterizes SEU. *Uniqueness of utility U up to unit and location* means that U can be replaced by U^* if and only if $U^* = \tau + \sigma U$ for a real τ and positive σ.

THEOREM 5 (CHARACTERIZATION OF SEU). *Assume that solvability holds. The following two statements are equivalent for $\succcurlyeq$ on X^n.*

(i) *SEU holds.*

(ii) *$\succcurlyeq$ satisfies*

  (a) *weak ordering*;

  (b) *weak monotonicity*;

  (c) *the Archimedean axiom*;

  (d) *tradeoff consistency.* □

OBSERVATION 6 (UNIQUENESS RESULTS FOR THEOREM 5). *Suppose that the assumptions and statements of Theorem* 5 *hold.*

(a) *In the trivial case where all states are null, utility is constant and the probabilities can be chosen arbitrarily.*

(b) *In the degenerate case where there is exactly one nonnull state, utility is unique up to a strictly increasing transformation. The probability of the nonnull state is one, and all other states have probability zero.*

(c) *In the nondegenerate case where there are two or more nonnull states, utility is unique up to unit and location, and the probabilities are uniquely determined.* □

The only nonnecessary condition in the above theorem, and in some results given later, is solvability. Examples in Appendix A will illustrate to what extent solvability is restrictive. Theorem 5, like several of its variations in the literature, can be interpreted as a modification of Savage's (1954) SEU characterization that allows for finite state spaces. Equally, it can be interpreted as a modification of Anscombe and Aumann's (1963) SEU characterization in which outcomes need not be lotteries over prizes to be evaluated linearly through an expected-utility formula, and in which multistage gambles and assumptions on dynamic optimization need not be used. Avoiding a commitment to expected-utility evaluation in a second stage and to dynamic-decision principles is especially desirable for the nonexpected utility models derived in later sections. The main motivation for Wakker (1984) to develop a predecessor of our result was yet another one, namely that it can be interpreted as an extension of de Finetti's book-making argument, in the sense of subjective expected value maximization, to nonlinear utility.

An interesting alternative characterization of SEU results if tradeoff consistency in (d) in Statement (ii) of Theorem 5 is replaced by transitivity of $\sim^*$, or by the requirement that $\sim^*$ is an equivalence relation (Observation 34). The adaptation of these results to nonexpected utility models is not very transparent. It can be seen that the $\sim^*$ relation need no longer apply to all equal utility differences under nonexpected utility, so that complex acyclicity conditions have to be used instead of transitivity. We do not elaborate on this point, and only consider tradeoff consistency in the main text.

4.2. The algebraic approach for Choquet expected utility. We next turn to Choquet expected utility, the rank-dependent generalization of SEU. Primarily, it allows for nonadditive probabilities. A *weighting function* W is a set function $W\colon 2^S \to [0, 1]$ such that $W(\varnothing) = 0$, $W(S) = 1$, and $A \subset B \Rightarrow W(A) \le W(B)$. Other terms are capacity or nonadditive probability. *Choquet expected utility* (CEU) holds if there exist a weighting function $W\colon 2^S \to [0, 1]$ and a *utility function* $U\colon X \to \mathbb{R}$ such that

$$\text{(6)} \qquad \text{CEU: } f \to \sum_{j=1}^{n} \pi_{\rho_j} U(f_{\rho_j})$$

represents preferences. The *decision weights* π_{ρ_j} are nonnegative and sum to one. They are derived from W and the act f as follows:

- choose a *rank-ordering permutation* ρ on $\{1, \ldots, n\}$ such that $f_{\rho_1} \succcurlyeq \cdots \succcurlyeq f_{\rho_n}$;
- define $\pi_{\rho_j} = W(\rho_1, \ldots, \rho_j) - W(\rho_1, \ldots, \rho_{j-1})$ for all j; in particular, $\pi_{\rho_1} = W(\rho_1)$.

The decision weights π_{ρ_j} depend on the act f through the rank-ordering ρ, a dependency that is not expressed in the notation.

The $n!$ sets $F \subset X^n$ of the form $\{f \in X^n\colon f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}\}$, for some rank-ordering permutation ρ, are called *comoncones*. A set of acts is *comonotonic* if it is a subset of a comoncone. Comoncones are maximal comonotonic sets; under CEU, they are cones when mapped into utility units. The following result is immediate.

OBSERVATION 7. *If CEU holds, then each comoncone is an SEU-set, with decision weights as probabilities.* □

Many results of CEU are obtained by applying SEU techniques within comoncones. The *comonotonic degenerate case* holds if on every comoncone exactly one state is nonnull. The *comonotonic Archimedean axiom* holds if every bounded standard sequence on F is finite for every comoncone F, and if, further, there exists a countable order-dense subset of X in the comonotonic degenerate case. We write $\alpha\beta \sim_c^* \gamma\delta$ if there exists a comoncone F such that $\alpha\beta \sim_F^* \gamma\delta$. The symmetries of Observation 1 hold also for $\sim_c^*$ instead of $\sim_F^*$. *Comonotonic tradeoff consistency* holds if improving an outcome in any $\sim_c^*$ relationship breaks that relationship. That is, $\alpha\beta \sim_c^* \gamma\delta$ and $\alpha' \succ \alpha$ preclude $\alpha'\beta \sim_c^* \gamma\delta$. Because of symmetry, similar conditions hold for β, γ, δ. This condition is slightly stronger than tradeoff consistency on every comoncone because it compares *-relations elicited from different comoncones. This reflects the restriction that the utility function should be the same across different comoncones. Figure 1 in Wakker and Tversky (1993) demonstrates, for two nonnull states, that tradeoff consistency within each comoncone is too weak to characterize CEU, so that a stronger condition is indeed required.

THEOREM 8 (CHARACTERIZATION OF CEU). *Assume solvability. The following two statements are equivalent for $\succcurlyeq$ on X^n.*

(i) *CEU holds.*
(ii) $\succcurlyeq$ *satisfies*
  (a) *weak ordering*;
  (b) *weak monotonicity*;
  (c) *the comonotonic Archimedean axiom*;
  (d) *comonotonic tradeoff consistency.* □

OBSERVATION 9 (UNIQUENESS RESULTS FOR THEOREM 8). *Suppose that the assumptions and statements of Theorem* 8 *hold.*

(a) *In the* (*trivial*) *case where at least one comoncone only has null states, all comoncones only have null states, the preference relation is trivial, utility is constant, and the weighting function can be chosen arbitrarily.*

(b) *In the degenerate case where there is exactly one nonnull state for each comoncone, utility is unique up to a strictly increasing transformation, and the weighting function is uniquely determined* (*only taking the values zero and one*) *by the requirement that, on each comoncone, the decision weight of the nonnull state is one and of all other states it is zero.*

(c) *In the nondegenerate case where at least one comoncone has two or more nonnull states, utility is unique up to unit and location, and the weighting function is uniquely determined.* □

4.3. The topological approach for prospect theory. This subsection assumes further, topological, structure (Mendelson 1962). All the following topological conditions are satisfied under the common assumptions of X being an interval in the reals or X being $\mathbb{R}_+^m$.

Readers not interested in general topology can simply assume any of these cases and ignore the following topological assumptions. First, we assume that X is a connected topological space and that the set of acts is endowed with the product topology. For the degenerate cases, we will further assume topological separability. Appendix A will demonstrate that the topological assumptions are more restrictive than the algebraic assumptions of the preceding sections, because the continuity conditions imply solvability and the Archimedean axioms.

Continuity with respect to a connected product topology holds if and only if it holds for the special case where X is endowed with the order topology. Luce (2000) used an equivalent Dedekind completeness condition. Before turning to prospect theory, we list a corollary of our earlier results. It characterizes continuous SEU and CEU. The only aspect that does not follow immediately from the algebraic results (and Appendix A) is continuity of the utility functions.

COROLLARY 10 (CHARACTERIZATION OF CONTINUOUS SEU (CEU)). *Assume that X is a connected topological space and that X^n is endowed with the product topology. For the degenerate case where there is exactly one nonnull state* (*on every comoncone*), *assume that X is also topologically separable. The following two statements are equivalent for $\succcurlyeq$ on X^n.*

(i) *SEU* (*CEU*) *holds with a continuous utility function.*

(ii) $\succcurlyeq$ *satisfies*

  (a) *weak ordering*;

  (b) *weak monotonicity*;

  (c) *continuity*;

  (d) (*comonotonic*) *tradeoff consistency.*

The uniqueness results are as in Observation 6 (*Observation* 9), *with only one modification*: *In the degenerate case, utility is unique up to strictly increasing transformations that, furthermore, have to be continuous.* □

For prospect theory, an additional structural assumption is added. It is assumed that the outcome set contains a special element, denoted r, and called *reference point. Gains* are outcomes strictly preferred to r, *losses* are outcomes strictly less preferred than r. For an act f we define the act f^+ by $f_j^+ = f_j$ if $f_j \succcurlyeq r$ and $f_j^+ = r$ otherwise, and the act f^- by $f_j^- = f_j$ if $f_j \preccurlyeq r$ and $f_j^- = r$ otherwise.

(*Cumulative*) *prospect theory* (*CPT*) holds if there exist weighting functions W^+, W^-: $2^S \to [0, 1]$, and a *utility function* U: $X \to \mathbb{R}$ with $U(r) = 0$ such that

$$\mathrm{CPT}(f) = \mathrm{CPT}^+(f^+) + \mathrm{CPT}^-(f^-)$$

represents preferences. Here *CPT*$^+$ denotes Choquet expected utility with respect to W^+, and *CPT*$^-$ denotes Choquet expected utility with respect to the *dual* of W^-, defined by $A \mapsto 1 - W^-(S - A)$.

The way of integration of prospect theory, symmetric about zero, is reminiscent of the Šipoš (1979) integral, which is the special case of $W^+ = W^-$. The Choquet integral is the special case in which W^- is dual to W^+. The prospect theory functional is more complex than the SEU or CEU functionals. Preliminary analyses suggest that its axiomatization for the algebraic approach is very complex, primarily because "conditional certainty equivalents," used throughout the topological proofs, need not exist. This axiomatization currently remains as an open problem. We, therefore, only consider the topological approach.

A *sign-comoncone* F is a subset of X^n for which there exist a rank-ordering permutation ρ, and a $0 \le k \le n$, such that $F = \{f \in X^n: f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(k)} \succcurlyeq r \succcurlyeq f_{\rho(k+1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}\}$. Acts from F yield only gains if $k = n$, and they yield only losses if $k = 0$. A set of acts is *sign-comonotonic* if it is a subset of a sign-comoncone. This holds if and only if the acts are comonotonic and, furthermore, there is no state for which one act yields a gain and another a loss.

OBSERVATION 11. *If CPT holds, then each sign-comoncone is an SEU-set.* □

We write $\alpha\beta \sim^*_{sc} \gamma\delta$ if there exists a sign-comoncone F such that $\alpha\beta \sim^*_F \gamma\delta$. The symmetries of Observation 1 also hold for $\sim^*_{sc}$. *Sign-comonotonic tradeoff consistency* holds if improving an outcome in any $\sim^*_{sc}$ relationship breaks that relationship. That is, $\alpha\beta \sim^*_{sc} \gamma\delta$ and $\alpha' \succ \alpha$ preclude $\alpha'\beta \sim^*_{sc} \gamma\delta$.

The taxonomy of degenerate cases for CPT is complex and space-consuming, with several combinations of gain- and loss-representations. For brevity, we restrict attention to the nondegenerate case in which genuine tradeoffs between gains and losses exist. $\succcurlyeq$ is *truly mixed* if there exists an act f with $f^+ \succ r$ and $f^- \prec r$. It is easily verified that this condition holds under CPT if and only if there exists an event $A \subset S$ such that $W^+(A) > 0$ and $W^-(S - A) > 0$, and the outcome set contains at least one gain and one loss.

To guarantee consistency in tradeoffs between gains and losses, one more condition is imposed. *Gain-loss consistency* holds if for all acts f, g with $f \sim r$, $f^+ \sim g^+$, and $f^- \sim g^-$ we have $g \sim f \sim r$.

THEOREM 12 (CHARACTERIZATION OF CONTINUOUS CPT). *Assume that X is a connected topological space and that X^n is endowed with the product topology. Let $\succcurlyeq$ on X^n be truly mixed. The following two statements are equivalent.*

(i) *CPT holds with a continuous utility function.*

(ii) *$\succcurlyeq$ satisfies*

- (a) *weak ordering*;
- (b) *weak monotonicity*;
- (c) *continuity*;
- (d) *gain-loss consistency*;
- (e) *sign-comonotonic tradeoff consistency.* □

OBSERVATION 13 (UNIQUENESS RESULTS FOR THEOREM 12). *Suppose that the assumptions and statements of Theorem* 12 *hold. Then the weighting functions of CPT in Statement* (i) *are unique and, given $U(r) = 0$, utility is unique up to unit.* □

5. Further applications and extensions.

This section sketches further applications of the tradeoff technique.

5.1. Infinite state spaces.

So far, we have assumed that the state space is a finite set $\{1, \ldots, n\}$. In this section, we briefly explain how the results of the preceding sections can be extended to arbitrary state spaces. Writing out all extensions would be lengthy, and would be routine; therefore, we only describe the extensions verbally for the nondegenerated case.

Assume a general *state space* S, endowed with an algebra, possibly a sigma-algebra, whose elements are called *events*. An *act* is a finite-valued function from S to X, where the inverse of each outcome is an event. An act is denoted by $(E_1{:}x_1, \ldots, E_n{:}x_n)$, where the E_js are events partitioning S, and the act assigns outcome x_j to each state in E_j.

The extension of the results of previous sections to this setup is obtained as follows. First, we restrict attention to a fixed partition of events $E_1, \ldots, E_n$. The corresponding set of acts is isomorphic to the sets studied in preceding sections, and here all theorems derived before can be applied. Next, the representations can be made to agree on all finite partitions for which the nondegenerate case applies. In this case, utility is unique up to unit and location. The agreement is obtained by choosing a common normalization of utility, a common refinement for each pair of partitions, and applying the uniqueness results of preceding sections to ensure that the representations coincide on their common domain. Because every pair of acts can be considered to be measurable with respect to a sufficiently fine partition, one representation results for the preferences between all acts.

The *-relations ($\sim^*$, $\sim^*_c$, or $\sim^*_{sc}$) in the preceding paragraph formally depend on the partitions chosen, and tradeoff-consistency conditions should be related to particular partitions.

By considering common refinements of partitions, it becomes obvious that these dependencies on partitions can be dropped. We can define a *-relation as the union of all the corresponding partition-dependent *-relations. Tradeoff consistency can be defined with respect to the partition-independent *-relation thus defined. Finally, extensions to acts with infinitely many outcomes can be obtained through the techniques of Wakker (1993b).

5.2. Restriction to binary acts. A generalization of the models presented before can be obtained by imposing the model only on binary acts. On this domain, the empirical evidence for CEU and CPT is most convincing, and the domain is rich enough to uniquely determine the utility function and the weighting function (Abdellaoui 2000; Bleichrodt and Pinto 2000; Luce 2000, Chapter 3; Miyamoto 1988; Pfanzagl 1959; Wakker and Deneffe 1996, p. 1143). Casadesus-Masanell et al. (2000) and Ghirardato and Marinacci (2001, 2002) used restrictions to binary acts to obtain definitions of ambiguity aversion, and to extend Gilboa and Schmeidler's (1989) and Chateauneuf's (1991) axiomatizations of the multiple priors model to continuous instead of linear utility. They used bisymmetry-like conditions, as did several other papers. These works are discussed further in §7.

Tradeoff consistency can be used to provide alternative characterizations of models for binary acts. Assume that all sets F considered above are restricted to binary acts. First, our theorems provide representations on each two-dimensional subspace of acts depending only on one event A. Next, tradeoff consistency, thus restricted, ensures that all these two-dimensional representations, for the various events A, have the same utility function. Monotonicity of the weighting function(s) follows from monotonicity of preference. Thus, one SEU, CEU, or CPT functional results that (through comparison with constant acts) is representing for all binary acts.

5.3. Decision under risk. Decision under risk can be considered a special case of decision under uncertainty (Wakker 1990). We consider an outcome set X as before. The domain of preference consists of lotteries, i.e., finite probability distributions $(p_1{:}x_1, \ldots, p_n{:}x_n)$ over X. These probabilities are not endogeneously derived from preferences but are available as prior information about the choice objects, in the same way as the outcomes and, for uncertainty, the events are. Such probabilities are sometimes called *exogeneous*.

To relate decision under risk to decision under uncertainty, we take the unit interval $[0, 1]$ as state space, with events being subintervals. Probabilities correspond to lengths of intervals. Each act generates a lottery over the outcomes, and for each lottery there exist several acts that generate it. Preferences between acts are determined by preferences between the corresponding lotteries. Obviously, different acts that generate the same probability distribution over outcomes are indifferent.

As in Chateauneuf and Wakker (1999, p. 144), the results of the preceding section can be used to axiomatize preferences between lotteries. Under SEU, CEU, and CPT, two intervals A, B of the same length (i.e., two events with the same probability) have the same weight. That is, $P(A) = P(B)$ under SEU, $W(A) = W(B)$ under CEU, and $W^+(A) = W^+(B)$ and $W^-(A) = W^-(B)$ under CPT. Consequently, those weights are transforms of the exogeneous probabilities. By additivity and nonatomicity, P of SEU must be identical to the exogeneous probabilities, and a regular expected utility representation results, alternative to, and obviously less appealing than, the one of von Neumann and Morgenstern (1944). The CEU model reduces to *rank-dependent utility*, with $W(A) = w(p)$ for p the exogeneous probability of A and w a probability transformation function. In this way, our techniques provide a tradeoff-consistency characterization of rank-dependent utility that is alternative, and we think preferable, to Wakker's (1994) Theorem 12, which was based on earlier preference-based tradeoff conditions and topological conditions. Under CPT for risk, $W^+(A) = w^+(p)$ and $W^-(A) = w^-(p)$, with interpretations similar to those for rank-dependent utility. A tradeoff-consistency characterization of CPT has been obtained that improves upon Chateauneuf and Wakker's (1999) result for reasons similar to those mentioned above.

5.4. Utility curvature. The equivalence tradeoff techniques give convenient tools for characterizing properties of utility in a way alternative to earlier characterizations based on preference-tradeoff techniques. For example, assume that outcomes are monetary with continuous strictly increasing utility. Then concave utility is characterized by the requirement that $\alpha > \beta$, $[\alpha; \beta] \sim_F^* [\alpha' + \epsilon; \beta + \epsilon]$, and $\epsilon \geq 0$ can only be true if $\alpha' \geq \alpha$, where F corresponds with the theory under consideration. This condition is alternative to Wakker (1994, Corollary 16) and Wakker and Tversky (1993, Corollary 9.3). A utility function U_2 being more concave than U_1, for decision makers $\succcurlyeq_1$ and $\succcurlyeq_2$, is characterized by the requirement that $\alpha > \beta > \gamma > \delta$, $[\alpha; \beta] \sim_{1,F}^* [\gamma; \delta]$, and $[\alpha'; \beta] \sim_{2,F'}^* [\gamma; \delta]$ can only be true if $\alpha' \geq \alpha$. The condition entails that U_2, when expressed in U_1 units, exhibits diminishing marginal utility. This result is alternative to Wakker (1994, Theorem 14). Schmidt (2003) used tradeoff techniques to study how variations of the reference point affect the curvature of utility under prospect theory. Abdellaoui (2002) used dual versions of such tools to characterize properties of probability transformations.

6. A review of axiomatizations of nonexpected utility. The next two sections review existing preference axiomatizations, and explain how our tradeoff technique can serve to generalize these results. Gilboa (1987), Sarin and Wakker (1992), and Savage (1954) used nonnecessary richness assumptions on the state space in their derivations of CEU and SEU for uncertainty. In decision under risk, such richness is naturally given through the richness of the unit probability interval. Many axiomatizations of expected utility for risk have used this richness (von Neumann and Morgenstern 1944, Fishburn 1982, and the references therein). To the best of our knowledge, only two derivations of rank-dependent utility, and no derivation of prospect theory, have used this richness for risk (Abdellaoui 2002, Nakamura 1995). Segal (1989, 1993a) used both richness of outcomes and richness of probabilities. There are no logical relations between these works and the large class of papers to which we now turn, that used richness of outcomes.

The only nonnecessary axiom in our characterizations of SEU and CEU is solvability. Only a few axiomatizations in decision under uncertainty and risk used algebraic structural assumptions as we did, with a nonnecessary solvability axiom, and, thus, reached a similar structural generality. Besides Miyamoto (1988) and Wakker (1991b, c), these include Nakamura (1990, 1992), Chew and Karni (1994), and some results in Luce (2000). The latter four references used a more restrictive version of solvability, requiring for every event what we required only for single states of nature i, and ruling out finite cases. Miyamoto (1988) ruled out the finite cases through his "densely spaced" Axiom 7.9 (p. 369).

All other axiomatizations in the richness-of-outcomes approach assumed connected topological spaces, or further restrictions. The algebraic results of this paper for Choquet expected utility and rank-dependent utility are therefore, in a structural sense, the most general presently available. That is, the conditions of our theorems are satisfied in every other axiomatization, but, conversely, for every other axiomatization some assumption need not be satisfied in our results. Within the topological approach, Corollary 10 is the most general CEU axiomatization presently available, because it does not assume topological separability except for degenerate cases. Several other papers assumed topological separability also for the nondegenerate case (Chateauneuf 1999, Remark 4; Quiggin 1982, through Axioms 3 and R.2; Ghirardato and Marinacci 2001, 2002). Many papers assumed, more restrictively, real-valued outcomes and continuity of utility (Casadesus-Masanell et al. 2000; Chew 1989; Gul 1992; Prelec 1998; Segal 1989, 1993a, b; Wakker 1984) or even linearity of utility (Chateauneuf 1991, de Finetti 1937, Schmidt and Zank 2001b, Yaari 1987, Weymark 1981, Theorem 3). For prospect theory, all characterizations known to us have assumed connected topological outcome spaces, often with Euclidean structures added. Bleichrodt and Miyamoto (2003) start from more general assumptions, but derive continuity with respect to a connected product-order topology.

Many papers used a stronger monotonicity condition than the weak monotonicity that we used. *Strong monotonicity* holds on $F \subset X^n$ if, for all $f, g \in F$, $f \succ g$ whenever $f_j \succcurlyeq g_j$ for all j and $f_j \succ g_j$ for a state j that is nonnull on F. As before, "on X^n" is usually omitted. *Comonotonic strong monotonicity* holds if strong monotonicity holds on each comoncone, and *sign-comonotonic strong monotonicity* holds if strong monotonicity holds on each sign-comoncone. These conditions are implied by the other conditions in all of our main theorems (Lemma 26).

Many papers on (Choquet) expected utility used the approach of Anscombe and Aumann (1963), where the outcome set consists of probability distributions over prizes, and preferences over outcomes are governed by expected utility (Schmeidler 1989). In a mathematical sense, these assumptions amount to the outcome space being a convex subset of a linear space and utility being linear. Again, these assumptions are more restrictive than continuity with respect to a connected topology. Wakker and Zank (1999) showed how SEU-axiomatizations for linear utility (Anscombe and Aumann 1963, de Finetti 1937) can be derived as corollaries from results for preference-tradeoff consistency. The following lemma adapts their Observation 5 to the tradeoff-consistency conditions of this paper. It is typical of the ways in which mixture axioms, such as those used by Schmeidler (1989), imply tradeoff consistency. Observation 3 of Wakker and Zank (1999), deriving tradeoff consistency from additivity axioms ($f \sim g \Rightarrow f + c \sim g + c$) (Chateauneuf 1991, Yaari 1987), can be adapted similarly.

LEMMA 14. *Assume that X is a convex subset of a linear space, and assume weak ordering, and weak and strong monotonicity. Further, assume "outcome monotonicity" for mixing (for all outcomes* α, α', β, $[\alpha' \succ \alpha \Rightarrow \frac{1}{2}\alpha' + \frac{1}{2}\beta \succ \frac{1}{2}\alpha + \frac{1}{2}\beta]$) *and "mixture independence": for all acts* f, f', g, $f' \sim f \Rightarrow \frac{1}{2}f' + \frac{1}{2}g \sim \frac{1}{2}f + \frac{1}{2}g$, *where the midpoint operation is applied statewise. Then tradeoff consistency holds. A similar implication holds under comonotonicity restrictions for* f, f', g, *and the strong monotonicity condition.* □

Studies that used preference-tradeoff consistency, and that can be generalized by means of the indifference-tradeoff consistency, include Abdellaoui (2002), Bleichrodt and Miyamoto (2003), Bleichrodt and Quiggin (1997, 1999), Ebert (2003), Karni (2003), Prelec (1998, Appendix 1), LeRoy and Werner (2000, §8.6), Schmidt (2003), Schmidt and Zank (2001a), and Tversky and Kahneman (1992). The next section considers another class of results that can be generalized by means of indifference-tradeoff consistency.

7. Comparing tradeoff consistency with bisymmetry, multisymmetry, and act-independence. Instead of tradeoff consistency, bisymmetry-based axioms have been used in the literature to axiomatize weighted averages of general outcome-transformation functions. These axioms assume that for each act f a *certainty-equivalent* $\mathrm{CE}(f)$ exists, i.e., an outcome equivalent to f. They use *CE-compositions* $\mathrm{CE}(\mathrm{CE}(f^1), \ldots, \mathrm{CE}(f^n))$, i.e., certainty equivalents of acts whose outcomes are certainty equivalents of other acts. The following elementary lemma, given without proof, shows that all clauses "for all certainty-equivalent functions CE" hereafter can be replaced by the clause "for some certainty-equivalent function CE." That is, the particular choice of certainty equivalent never matters.

LEMMA 15. *Assume weak ordering and weak monotonicity. Then* $\mathrm{CE}(\mathrm{CE}(f^1), \ldots, \mathrm{CE}(f^n)) \sim \mathrm{CE}'(\mathrm{CE}'(f^1), \ldots, \mathrm{CE}'(f^n))$ *whenever both* CE *and* CE′ *are certainty equivalent functions.* □

For event $A \subset S$, $(A{:}x, A^c{:}y)$ denotes the act yielding outcome x for each $s \in A$ and outcome y for each $s \in A^c$. A is *nondegenerate* if $(A{:}x, A^c{:}y) \succ (A{:}y, A^c{:}y)$ and $(A{:}x, A^c{:}x) \succ (A{:}x, A^c{:}y)$ for some $x \succ y$. This condition implies that A and A^c are nonnull, and serves to rule out degenerate cases. We first present results for expected utility, and then consider comonotonic extensions. *Multisymmetry* holds *for event A* if A is nondegenerate and

$$\text{(7)} \qquad \mathrm{CE}(A{:}\mathrm{CE}(f_1,\dots,f_n), A^c{:}\mathrm{CE}(g_1,\dots,g_n)) \sim \mathrm{CE}(\mathrm{CE}(A{:}f_1, A^c{:}g_1),\dots,\mathrm{CE}(A{:}f_n, A^c{:}g_n))$$

for all acts f, g, and certainty-equivalent functions CE. Event A is called the *mixing event*. The condition is illustrated in Figure 1(c). The circles can be interpreted as certainty-equivalent operations (Figures 1(a) and 1(b)). A special case of multisymmetry is depicted in Figure 1(d), where the acts depend only on event A.

An interesting interpretation of multisymmetry results if we interpret the CE-compositions in Figure 1 as multistage options evaluated through folding back, with certainty equivalents substituted at each circle ("chance node"). This interpretation is possible only if we can assume that events can be repeated, at least as a thought experiment, and that the repetitions are "subjectively independent" in the sense that certainty equivalents are not affected by the events upon which they are conditioned (Ghirardato et al. 2001; Luce 1988, Equations 22 and 23; Pfanzagl 1959, p. 288; Segal's 1993b "order indifference"). Multisymmetry then implies that the order in which the uncertainty is resolved does not matter for the preference value of the CE-composition.

Because of weak and strong monotonicity and nonnullness of A, the n-tuple $(f_1,\dots,f_n)$ is "separable" in $\mathrm{CE}(A{:}\mathrm{CE}(f_1,\dots,f_n), A^c{:}\mathrm{CE}(g_1,\dots,g_n))$ which, by multisymmetry, implies the same separability in $\mathrm{CE}(\mathrm{CE}(A{:}f_1, A^c{:}g_1),\dots,\mathrm{CE}(A{:}f_n, A^c{:}g_n))$. More precisely, replacing the n-tuple $(f_1,\dots,f_n)$ in the latter CE-composition by a weakly (strictly) preferred n-tuple $(f_1',\dots,f_n')$ leads to a weakly (strictly) preferred composition $\mathrm{CE}(\mathrm{CE}(A{:}f_1', A^c{:}g_1),\dots,\mathrm{CE}(A{:}f_n', A^c{:}g_n))$, irrespective of what the common outcomes g_j are. Similarly, multisymmetry implies that replacing the n-tuple $(g_1,\dots,g_n)$ in the CE-composition $\mathrm{CE}(\mathrm{CE}(A{:}f_1, A^c{:}g_1),\dots,\mathrm{CE}(A{:}f_n, A^c{:}g_n))$ by a weakly (strictly) preferred n-tuple $(g_1',\dots,g_n')$ leads to a weakly (strictly) preferred composition $\mathrm{CE}(\mathrm{CE}(A{:}f_1, A^c{:}g_1'),\dots,\mathrm{CE}(A{:}f_n, A^c{:}g_n'))$, irrespective of what the common outcomes f_j are. The latter implication uses the assumption that A^c is nonnull. A weak version of the aforementioned separability, imposed only on indifferences, is illustrated in Figure 1(e): *Act-independence* holds *for*

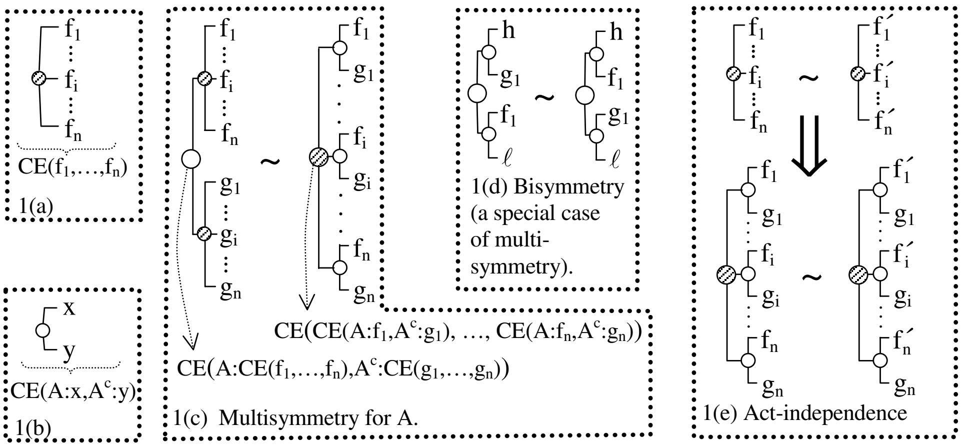

FIGURE 1 (BISYMMETRY-BASED AXIOMS). Dashed circle nodes from which three (reflecting n) branches emanate, designate certainty equivalents of general acts (a). For a given event A, open circle nodes, from which two branches emanate, designate certainty equivalents of acts depending only on A (b). The other figures display CE compositions with A as mixing event. Figure (d) is the special case of (c) where all acts depend only on A (notation chosen compatible with Figure 5(c)). Figure (e) illustrates a separability implication of (c).

(mixing) event A if (a) A is nondegenerate; (b) for all certainty-equivalent functions CE the implication of Figure 1(e) holds; and (c) a similar implication holds for A^c instead of A. The latter condition (c), for A^c, means that replacing $(g_1, \ldots, g_n)$ in the lower-left CE-composition of Figure 1(e) by an equivalent $(g_1', \ldots, g_n')$ does not affect the preference value of the CE-composition. Act-independence, introduced by Gul (1992, Assumption 2), is appealing because it resembles the independence condition for preferences over (mixtures of) probability distributions. Our condition, imposed only on indifferences, generalizes earlier formulations in the literature.

Under regularity conditions, SEU holds if and only if multisymmetry holds for all mixing events A (Krantz et al. 1971, §6.9, for $n = 2$; Pfanzagl, 1959, pp. 287–288, for all binary acts; Münnich et al. 2000, for all acts). Also, SEU holds if and only if (a preference version of) act-independence holds for all mixing events (Chew and Karni 1994, Gul 1992). These conditions are, therefore, alternatives to tradeoff consistency. We next demonstrate that the aforementioned conditions imply tradeoff consistency in an elementary manner, i.e., without using technical assumptions such as continuity or solvability. Thereby, the results based on these conditions can be obtained as corollaries of the results based on tradeoff consistency, without the need for separate proofs.

THEOREM 16. *Assume weak ordering, weak and strong monotonicity, and a certainty equivalent for every act. Then multisymmetry for an event A implies act-independence for event A, and the latter implies tradeoff consistency.* □

We next turn to rank-dependent theories, with comonotonic generalizations of the above conditions. Note that nondegeneracy of event A implies that A and A^c are nonnull on the comonotonic set of acts $(A{:}x, A^c{:}y)$ with $x \succcurlyeq y$. *Comonotonic multisymmetry* holds for event A if A is nondegenerate and Equation (7) holds whenever f and g are comonotonic, and $f_j \succcurlyeq g_j$ for all j. Similarly, *comonotonic act-independence* holds for event A if: (a) A is nondegenerate; (b) the implication in Figure 1(e) holds whenever f, g, f' are comonotonic, and $f_j \succcurlyeq g_j$ and $f_j' \succcurlyeq g_j$ for all j; (c) the corresponding implication holds for A^c instead of A. That is, replacing $(g_1, \ldots, g_n)$ in the lower-left CE-composition of Figure 1(e) by an equivalent $(g_1', \ldots, g_n')$ does not affect the preference value of the CE-composition whenever f, g, g' are comonotonic, and $f_j \succcurlyeq g_j$ and $f_j \succcurlyeq g_j'$ for all j.

THEOREM 17. *Assume weak ordering, weak and comonotonic strong monotonicity, and a certainty equivalent for every act. Then comonotonic multisymmetry for an event A implies comonotonic act-independence for that event, and the latter implies comonotonic tradeoff consistency.* □

Several authors have considered the restrictions of axioms and representations to binary acts. The above results, as well as their proofs, hold on this restricted domain without any modification.

OBSERVATION 18. *Theorems* 16 *and* 17 *also hold true if restricted to binary acts.* □

For axiomatizations of rank-dependent theories, based on multisymmetry, see Chew (1989, for decision under risk), Chew and Karni (1994), Nakamura (1990, 1992), and Quiggin (1982, for decision under risk). For binary acts, see Nakamura (1990, Proposition 1) and Luce (2000, Theorem 3.7.3). Act-independence was used by Chew and Karni (1994) for general acts, and by Ghirardato and Marinacci (2001) for binary acts. All of these results become corollaries of the results of this paper, where, furthermore, a number of generalizations are obtained. First, the preference conditions need to be imposed only on one mixing event. With the exception of Quiggin (1982), all the works mentioned imposed the preference conditions on all mixing events. Second, act-independence needs to be imposed only on indifferences, whereas in the literature it has always been imposed on all preferences. Third, topological separability need not be assumed (note that existence of a mixing event rules out the degenerate cases). With the exception of Pfanzagl's works, all topological derivations in the literature have assumed topological separability.

Bisymmetry-based axioms cannot be adapted (at least not in an obvious way) to prospect theory and sign-dependence. The reason is that these axioms use certainty equivalents, which are not contained in most cosigned sets (Chew and Tversky 1990).

Ghirardato et al. (2002) proposed a method for eliciting utility midpoints from observed preferences, using a bisymmetry-based approach. They used this midpoint technique as an alternative to our tradeoff technique to analyze cardinal utility. As an aside, tradeoffs can also elicit midpoints β of α and γ, through $\alpha\beta \sim^* \beta\gamma$ or $\alpha\beta \sim_c^* \beta\gamma$. Ghirardato et al. (2002) assumed a rank-dependent representation for all binary acts, derived from other axioms, and then used their midpoint technique to define a comonotonic independence axiom and extend the rank-dependent representation to nonbinary acts. Their independence axiom implies tradeoff consistency, which is proved in the same way as Lemma 14. Therefore, their characterization of rank-dependent models follows as a corollary of the results of this paper. Ghirardato et al. (2002) also use their mixture operation to derive axiomatizations of the multiple priors model and of Bewley's (1986) incomplete-preference model.

Besides the greater mathematical generality of the tradeoff approach, including its applicability to prospect theory, and the natural role of tradeoffs in decision processes and utility measurements, we mention two other reasons why we prefer this approach to the bisymmetry-based approach. First, the bisymmetry-based axioms use many CE substitutions, which complicate their empirical testing. Second, subjective independence for repetitions of events is hard to imagine, even as a thought experiment, for events with unknown probabilities. For criticisms of tradeoff consistency, see Luce (2000, §§3.7.3 and 5.3.1.1).

For preference axiomatizations of weighted averages of general outcome-transformation functions in a static model with richness of outcomes, the only result that, to the best of our knowledge, is not a direct corollary of our results, is Chateauneuf (1999). He considers decision under risk and his central new axiom, A.4, imposes a kind of tradeoff consistency for nested probabilities. This axiom A.4 is appealing because it is a direct weakening of the independence condition for probability mixtures.

8. Conclusion. This paper has presented an improved technique for deriving preference foundations for decision models based on weighted averages, and their rank- and sign-dependent generalizations. We have formulated our results for decision under risk and uncertainty, but they apply as well to the measurement of inequality and welfare, and intertemporal decisions. Our results are both more accessible and more general than earlier results, and this claim holds both for the intuitive and for the technical axioms used.

Appendix A: The generality of the new version of tradeoff consistency and of the algebraic approach. Preference conditions similar to tradeoff consistency can be found in some early papers co-authored by Tversky (Krantz et al. 1971, §6.11.2: standard-sequence invariance; Tversky 1977, Appendix: invariance; Tversky and Kahneman 1991, reference interlocking; Tversky et al. 1988, the interlocking condition), and also in Miyamoto (1988, Definition 7.10), Miyamoto (1999, Definition 12), and Skiadas (1997, Axiom A10). All of these papers used additional conditions to ensure the existence of additive representations, with their tradeoff conditions serving only to imply proportionality across different dimensions.

The generality of tradeoff consistency. Many axiomatizations in the literature have adopted different versions of tradeoff consistency, using preferences instead of indifferences ("preference-tradeoff consistency"). These conditions are as follows: No four preferences should exist of the form

$$\alpha_i f \succcurlyeq \beta_i g, \qquad \gamma_i f \preccurlyeq \delta_i g, \qquad \alpha_j v \preccurlyeq \beta_j w, \qquad \gamma_j v \succ \delta_j w, \tag{8}$$

with the acts of the first two preferences contained in a set $F \subset X^n$ on which state i is nonnull, and the acts of the second two preferences contained in a set $F' \subset X^n$ on which state j is nonnull. For *preference-tradeoff consistency*, $F = F' = X^n$; for *comonotonic preference-tradeoff consistency*, F and F' are comoncones; finally, for *sign-comonotonic preference-tradeoff consistency*, F and F' are sign-comoncones. Some papers used the notation $\alpha\beta \succcurlyeq^* \gamma\delta$ for the first two preferences in Equation (8), and the notation $\gamma\delta \succ^{**} \alpha\beta$ for the latter two preferences.

Wakker (1984) introduced this kind of condition, primarily for the purpose of normative defenses of expected utility. Wakker (1989, endnote rf13 on p. 168) refers to similar earlier ideas, in particular, standard sequence invariance of Krantz et al. (1971). As argued by Köbberling and Wakker (2001) in a normative defense of expected utility, we now prefer the indifference-tradeoff consistencies. They are more transparent because of symmetries of indifference. They also yield more general mathematical results. The reason is that assuming indifferences, as is done in our conditions, is more restrictive than assuming preferences. More restrictive premises lead to less restrictive axioms and, hence, to more general theorems.

PROPOSITION 19. *Under weak ordering,*

- *preference-tradeoff consistency implies weak monotonicity and tradeoff consistency*;
- *comonotonic preference-tradeoff consistency implies weak monotonicity and comonotonic tradeoff consistency*; *and*
- *sign-comonotonic preference-tradeoff consistency implies weak monotonicity and sign-comonotonic tradeoff consistency.* □

The proposition is elementary in the sense that it only uses weak ordering and no technical richness assumptions such as solvability or continuity. Results using the earlier versions of tradeoff consistency can, therefore, be obtained as elementary corollaries of the results based on the tradeoff consistencies of this paper. Under weak ordering, any set of data (finite in practice) that reveals a violation of our indifference versions of tradeoff consistency, automatically implies a violation of the preference versions of tradeoff consistency. The reverse implication need not hold (Example 23 in Appendix B).

The generality of the algebraic approach. Luce et al. (1990, p. 49, l. 10), and Wakker (1988; 1989, end of §III.6) argued that the algebraic approach is preferable to the topological approach. Most of Wakker's papers used the topological approach only because it is closer to the common conventions in the economic literature.

We first state two lemmas demonstrating that the topological assumptions imply the algebraic assumptions, so that the latter lead to more general results. Then we give some examples where the algebraic approach applies but the topological does not, demonstrating that the algebraic approach leads to strictly more general results.

LEMMA 20. *Assume that $\succcurlyeq$ is a continuous weak order with respect to a connected product topology. Then solvability holds.* □

LEMMA 21. *Assume that $\succcurlyeq$ is a continuous weak order with respect to a connected product topology on X^n, that X is topologically separable in the degenerate case, and that weak and* (*comonotonic*) *strong monotonicity hold. Then the* (*comonotonic*) *Archimedean axiom holds.* □

The following examples modify examples of Krantz et al. (1971, §§4.8 and 6.11.1) and Wakker (1989, Example III.6.8).

EXAMPLE 22. Assume that $n = 2$, and that SEU holds with probabilities 0.5 for each state and with the identity function as utility. Consider the following three cases.

(a) X is a finite set of consecutive integers, say $\{-10, \ldots, 10\}$.
(b) X is the set of rationals.

(c) X contains all real numbers of the form $i+j\sqrt{2}$ with i and j, possibly negative, integers.

Solvability easily follows in each case; in (b) and (c), it follows because X is closed under addition and subtraction. In each case, $\succcurlyeq$ is not continuous with respect to a connected product topology: Because $\sqrt{3}$ is not contained in any of the sets X, the sets $\{\alpha \in X\colon \alpha > \sqrt{3}\}$ and $\{\alpha \in X\colon \alpha < \sqrt{3}\}$ constitute a nontrivial partition of X into sets that are open with respect to the order topology. Neither the order topology nor any other topology for which $\succcurlyeq$ is continuous on X can be connected.

In Example 22(c), there do not exist *certainty equivalents* for each act, i.e., outcomes equivalent to the act. For the act $(0, \sqrt{2})$, the certainty equivalent would have to be $\sqrt{2}/2$, but this real number is not contained in X ($i+j\sqrt{2} = \sqrt{2}/2$ would require that $\sqrt{2} = i/(1/2-j)$, i.e., that $\sqrt{2}$ be rational, which it is not). Similar reasonings show that there need not exist midpoints, in terms of utility units, between each pair of outcomes (0 and $\sqrt{2}$) or between each pair of equivalence classes (e.g., those of $(0,0)$ and $(0,\sqrt{2})$). □

Example 22(a) shows that our axioms can be satisfied in finite models. Such models must be *equally spaced*, i.e., the outcome set consists of one standard sequence, or a "two-sided" extension thereof, such as $\mathbb{N}$, plus other outcomes equivalent to elements of the (two-sided) standard sequence (Krantz et al. 1971, §4.8; Wakker 1991a). In this case, different comoncones, or sign-comoncones, all have the same number, say k, of equally weighted nonnull states, and can differ only regarding the k states that are nonnull. With X the set of rationals, as in Example 22(b), all nonexpected utility models of this paper can be applied, as long as all decision weights are rational too. Example 22(c) demonstrates that the common proof techniques, which extensively use midpoints (Casadesus-Masanell et al. 2000, Ghirardato et al. 2002, Vind 1991, Wakker 1989, p. 60) and (conditional) certainty equivalents, cannot be used in the algebraic approach. Our proofs are primarily based on the algebraic techniques of Krantz et al. (1971), and on some results derived therefrom (Wakker 1991b, c).

We prefer the algebraic approach to the topological approach not only because its axioms are more general, but also because we consider these axioms to be more natural. The primary empirical meaning of continuity with respect to a connected product topology is precisely the solvability condition that it implies. It is more natural to state this empirical meaning explicitly, than to reinforce it into a stronger condition, of which the further empirical implications for finite data sets are not clearly identified.

A fundamental problem of continuity and solvability assumptions, and thereby of all preference axiomatizations in the literature, is that these assumptions are not merely technical. They also add empirical implications to the other axioms, and (and this is the problem) it is not clear precisely what these empirical implications are. Proposition 30 and Example 23 hereafter will illustrate the point: The example will show that a finite data set can violate preference-tradeoff consistency without violating indifference-tradeoff consistency. As the proposition shows, the two versions of tradeoff consistency become equivalent only under solvability or continuity assumptions. These latter assumptions, therefore, do add empirical implications to indifference-tradeoff consistency, such as preference-tradeoff consistency.

The phenomenon of quasitechnical axioms adding empirical content to other empirical axioms is not typical of the particular axioms studied here, but applies to continuity and solvability axioms in general. Krantz et al. (1971, §9.1) and Wakker (1988) further discussed this issue. The Archimedean axiom is not problematic in this respect because it does not add empirical implications (Krantz et al. 1971, §6.5.1; Luce et al. 1990, Theorem 21.21; Pfanzagl 1968, §§6.6 and 9.5), or only infinitesimally so (Narens 1985, Theorem 2.8.3), to the other axioms. The axiom merely reflects the mathematical convention of excluding nonstandard real numbers and does not affect interpretations of finite data sets. Separating out this empirically harmless technical axiom is another virtue of the algebraic approach.

Appendix B: Preference-tradeoff consistency, and other implications of tradeoff consistency. Self-contained proofs of the theorems in the main text could be obtained, adapting earlier proofs to indifference-tradeoff conditions. Wakker's (1989) Remark III.7.3, which shows that many derivations of additive conjoint measurement only need indifference versions of the additivity conditions plus monotonicity, would be used extensively, and several other modifications would be required. These proofs would be very lengthy. We have, therefore, chosen an alternative approach. It consists of deriving preference-tradeoff conditions from the indifference conditions used in the main text, and then using existing theorems. That is, we reverse the implications of Proposition 19. The derivation of this reversal is not elementary, and needs the technical assumptions. The reversal does not hold in general, e.g., for finite models, and, therefore, data violating preference-tradeoff consistency need not violate indifference-tradeoff consistency, as illustrated by the following example. This fact can obviously be a reason to verify not only indifference versions of tradeoff consistency, but also preference versions, in data.

EXAMPLE 23. Assume two states (healthy or ill) and four outcomes, α (airplane flight), β (travel by boat), γ (travel by Greyhound bus), and λ (long delay). Preferences are represented by $(f_1, f_2) \mapsto V_1(f_1) + V_2(f_2)$, with V_1 and V_2 defined in the following table.

	α	β	γ	λ
V_1	1	0.9	0.3	0
V_2	0.99	0.44	0.33	0

With always λ and γ as gauge outcomes, $(\lambda, \alpha) \succcurlyeq (\gamma, \beta)$ and $(\lambda, \beta) \preccurlyeq (\gamma, \gamma)$ imply $\alpha\beta \succcurlyeq^* \beta\gamma$, whereas $(\alpha, \lambda) \preccurlyeq (\beta, \gamma)$ and $(\beta, \lambda) \succ (\gamma, \gamma)$ imply $\beta\gamma \succ^{**} \alpha\beta$. Preference-tradeoff consistency (Equation (8)) is violated, as well as its [sign-]comonotonic versions. Indifference-tradeoff consistency holds vacuously, because there are no nonreflexive indifferences. This follows because no nonzero V_1 difference for the first state can be matched exactly by a V_2 difference for the second state, the former always being a multiple of 0.1, and the latter never. □

In the rest of the proofs, we will often use weak monotonicity conditions without explicit mention. We first sketch the main idea of the derivation of preference-tradeoff consistency from tradeoff consistency. Assume that we have a violation of preference-tradeoff consistency, i.e., four preferences $\alpha_i f \succcurlyeq \beta_i g$, $\gamma_i f \preccurlyeq \delta_i g$, $\alpha_j v \preccurlyeq \beta_j w$, and $\gamma_j v \succ \delta_j w$ hold as in Equation (8). Assume that we can worsen the f_k for $k \neq i$ so much and improve the g_k for $k \neq i$ so much that, after these changes, $\alpha_i f \sim \beta_i g$ (Lemma 31). The preference $\gamma_i f \preccurlyeq \delta_i g$ will only be intensified by such changes. Imagine that, next, we can improve outcome γ and worsen outcome δ so much that, after these changes, the indifference $\gamma_i f \sim \delta_i g$ results (Step 2 in the main proof of Proposition 30). The preference $\gamma_j v \succ \delta_j w$ will only be intensified by such changes. Imagine that, next, we can improve the v_k for $k \neq j$ so much, and worsen the w_k for $k \neq j$ so much, that after these changes, $\alpha_j v \sim \beta_j w$ (Step 3 of proof of Proposition 30). The preference $\gamma_j v \succ \delta_j w$ will only be intensified by such changes. After such manipulations, the special case of Equation (8) has resulted where the first three preferences are indifferences. It can be demonstrated that such cases are excluded by tradeoff consistency (Corollary 28). The above manipulations extensively used technical conditions such as solvability. Changes as suggested may not always be possible, in which case alternative similar changes have to be devised, leading to a complex taxonomy of different cases.

To begin our formal analysis, we first derive some preparatory results that may be of interest on their own. Proposition 30 will then complete the proof. Henceforth, statements will often be combined by means of braces and square brackets, as in {[sign-]comonotonic} strong monotonicity, etc. If this notation is used, then, for the sign-comonotonic case, all texts, both those between square brackets and those between braces, are to be read. For the comonotonic case, the texts between braces are to be read but those between square brackets are to be skipped. For the case without sign- or comonotonicity restrictions, all texts between square brackets and braces (except when indicating sets) are to be skipped.

The following condition provides a useful tool in the subsequent proofs. It can also serve as an alternative formulation of the tradeoff-consistency conditions defined in the main text, because it is equivalent under appropriate assumptions (Corollary 28) and is stated directly in terms of preferences instead of the derived concept $\sim^*$. {[*Sign-*]*comonotonic*} *strong indifference-tradeoff consistency* holds if $\alpha_i f \sim \beta_i g$, $\gamma_i f \sim \delta_i g$, and $\alpha_j x \sim \beta_j y$ imply $\gamma_j x \sim \delta_j y$, whenever $\{\alpha_i f, \beta_i g, \gamma_i f, \delta_i g \in F$ and $\alpha_j x, \beta_j y, \gamma_j x, \delta_j y \in F'$ for some [sign-]comoncones F and F', where$\}$ i and j are nonnull {on F and F', respectively}.

LEMMA 24. *Assume weak ordering, weak and* {[*sign-*]*comonotonic*} *strong monotonicity. Then* {[*sign-*]*comonotonic*} *strong indifference-tradeoff consistency implies* {[*sign-*]*comonotonic*} *tradeoff consistency.*

PROOF. Suppose that the assumptions of the lemma hold, and that {[sign-]comonotonic} tradeoff consistency is violated. We derive a violation of {[sign-]comonotonic} strong indifference-tradeoff consistency. There exist four indifferences $\alpha_i f \sim \beta_i g$, $\gamma_i f \sim \delta_i g$, $\alpha_j x \sim \beta_j y$, and $\gamma_j x \sim \delta'_j y$, with i and j nonnull {on appropriate [sign-]comoncones containing the relevant acts} and $\delta' \nsim \delta$, say, $\delta' \prec \delta$. Because of symmetries, other cases are similar.

For $F = F' = X^n$ (no sign- or comonotonic restrictions), we replace δ' by δ in the last indifference, changing it into a strict preference by strong monotonicity, and, thus, generating a violation of strong indifference-tradeoff consistency. Under the nonexpected utility theories considered in this paper, we have to ensure the appropriate sign- and comonotonicity restrictions. We then proceed as follows. In each case hereafter, the changed acts will be in the same [sign-]comoncone as the original ones because the new outcomes are always between the original ones and β, thus satisfying all required sign- and comonotonicity restrictions as do β and the original outcome.

(a) If $\delta' \prec \delta \preccurlyeq \beta$, then we replace δ' by δ in the last indifference, turning it into a strict preference $\gamma_j x \prec \delta_j y$ by {[sign-]comonotonic} strong monotonicity. Thus, a violation of {[sign-]comonotonic} strong indifference-tradeoff consistency results.

(b) If $\delta' \prec \beta \prec \delta$, then we have $\gamma_i f \sim \delta_i g \succ \beta_i g \sim \alpha_i f$ implying $\gamma \succ \alpha$, but we also have $\gamma_j x \sim \delta'_j y \prec \beta_j y \sim \alpha_j x$ implying $\alpha \succ \gamma$. Hence, a contradiction has resulted.

(c) If $\beta \preccurlyeq \delta' \prec \delta$, then we replace δ by δ' in the second indifference, to obtain $\alpha_i f \sim \beta_i g$, $\gamma_i f \succ \delta'{}_i g$, $\alpha_j x \sim \beta_j y$, and $\gamma_j x \sim \delta'_j y$. {[Sign-]comonotonic} strong indifference-tradeoff consistency is violated by relabeling and symmetry. □

In the remainder of Appendix B, we will often use the following assumption.

ASSUMPTION 25. Weak ordering, weak monotonicity, and {[sign-]comonotonic} tradeoff consistency hold. □

LEMMA 26. *Under Assumption* 25, {[*sign-*]*comonotonic*} *strong monotonicity holds.*

PROOF. Let F be a [sign-]comoncone, or X^n, as the case may be. For $f, g \in F$, assume that $f_j \succcurlyeq g_j$ for all states j, and $f_i \succ g_i$ for some state i that is nonnull on F. The following reasoning proves $f \succ g$. {We assume, without loss of generality, that the acts in F are rank-ordered from best to worst by the identity ordering, e.g., $f_1 \succcurlyeq \cdots \succcurlyeq f_n$, etc.}

Define $h = (f_1, \ldots, f_i, g_{i+1}, \ldots, g_n)$. Then $f \succcurlyeq h \succcurlyeq (g_i)_i h \succcurlyeq g$ {with all acts contained in F}. It suffices to show $h = (f_i)_i h \succ (g_i)_i h$. Assume, for contradiction, that $(f_i)_i h \sim (g_i)_i h$. Combining this indifference with $(f_i)_i h \sim (f_i)_i h$ implies $f_i g_i \sim^*_F f_i f_i$, and writing $(f_i)_i h \sim (f_i)_i h$ twice gives $f_i f_i \sim^*_F f_i f_i$. {[Sign-]comonotonic} tradeoff consistency implies $f_i \sim g_i$, contradicting the assumed $f_i \succ g_i$. □

LEMMA 27. *Under Assumption* 25 *and solvability,* {[*sign-*]*comonotonic*} *strong indifference-tradeoff consistency holds.*

PROOF. We will only consider the "most complex" sign-comonotonic case. Assume, for contradiction, a violation of sign-comonotonic strong indifference-tradeoff consistency. We have $\alpha_i f \sim \beta_i g$, $\gamma_i f \sim \delta_i g$, and $\alpha_j x \sim \beta_j y$, and, say, $\gamma_j x \prec \delta_j y$, with all acts in appropriate sign-comoncones, and with i and j nonnull on the relevant sign-comoncones.

First, assume $\alpha \succcurlyeq \gamma$. Then $\beta_i g \sim \alpha_i f \succcurlyeq \gamma_i f \sim \delta_i g$. Transitivity and sign-comonotonic strong monotonicity (Lemma 26) imply $\beta \succcurlyeq \delta$. That is, $\alpha_j x \sim \beta_j y \succcurlyeq \delta_j y \succ \gamma_j x$. There exists, because of solvability, an ε with $\varepsilon_j x \sim \delta_j y$ and, further, $\alpha \succcurlyeq \varepsilon \succ \gamma$. (It can, but need not, be seen that, in general, $\varepsilon \succ \alpha$ is possible if state j, which is nonnull on the set now considered, is null elsewhere. In this case, we can always take $\varepsilon = \alpha$.) We have $\varepsilon_j x \in F'$ because $\alpha_j x \in F'$ and $\gamma_j x \in F'$. Since j is nonnull on F', the equivalences $\alpha_j x \sim \beta_j y$ and $\varepsilon_j x \sim \delta_j y$ imply $\alpha\beta \sim^*_{sc} \varepsilon\delta$. Also $\alpha\beta \sim^*_{sc} \gamma\delta$ and $\gamma \nsim \varepsilon$, which violates sign-comonotonic tradeoff consistency.

Second, assume $\alpha \preccurlyeq \gamma$, i.e., $\beta_j y \sim \alpha_j x \preccurlyeq \gamma_j x \prec \delta_j y$. As before, there is an ε with $\gamma_j x \sim \varepsilon_j y$ and $\beta \preccurlyeq \varepsilon \prec \delta$, so that $\varepsilon_j y \in F'$. The equivalences $\alpha_j x \sim \beta_j y$ and $\gamma_j x \sim \varepsilon_j y$ imply $\alpha\beta \sim^*_{sc} \gamma\varepsilon$, but also $\alpha\beta \sim^*_{sc} \gamma\delta$ and $\delta \nsim \varepsilon$, which violates sign-comonotonic tradeoff consistency.

Both cases lead to a contradiction and, therefore, $\gamma_j x \sim \delta_j y$ must hold. □

COROLLARY 28. *Under weak ordering, weak and* {[*sign-*]*comonotonic*} *strong monotonicity,* {[*sign-*]*comonotonic*} *strong indifference-tradeoff consistency implies* {[*sign-*]*comonotonic*} *tradeoff consistency. Under weak ordering, weak monotonicity, and solvability,* {[*sign-*]*comonotonic*} *tradeoff consistency implies* {[*sign-*]*comonotonic*} *strong indifference-tradeoff consistency.* □

The following corollary demonstrates that tradeoff consistency implies a restriction of the sure-thing principle for indifferences, corresponding to the theory under consideration.

COROLLARY 29. *Under Assumption* 25 *and solvability,* $(\alpha_i f \sim \alpha_i g) \Leftrightarrow (\beta_i f \sim \beta_i g)$ {*whenever the acts are* [*sign-*]*comonotonic*}.

PROOF. {Let the four acts in the corollary be contained in a [sign-]comonotonic set F.} If i is null {on F}, then $\alpha_i f \sim \beta_i f$ and $\alpha_i g \sim \beta_i g$, from which everything follows. Next suppose that i is not null {on F}. Then, by Lemma 27, $\alpha_i f \sim \alpha_i f$, $\beta_i f \sim \beta_i f$, and $\alpha_i f \sim \alpha_i g$ imply $\beta_i f \sim \beta_i g$. □

PROPOSITION 30. *Under Assumption* 25 *and solvability,* {[*sign-*]*comonotonic*} *preference-tradeoff consistency holds.*

PROOF. We derive one final preparatory lemma within this proof. Ends-of-proofs within proofs are indicated by QED, not by □.

LEMMA 31. *Assume weak ordering, weak monotonicity, and solvability. Let F be either X^n, or a comoncone, or a sign-comoncone. Let $\alpha_i f \succcurlyeq \beta_i g$ and $\gamma_i f \preccurlyeq (\prec)\delta_i g$, where all four acts are contained in F. Then there exist $\bar{f}$, $\bar{g}$ with $\alpha_i \bar{f}, \beta_i \bar{g}, \gamma_i \bar{f}, \delta_i \bar{g} \in F$, such that either $(\alpha_i \bar{f} \sim \beta_i \bar{g}$ and $\gamma_i \bar{f} \preccurlyeq (\prec)\delta_i \bar{g})$ or $(\alpha_i \bar{f} \succ \beta_i \bar{g}$, $\gamma_i \bar{f} \preccurlyeq (\prec)\delta_i \bar{g}$, and $\bar{f}_j \preccurlyeq \bar{g}_j$ for all $j \neq i)$. In the latter case, $\alpha \succ \beta$. Furthermore, in any case, we can have $\bar{f}_j \preccurlyeq f_j$ and $\bar{g}_j \succcurlyeq g_j$ for all $j \neq i$.*

PROOF. Let $\alpha_i f, \beta_i g, \gamma_i f, \delta_i g \in F$ be such that $\alpha_i f \succcurlyeq \beta_i g$ and $\gamma_i f \preccurlyeq (\prec)\delta_i g$. As before, we will only consider the most complex case, where F is a sign-comoncone. Without loss of generality, it can be assumed that the acts in F are rank-ordered from best to worst by the identity ordering. Furthermore, we may assume that $f = \alpha_i f$ and $g = \beta_i g$.

Step 1. We will "push the g_j with $g_j \prec f_j$ up towards f_j," and next "push the f_j with $f_j \succ g_j$ down towards g_j," until either an indifference $\alpha_i f \sim \beta_i g$ results, or $f_j \preccurlyeq g_j$ for all $j \neq i$ and still $\alpha_i f \succ \beta_i g$ (then α must be much better than β). If one of these cases already holds, then the rest of Step 1 can be skipped, and the proof continues with Step 2. To maintain the appropriate sign- and comonotonicity requirements throughout, we move the g_js up for j rank-ordered before i, starting with the first-rank-ordered j (farthest away from i, and associated with the best outcome), and we move the f_js down for j rank-ordered after i, starting with the last-rank-ordered j (farthest away from i, and associated with the worst outcome). Assume $\alpha_i f \succ \beta_i g$ and $f_j \succ g_j$ for at least one $j \neq i$.

Step 1.1 (*For* $j < i$, *pushing the* $g_j \prec f_j$ *up towards* f_j). Assume that there is a $j < i$ with $f_j \succ g_j$ (otherwise, go to Step 1.2). Let l be the minimal (rank-ordered first) index for which $f_l \succ g_l$. Consider the act $(f_l)_l g$. This act is an element of F, because $g_{l-1} \succcurlyeq f_{l-1} \succcurlyeq f_l \succ g_l \succcurlyeq g_{l+1}$. (Ignore, as usual, restrictions referring to undefined subscripts, such as $l-1$ if $l = 1$.) If $(f_l)_l g \preccurlyeq f$, replace the old act g by $g = (f_l)_l g$, and start again with Step 1. Otherwise, i.e., if $(f_l)_l g \succ f \succ g$, there exists, because of solvability, an ε with $\varepsilon_l g \sim f$. We have $f_l \succ \varepsilon \succ g_l$, so that $\varepsilon_l g \in F$. Replace the original act g by $g = \varepsilon_l g$, and go to Step 2.

Step 1.2 (*For* $j < i$, *pushing* $f_j \succ g_j$ *down towards* g_j). Assume that there is no $j < i$ with $f_j \succ g_j$. Then there is a $j > i$ with $f_j \succ g_j$. Let l be the maximal (rank-ordered last) index for which $f_l \succ g_l$. Consider the act $(g_l)_l f$. This act is an element of F, because $f_{l-1} \succcurlyeq f_l \succ g_l \succcurlyeq g_{l+1} \succcurlyeq f_{l+1}$. If $(g_l)_l f \succcurlyeq g$, replace the old act f by $f = (g_l)_l f$, and start again with Step 1. Otherwise, i.e., if $f \succ g \succ (g_l)_l f$, there exists, because of solvability, an ε with $\varepsilon_l f \sim g$. We have $f_l \succ \varepsilon \succ g_l$ and $\varepsilon_l f \in F$. Replace the original act f by $f = \varepsilon_l f$, and go to Step 2.

Step 2. Because there are only finitely many states, the procedure in Step 1 ends after finitely many repetitions. Define $\bar{f}$ and $\bar{g}$ as the newly constructed acts f and g resulting from Step 1. We have either $\alpha_i \bar{f} \sim \beta_i \bar{g}$ or $[\alpha_i \bar{f} \succ \beta_i \bar{g}$ and $\bar{f}_j \preccurlyeq \bar{g}_j$ for all $j \neq i]$. In what follows, f and g refer again to the original acts f and g. For the act $\bar{f}$ we have $\bar{f}_l = f_l$ if $l < i$, and $\bar{f}_l \preccurlyeq f_l$ otherwise. Then $\gamma_i \bar{f} \preccurlyeq \gamma_i f$. For the act $\bar{g}$ we have $\bar{g}_l = g$ if $l > i$, and $\bar{g}_l \succcurlyeq g_l$ otherwise. Then $\delta_i \bar{g} \succcurlyeq \delta_i g$. Because the f_js for $j < i$ have remained unaffected, and the f_js for $j > i$ have been moved down in preference, still $f_{i-1} \succcurlyeq \gamma \succcurlyeq \underline{f}_{i+1}$ and $\gamma_i \bar{f} \in F$. Similarly, $\delta_i \bar{g} \in F$. We have $\gamma_i \bar{f} \preccurlyeq \gamma_i f \preccurlyeq (\prec) \delta_i g \preccurlyeq \delta_i \bar{g}$ and, hence, $\gamma_i \bar{f} \preccurlyeq (\prec) \delta_i \bar{g}$. This completes the proof of Lemma 31. QED

In the rest of the proof of Proposition 30, we again consider only the most complex case, the sign-comonotonic case. Let $\succcurlyeq$ satisfy sign-comonotonic tradeoff consistency. Let F, F' be two sign-comoncones. Assume that there are acts

$$\alpha_i f \succcurlyeq \beta_i g, \qquad \gamma_i f \preccurlyeq \delta_i g, \qquad \alpha_j x \preccurlyeq \beta_j y, \qquad \gamma_j x \succ \delta_j y,$$

such that $\alpha_i f, \beta_i g, \gamma_i f, \delta_i g \in F$, i is nonnull on F, and $\alpha_j x, \beta_j y, \gamma_j x, \delta_j y \in F'$ with j nonnull on F'.

Step 1. In this step, we show that either $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \delta$, or $\alpha \succcurlyeq \gamma$ and $\beta \succcurlyeq \delta$. Assume $\alpha \prec \gamma$. Sign-comonotonic strong monotonicity (Lemma 26) implies $\delta_i g \succcurlyeq \gamma_i f \succ \alpha_i f \succcurlyeq \beta_i g$. We have $\beta \prec \delta$. Next assume $\alpha \succcurlyeq \gamma$. Then $\beta_j y \succcurlyeq \alpha_j x \succcurlyeq \gamma_j x \succ \delta_j y$ and $\beta \succ \delta$.

Step 2. In this step, we show that there exist $\bar{\alpha}_i f \sim \bar{\beta}_i g$, $\bar{\gamma}_i f \sim \bar{\delta}_i g$, $\bar{\alpha}_j x \prec \bar{\beta}_j y$, and $\bar{\gamma}_j x \succ \bar{\delta}_j y$ with $\bar{\alpha}_i f, \bar{\beta}_i g, \bar{\gamma}_i f, \bar{\delta}_i g \in F$, and $\bar{\alpha}_j x, \bar{\beta}_j y, \bar{\gamma}_j x, \bar{\delta}_j y \in F'$. In each of the following two cases, $\bar{\alpha}$ and $\bar{\gamma}$ are between α and γ, and $\bar{\beta}$ and $\bar{\delta}$ are between β and δ, in preference. They will, therefore, satisfy all the required sign- and comonotonicity restrictions.

First, assume $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \delta$. See Figure 2(a). We have $\delta_i g \succcurlyeq \gamma_i f \succcurlyeq \alpha_i f \succcurlyeq \beta_i g$. Solvability implies that there is a $\bar{\beta}$ with $\alpha_i f \sim \bar{\beta}_i g$, and a $\bar{\delta}$ with $\gamma_i f \sim \bar{\delta}_i g$. We can get $\delta \succcurlyeq \bar{\delta} \succcurlyeq \bar{\beta} \succcurlyeq \beta$, from which it follows that $\bar{\beta}_i g, \bar{\delta}_i g \in F$, $\bar{\beta}_j y, \bar{\delta}_j y \in F'$, $\alpha_j x \preccurlyeq \bar{\beta}_j y$, and $\gamma_j x \succ \bar{\delta}_j y$. In this case, let $\bar{\alpha} = \alpha$ and $\bar{\gamma} = \gamma$. We must have $\bar{\alpha}_j x \prec \bar{\beta}_j y$ because the other case, $\bar{\alpha}_j x \sim \bar{\beta}_j y$, would imply, by Lemma 27, that $\bar{\gamma}_j x \sim \bar{\delta}_j y$, which is a contradiction.

Second, assume $\alpha \succcurlyeq \gamma$ and $\beta \succcurlyeq \delta$. See Figure 2(b). Similarly as in the first case, solvability implies that there is an $\bar{\alpha}$ with $\alpha \succcurlyeq \bar{\alpha} \succcurlyeq \gamma$ and $\bar{\alpha}_i f \sim \beta_i g$, and a $\bar{\gamma}$ with $\bar{\alpha} \succcurlyeq \bar{\gamma} \succcurlyeq \gamma$ and $\bar{\gamma}_i f \sim \delta_i g$. Furthermore, $\bar{\alpha}_i f, \bar{\gamma}_i f \in F$, $\bar{\alpha}_j x, \bar{\gamma}_j x \in F'$, $\bar{\alpha}_j x \preccurlyeq \beta_j y$, and $\bar{\gamma}_j x \succ \delta_j y$. In this case, let $\bar{\beta} = \beta$ and $\bar{\delta} = \delta$. Lemma 27 implies that we must have $\bar{\alpha}_j x \prec \beta_j y$.

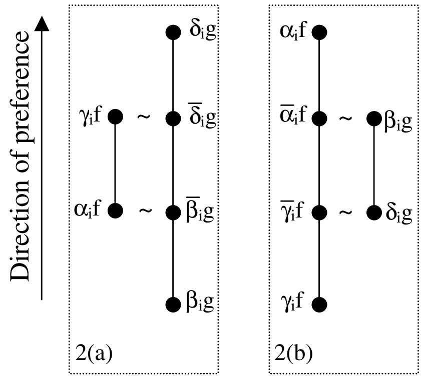

FIGURE 2. Configurations of acts.

As a preparation for the following step, we rename, for notational convenience, $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$, $\gamma = \bar{\gamma}$ and $\delta = \bar{\delta}$, so that we have $\alpha_i f \sim \beta_i g$, $\gamma_i f \sim \delta_i g$, $\alpha_j x \prec \beta_j y$, and $\gamma_j x \succ \delta_j y$. Furthermore, as before, either $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \delta$, or $\alpha \succcurlyeq \gamma$ and $\beta \succcurlyeq \delta$. Because of symmetry, we may assume hereafter $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \delta$.

Step 3. We will rule out all cases except $\gamma \succ \delta \succcurlyeq \beta \succ \alpha$. By Lemma 31, we can find $\tilde{x}$, $\tilde{y}$ ("pushing the x_js up and the y_js down") such that either $\alpha_j \tilde{x} \sim \beta_j \tilde{y}$ and $\gamma_j \tilde{x} \succ \delta_j \tilde{y}$, or $\alpha_j \tilde{x} \prec \beta_j \tilde{y}$ and $\gamma_j \tilde{x} \succ \delta_j \tilde{y}$ with $\tilde{x}_l \succcurlyeq \tilde{y}_l$ for all $l \neq j$. Furthermore, $\alpha_j \tilde{x}, \beta_j \tilde{y}, \gamma_j \tilde{x}, \delta_j \tilde{y} \in F'$. The first case leads to a contradiction because of Lemma 27. Therefore, we assume that the second case holds. This implies, in particular, $\alpha \prec \beta$, because $\tilde{x}_l \succcurlyeq \tilde{y}_l$ for all $l \neq j$.

Again by Lemma 31, we can find $\bar{x}, \bar{y}$ ("pushing" the x_js down and the y_js up) such that either $\gamma_j \bar{x} \sim \delta_j \bar{y}$ and $\alpha_j \bar{x} \prec \beta_j \bar{y}$, or $\gamma_j \bar{x} \succ \delta_j \bar{y}$ and $\alpha_j \bar{x} \prec \beta_j \bar{y}$ with $\bar{x}_l \preccurlyeq \bar{y}_l$ for all $l \neq j$. Furthermore, $\alpha_j \bar{x}, \beta_j \bar{y}, \gamma_j \bar{x}, \delta_j \bar{y} \in F'$. The first case leads to a contradiction because of Lemma 27; we therefore assume that the second case holds. This implies, in particular, $\gamma \succ \delta$, because $\bar{x}_l \preccurlyeq \bar{y}_l$ for all $l \neq j$.

Step 4. We have $\gamma \succ \delta \succcurlyeq \beta \succ \alpha$. This will lead to the final contradiction. Sign-comonotonic strong monotonicity (Lemma 26) implies $\beta_i f \succ \alpha_i f \sim \beta_i g$ and $\delta_i f \prec \gamma_i f \sim \delta_i g$ with $\beta_i f, \delta_i f \in F$. Lemma 31, applied to $\beta_i f \succ \beta_i g$ and $\delta_i f \prec \delta_i g$, implies that we can find $\bar{f}$, $\bar{g}$ such that $\beta_i \bar{f} \sim \beta_i \bar{g}$ and $\delta_i \bar{f} \prec \delta_i \bar{g}$ (with $\bar{f}_j \preccurlyeq f_j$ and $\bar{g}_j \succcurlyeq g_j$ for all $j \neq i$), because the second case in that lemma, implying the impossible $\beta \succ \beta$, cannot occur here. Furthermore, $\beta_i \bar{f}, \beta_i \bar{g}, \delta_i \bar{f}, \delta_i \bar{g} \in F$. A contradiction with Corollary 29 has resulted. The proof is complete. □

Appendix C: Further proofs of results except those in §7. This appendix completes the proofs of the results, except those in §7. We give one more elementary lemma. The proof is omitted.

LEMMA 32. *Assume weak ordering and weak and {[sign-]comonotonic} strong monotonicity. If a {[sign-]comonotonic} standard sequence* $\alpha^0, \alpha^1, \ldots$ *is increasing, then* $\alpha^{k+1} \succ \alpha^k$ *for all k. If it is decreasing, then* $\alpha^{k+1} \prec \alpha^k$ *for all k.* □

PROOF OF THEOREMS 5 AND 8 AND OBSERVATIONS 6 AND 9. For the implications (i) ⇒ (ii) in both theorems, necessity of the Archimedean and the tradeoff-consistency axioms follows from Equation (5) as explained in the main text, and the other conditions are obvious. We therefore assume Statement (ii) and derive the corresponding Statement (i) and the uniqueness results in the Observations. Statement (ii) and Proposition 30 imply the preference versions of the tradeoff-consistency axioms.

We first consider the SEU model of Theorem 5, and the uniqueness results of Observation 6. These follow from Theorem 5 in Wakker (1991c) for two or more nonnull states.

For exactly one nonnull state, the existence of a representing function on the outcomes follows from Theorem 2.2 of Krantz et al. (1971), and then everything follows. For the trivial case of a trivial preference relation, all results are trivial.

We next turn to the CEU model of Theorem 8 and Observation 9. These follow from Theorem 8 in Wakker (1991c) for the nondegenerate case. If one comoncone only has null states, then all outcomes are equivalent, the corresponding constant acts being contained in this one comoncone. By weak monotonicity, the preference relation is trivial, and so are the results for this case. The only remaining case is the degenerate case of exactly one nonnull state in every comoncone. Then, as before, Theorem 2.2 of Krantz et al. (1971) implies the existence of a representing function U on the outcomes. For every act f in a comoncone with nonnull state i, $U(f_i)$ determines the preference value of act f. From this observation, everything follows. □

We next turn to the claims at the end of §4.1. The following lemma prepares for Observation 34.

LEMMA 33. *Assume weak ordering, at least two nonnull states, weak monotonicity, and solvability. Then, for all outcomes α, β, there exist acts f, g and a nonnull state j such that $\alpha_j f \sim \beta_j g$. This implies $\alpha\beta \sim^* \alpha\beta$ and, hence, reflexivity of $\sim^*$.* □

PROOF. Assume that states 1 and 2 are nonnull. Let α, β be outcomes, and h an arbitrary fixed act. W.l.o.g. assume $\alpha \succcurlyeq \beta$ and $\beta_1\alpha_2 h \succcurlyeq \alpha_1\beta_2 h$. We may add "$\succcurlyeq \beta_1\beta_2 h$." By solvability, there exists a δ with $\alpha_1\beta_2 h \sim \beta_1\delta_2 h$. Writing the indifference twice and applying Equation (4) with $j = 1$, $\gamma = \alpha$, $\delta = \beta$ implies $\alpha\beta \sim^* \alpha\beta$. □

The following observation, formalizing the claim at the end of §4.1, was stated for real-valued outcomes, and without formal proof, by Köbberling and Wakker (2001).

OBSERVATION 34. *In Statement* (ii) *of Theorem* 5, *tradeoff consistency* (d) *can be replaced by the requirement that strong monotonicity holds and that $\sim^*$ is transitive. If there are at least two nonnull states, transitivity of $\sim^*$ can be replaced by the requirement that $\sim^*$ is an equivalence relation.* □

PROOF. It is obvious that SEU implies strong monotonicity. If $\succcurlyeq$ is trivial, then all states are null and $\sim^*$ is empty, and everything follows. In the degenerate case of exactly one nonnull state, [$\alpha\beta \sim^* \gamma\delta$ if and only if $\alpha \sim \beta$ and $\gamma \sim \delta$] for all outcomes α, β, γ, δ, and everything follows. Note that, in the trivial and the degenerate cases, $\sim^*$ is transitive and symmetric but not reflexive and, hence, is not an equivalence relation. The rest of this proof assumes two or more nonnull states.

To demonstrate that SEU implies transitivity of $\sim^*$, assume $\alpha\beta \sim^* \gamma\delta$ and $\gamma\delta \sim^* \sigma\tau$. The latter $\sim^*$ implies that $\gamma_j f \sim \delta_j g$ and $\sigma_j f \sim \tau_j g$ for some f, g, and nonnull j. $\alpha\beta \sim^* \gamma\delta$ implies $U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$, by Equation (5). This equality, $\gamma_j f \sim \delta_j g$, and substitution of expected utility imply $\alpha_j f \sim \beta_j g$. This indifference and $\sigma_j f \sim \tau_j g$ imply $\alpha\beta \sim^* \sigma\tau$. Transitivity of $\sim^*$ follows. Symmetry of $\sim^*$ was demonstrated in Observation 1. Reflexivity is implied by Lemma 33. Note that the latter requires solvability. We conclude that $\sim^*$ is an equivalence relation.

Conversely, assume that Statement (ii) in Theorem 5 holds, but with (d) replaced by strong monotonicity and transitivity of $\sim^*$. We derive tradeoff consistency. Assume $\alpha\beta \sim^* \gamma\delta$ and $\alpha'\beta \sim^* \gamma\delta$. $\sim^*$ is symmetric (Observation 1) and transitive, so that $\alpha\beta \sim^* \alpha'\beta$. Therefore, there exist acts f and g and a nonnull state i such that $\alpha_i f \sim \beta_i g$ and $\alpha'_i f \sim \beta_i g$, and, hence, $\alpha_i f \sim \alpha'_i f$. By strong monotonicity, $\alpha \sim \alpha'$ follows. Tradeoff consistency holds. □

Because Lemmas 20 and 21 are used in the proof of Corollary 10, we give their proofs prior to that of Corollary 10.

PROOF OF LEMMA 20. This result was proved in Wakker (1989, Lemma III.3.3). It can also be inferred from Krantz et al. (1971, §6.2.13), where it is stated for three or more dimensions. □

The following lemma prepares for the proof of Lemma 21.

LEMMA 35. *Assume continuous weak ordering with respect to a connected product topology on X^n. The restriction of $\succcurlyeq$ to X (through identification with constant acts) is continuous.*

PROOF. Consider a set $\{\alpha \in X: \alpha \succ \beta\}$, and an element γ thereof. The set $T = \{x \in X^n: x \succ (\beta, \ldots, \beta)\}$ is open in X^n. For $(\gamma, \ldots, \gamma)$ (as for all other elements of T), there exists a product set $A_1 \times \cdots \times A_n \subset T$ of open sets $A_j \subset X$ that contains $(\gamma, \ldots, \gamma)$. The intersection $A_1 \cap \cdots \cap A_n$ is a neighborhood of γ in the set $\{\alpha \in X: \alpha \succ \beta\}$, proving that the latter set is open. Similarly, the set $\{\alpha \in X: \alpha \prec \beta\}$ is open. □

PROOF OF LEMMA 21. In the degenerate case of exactly one nonnull state in Lemma 21, continuous weak ordering with respect to a separable connected product topology implies, first, that $\succcurlyeq$ on X is continuous (Lemma 35); second, that there exists a function representing it (well known, e.g., Wakker 1989, Theorem III.3.6); and, third, that there exists a countable order-dense subset of X (Krantz et al. 1971, Theorem 2.2). Therefore, the proof of Lemma 21 is complete if the following lemma is proved.

LEMMA 36. *Assume continuous weak ordering with respect to a connected product topology. Under weak and {comonotonic} strong monotonicity, every bounded {comonotonic} standard sequence is finite.*

PROOF. This proof can also be inferred from Krantz et al. (1971, §6.12.3) for the noncomonotonic case, and from Wakker (1991b, Lemma 4.2) for the comonotonic case. Assume that $(\alpha^{k+1})_i g_j f \sim (\alpha^k)_i G_j f$ for all k {with all acts contained in a comoncone F and} with i and j nonnull {on F}. Assume that the standard sequence is increasing, i.e., $G \succ g$. (The other case is similar.) By Lemma 32, $\alpha^k \succ \alpha^{k-1}$ for all k. It is obvious that α^0 can serve as an outcome less preferred than all elements of the standard sequence. Let σ be an outcome more preferred than all α^k and assume, for contradiction, that the standard sequence is infinite. The set $\{\beta: \beta \prec \alpha^k \text{ for some } k\}$ is, by Lemma 35, a union of open sets and, hence, is open. It contains α^0 and is nonempty. Its complement is the set $\{\beta: \beta \succcurlyeq \alpha^k \text{ for all } k\}$, which is nonempty because it contains σ. For contradiction with connectedness, it suffices to prove openness of the latter set.

{Comonotonic restrictions for F may require that all α^k are less preferred than some fixed outcome σ', for $(\alpha^k)_i g_j f$ and $(\alpha^{k-1})_i G_j f$ to be contained in F. For example, if the rank-ordering for F is the identity, if $i = n$, and $j < n-1$, then $\sigma' = f_{n-1}$. We can then take $\sigma = \sigma'$. In this case, for any outcome β with $\alpha^0 \preccurlyeq \beta \preccurlyeq \sigma$, it follows that $\beta_i g_j f$ and $\beta_i G_j f$ are contained in F.} Consider the preferences $\sigma_i G_j f \succ \sigma_i g_j f \succcurlyeq (\alpha^{k+1})_i g_j f \succ (\alpha^k)_i g_j f \sim (\alpha^{k-1})_i G_j f$, where the strict preferences follow from {comonotonic} strong monotonicity. From $\sigma_i G_j f \succ \sigma_i g_j f \succ (\alpha^{k-1})_i G_j f$ and solvability (Lemma 20) we conclude that $\gamma_i G_j f \sim \sigma_i g_j f$ for some γ, by weak monotonicity $\sigma \succ \gamma \succ \alpha^{k-1}$. This holds for all k.

Similarly, for any τ instead of σ such that $\tau \succ \alpha^{k-1}$ for all k, there exists a γ' such that $\tau \succ \gamma' \succ \alpha^{k-1}$ for all k. {$\gamma' = \sigma'$ if $\tau \succ \sigma'$, γ' is constructed as γ above if $\tau \preccurlyeq \sigma'$.} By Lemma 35, the set $\{\delta \in X: \delta \succ \gamma'\}$ is open, by transitivity it is contained in the set $\{\beta: \beta \succcurlyeq \alpha^k \text{ for all } k\}$, and it contains τ; in short, it is a neighborhood of τ within the set $\{\beta: \beta \succcurlyeq \alpha^k \text{ for all } k\}$. The latter set is open. A contradiction has resulted and, hence, the standard sequence must be finite. QED

The proof of Lemma 21 is now complete. □

PROOF OF COROLLARY 10. The implication (i) ⇒ (ii) is elementary. Next assume (ii). The conditions of Theorems 5 and 8 have been established in the main text and in Lemmas 20 and 21 (the latter can be used because of Lemma 26). Theorems 5 and 8 imply Statement (i), except for continuity of the utility functions. Continuity of utility is proved as in Wakker (1988, Theorem 3.1) and Wakker (1993a, Lemma A.1). The extra continuity requirement in the uniqueness result is obviously satisfied. An alternative proof of this corollary can be obtained from Proposition 30 and existing results in the literature for the topological approach, i.e., Wakker (1989, Theorems IV.2.7 and VI.5.1 and Remark A3.1). □

PROOF OF OBSERVATION 11. For a sign-comonotonic set, we can take one fixed n-tuple of decision weights that are all nonnegative. For the states of nature with gain-outcomes, they are derived from W^+. For the states of nature with loss-outcomes, they are derived from the dual of W^-. If at least one of these weights is positive, then we normalize by dividing each weight by the total sum of decision weights, leading to probabilities and an SEU-model. If all decision weights are zero, then the preference relation is trivial on the particular set, and we take the probabilities arbitrary and utility constant. □

PROOF OF THEOREM 12 AND OBSERVATION 13. These results follow from Proposition 30 and Theorem 6.3 in Wakker and Tversky (1993). □

PROOF OF LEMMA 14. Under the assumptions of the lemma, suppose that $\alpha\beta \sim^*_{\{c\}} \gamma\delta$, and $\alpha_i f \sim \beta_i g$ and $\gamma_i f \sim \delta_i g$ {with all acts comonotonic and} with i nonnull {on the corresponding comoncone}. Twofold application of mixture independence implies $\frac{1}{2}\alpha_i f + \frac{1}{2}\delta_i g \sim \frac{1}{2}\beta_i g + \frac{1}{2}\delta_i g \sim \frac{1}{2}\beta_i g + \frac{1}{2}\gamma_i f$. By {comonotonic} strong monotonicity and nonnullness of i, $\frac{1}{2}\alpha + \frac{1}{2}\delta \sim \frac{1}{2}\beta + \frac{1}{2}\gamma$. Deriving a similar implication $\frac{1}{2}\alpha' + \frac{1}{2}\delta \sim \frac{1}{2}\beta + \frac{1}{2}\gamma$ from $\alpha'\beta \sim^*_c \gamma\delta$, we see that, by outcome monotonicity for mixing, $\alpha' \sim \alpha$ must hold. Because of symmetry in outcomes, {comonotonic} tradeoff consistency follows. □

PROOF OF PROPOSITION 19. The preference-tradeoff consistency conditions imply not only weak monotonicity, but also the corresponding strong monotonicities (Wakker 1989, Lemma VI.4.10; Wakker and Tversky 1993, Lemma A2.2). They also imply the corresponding strong equivalence-tradeoff conditions, because indifferences $\alpha_i f \sim \beta_i g$, $\gamma_i f \sim \delta_i g$, and $\alpha_j x \sim \beta_j y$ imply both the triple $\alpha_i f \preccurlyeq \beta_i g$, $\gamma_i f \succcurlyeq \delta_i g$, and $\alpha_j x \succcurlyeq \beta_j y$ and the triple $\alpha_i f \succcurlyeq \beta_i g$, $\gamma_i f \preccurlyeq \delta_i g$, and $\alpha_j x \preccurlyeq \beta_j y$. Therefore, both $\gamma_j x \succcurlyeq \delta_j y$ and $\gamma_j x \preccurlyeq \delta_j y$ follow. We conclude that the corresponding strong equivalence-tradeoff consistency holds. The proposition follows from Lemma 24. □

Appendix D: Proofs of results in Section 7.

PROOF OF THEOREMS 16 AND 17 AND OBSERVATION 18. All reasonings hereafter hold both on the domain of all acts and on the domain of binary acts, because none of the constructions generates a nonbinary act from a binary one. The proof of Observation 18, therefore, follows immediately from the following reasonings and is not discussed further.

As before, the texts between braces are to be read for the comonotonic results, but are to be skipped for the results without comonotonicity restrictions. That {comonotonic} multisymmetry implies {comonotonic} act-independence was explained in the main text. Throughout this proof, event A denotes the mixing event.

{A will always be associated with better outcomes than A^c. The comonotonicity restrictions for Figure 1(c) correspond with those for Figure 1(e). In Figure 1(c) we have, because of weak monotonicity, that $(\mathrm{CE}(A{:}f_1, A^c{:}g_1), \dots, \mathrm{CE}(A{:}f_n, A^c{:}g_n))$ is in the same comoncone as f and g, and $\mathrm{CE}(f_1, \dots, f_n) \succcurlyeq \mathrm{CE}(g_1, \dots, g_n)$, so that event A is indeed associated with the best outcome.} We assume {comonotonic} act-independence, and derive {comonotonic} tradeoff consistency. {By weak monotonicity, $(\mathrm{CE}(A{:}f_1, A^c{:}g_1), \dots, \mathrm{CE}(A{:}f_n, A^c{:}g_n))$ and $(\mathrm{CE}(A{:}f_1', A^c{:}g_1), \dots, \mathrm{CE}(A{:}f_n', A^c{:}g_n))$ in Figure 1(e) are in the same comoncone as f, g, f'.} We also make the following assumption in the rest of this appendix.

ASSUMPTION 37. There exists a certainty equivalent function CE, and weak ordering, weak and {comonotonic} strong monotonicity, and {comonotonic} act-independence for an event A hold. Q.E.D.

The only part of {comonotonic} multisymmetry that we will use in this proof and, therefore, derive from {comonotonic} act-independence, concerns binary acts for the mixing event A {with the same ranking position in all mixtures. In this manner, we avoid the technical problems and assumptions that were needed in Lemma 3 of Chew and Karni (1994) and were explained there before the lemma. Therefore, our derivations can remain elementary and can avoid the use of continuity or solvability}.

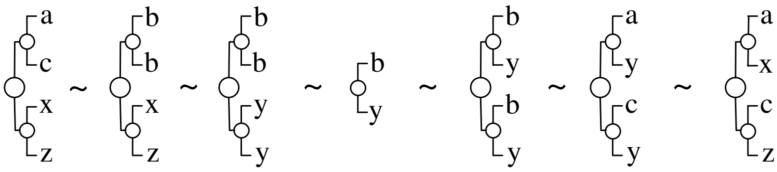

FIGURE 3. Deriving bisymmetry from act-independence.

LEMMA 38. *The most left and most right CE-compositions in Figure* 3 *are indifferent* {*under the following rank-ordering conditions*: $a \succcurlyeq c$, $x \succcurlyeq z$, $a \succcurlyeq x$, $c \succcurlyeq z$}.

PROOF. See Figure 3. {We assume the rank-ordering of outcomes displayed in the lemma. We may further assume that $c \succcurlyeq x$. Otherwise, interchange c and x in Figure 3 and in what follows, so that now b defined hereafter will be the CE of $(A{:}a, A^c{:}x)$, etc.} The first two indifferences in Figure 3 follow from defining appropriate certainty equivalents b and y and weak monotonicity. The third and fourth indifferences follow from reflexivity/idempotency (i.e., $(A{:}d, A^c{:}d) \sim d$). The next-to-last indifference follows from the definition of b and {comonotonic} act-independence. The last indifference follows from the definition of y and {comonotonic} act-independence. {From the ranking requirements $a \succcurlyeq c$, $x \succcurlyeq z$, $a \succcurlyeq x$, $c \succcurlyeq z$ it follows that all outcomes associated with A in Figure 3 dominate their counterparts associated with A^c, except possibly $c \prec y$. The latter preference is excluded nevertheless because of our additional assumption $c \succcurlyeq x$ and weak monotonicity. The latter step explains why we can escape the technical complications of Lemma 3 of Chew and Karni (1994).} QED

A comonotonic version of the following reformulation of act-independence, also with indifferences, can be recognized in Axiom 5 of Nakamura (1995). Lemma 39, illustrated in Figure 4, follows immediately from twofold application of act-independence: Replacing both acts in a CE-composition by equivalent other acts does not affect the preference value of the composition.

LEMMA 39. *The two indifferences preceding the implication sign in Figure* 4 *imply equivalence of the left and right CE-compositions after the implication sign* {*whenever* f, g, f', g' *are comonotonic and, for all* j, (a) $f_j \succcurlyeq g_j$; (b) $f'_j \succcurlyeq g_j$; *and* (c) $f'_j \succcurlyeq g'_j$}. QED.

Lemma 40, the key step in this proof, shows that $\alpha\beta \sim^*_{\{c\}} \gamma\delta$ has an implication independent of the particular state i and acts f, g used to elicit the $\sim^*_{\{c\}}$ relationship. It generalizes the main step in the proof of Lemma 14 to the case of mixing through an event A, using, in Figure 5(d), the basic configuration underlying the midpoint operation of Ghirardato et al. (2002). The subscript $_{\{c\}}$ is to be read only for the comonotonic results.

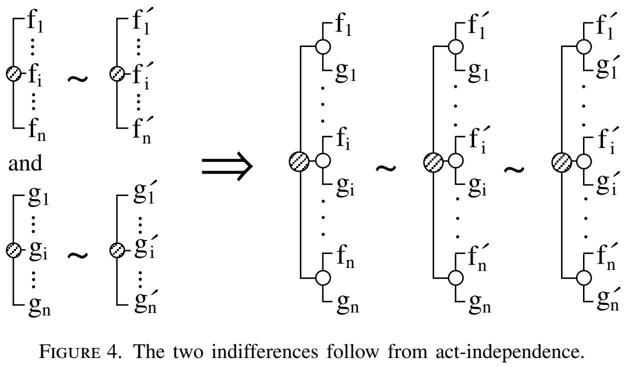

FIGURE 4. The two indifferences follow from act-independence.

LEMMA 40. *$\alpha\beta \sim^*_{\{c\}} \gamma\delta$ implies that, for all outcomes h, ℓ, the indifference in Figure* 5(d) *holds* {*whenever the $*$ relation is elicited as in Figure* 5(a) (*so that all acts in the figure are comonotonic and i is nonnull on the corresponding comoncone*), *ℓ is less preferred than all other outcomes in Figure* 5, *and h is more preferred than all other outcomes in the figure*}.

PROOF. See Figure 5. Assume Figure 5(a) {with the four acts comonotonic and} state i nonnull {on the corresponding comoncone}. By {comonotonic} act-independence, the upper (lower) indifference in Figure 5(b) follows from the upper (lower) one in Figure 5(a).

The first indifference in Figure 5(c) follows from the two in Figure 5(b) and Lemma 39. In this argument, the "smallest-circle" CEs in Figures 5(b) and 5(c) play no role other than as outcomes. For example, the CE of h and f_1 in Figures 5(b) and 5(c) plays the same role as f_1 in Lemma 39, etc.

{Furthermore, the following rank-ordering conditions, required for application of the comonotonic version of Lemma 39, are satisfied.

(a) The four n-outcome acts in Figure 5(b) (resulting from interpreting the smallest-circle-CEs there, for binary acts depending on A, as outcomes) are comonotonic, being in the same comoncone as those in Figure 5(a).

(b) For all binary acts depending on A in the figure, A is associated with the best outcome and A^c with the worst. This follows because h (ℓ) are better (worse) than all other outcomes, and $(A{:}h, A^c{:}x) \succcurlyeq (A{:}y, A^c{:}\ell)$ for all outcomes x, y in the figure. The latter preference implies requirements (a), (b), and (c) in Lemma 39.}

The second indifference in Figure 5(c) follows from Lemma 38 and weak monotonicity. The indifference in Figure 5(d) follows from indifference between the first and third ("dashed-circle") CE-compositions in Figure 5(c), agreement of these two compositions everywhere outside of state i, nonnullness of state i {on the relevant comoncone}, and {comonotic} strong monotonicity. QED

To establish {comonotonic} tradeoff consistency, assume $\alpha\beta \sim^*_{\{c\}} \gamma\delta$ and $\alpha'\beta \sim^*_{\{c\}} \gamma\delta$. Then the equivalence in Figure 5(d) holds with α and also with α' instead of α. Because of nonnullness of events A and A^c and {comonotonic} strong monotonicity, this can hold only

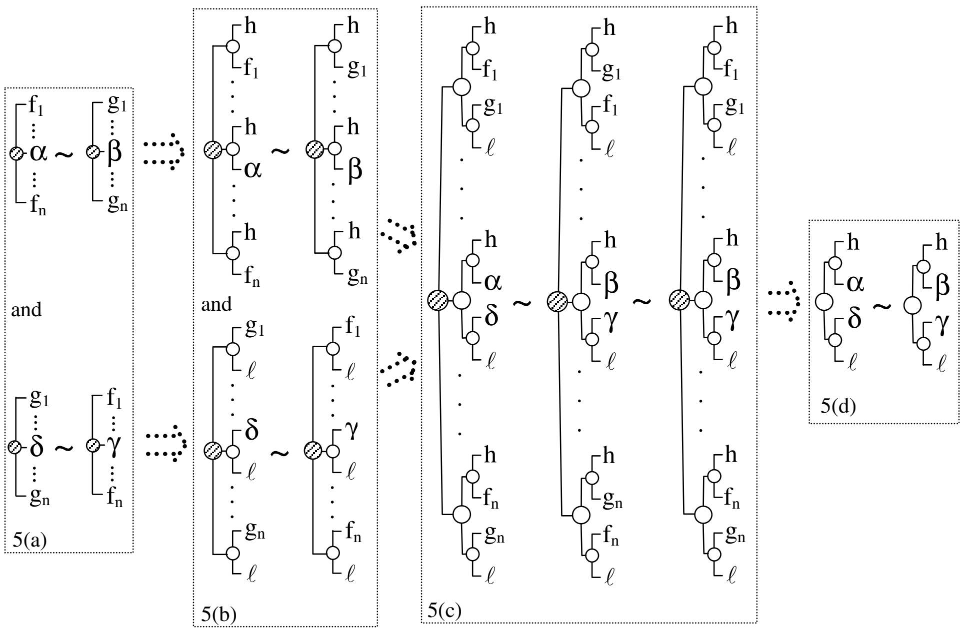

FIGURE 5. Proof of Lemma 40.

if $\alpha \sim \alpha'$. Tradeoff consistency follows because the condition is symmetric in the outcomes (Observation 1). The proof of Theorems 16 and 17, and Observation 18, is complete. □

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