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NONADDITIVE PROBABILITIES AND DERIVED STRENGTHS OF PREFERENCES

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SUMMARY

A method is introduced to derive strength of preference revelations on consequences from ordinal preferences on acts. By means of this a behavioural foundation is given for the maximization of subjective expected utility with continuous utility, where 'probabilities' may be nonadditive. Further, utility may be nonlinear, no lottery mechanisms are needed, and the state space may be arbitrary. An alternative interpretation of the resulting approach is given according to which the deviation from expected utility is not so much the nonadditivity of the probabilities, but rather is act-dependence of the probabilities.

KEYWORDS: nonadditive probability, expected utility generalized, strengths of preferences, uncertainty, risk.

1. INTRODUCTION.

This paper considers Schmeidler's approach to decision making under uncertainty. Schmeidler's approach generalizes expected utility by dealing with (capacities =) nonadditive probabilities; it uses the 'Choquet integral' to derive decisions from these. Nonadditivity of probabilities of decision makers is often taken to reflect *uncertainty about* the quantification of uncertainty. If probabilities of elementary outcomes add up to less than one, the difference with one is taken to measure the uncertainty about the probabilities.

Mainly Savage(1954) provided the behavioural foundation for *additive* probabilities, using expected utility to derive decisions from these additive probabilities. Only after publication of Savage's result did subjective expected utility theory become full-blown. Schmeidler(1984a) was the first to provide (in the spirit of Savage) a behavioural foundation for the use of *nonadditive* probabilities. In this behavioural foundation Schmeidler required the availability of ('objective') lottery mechanisms on consequences. That requirement is removed by the Main Theorem of this paper, Theorem 6.8, which requires continuity instead.

In section 2 we shall sketch the historical background of nonadditive probabilities. Section 3 gives some methodological observations concerning generalizations of expected utility. Section 4 gives elementary definitions and notations on preference relations. Section 5 presents the Choquet integral. This generalizes the usual integral to cope also with nonadditive probabilities. Within 'comonotonic' sets the Choquet integral will behave as a usual additive integral.

The main new ideas of this paper are presented in section 6, where the extension of expected utility to nonadditive probabilities is characterized with continuity of utility as only restriction, for 'strongly bounded' acts. This should make matters suited for economic applications. The main new tool for our result is given in Definitions 6.1 and 6.3, where strength-of-preference revelations over consequences are derived from solely an 'ordinal' preference relation over the acts, by means of a tradeoff-idea from multiattribute utility theory. A condition to exclude 'comonotonic-contradictory' (revelations of) strengths of preferences will then characterize the desired representation. Also this way to derive strengths of preferences is used to reformulate the characterization of subjective expected utility maximization of Wakker(1984), by reformulating 'cardinal coordinate independence' as the exclusion of contradictory (revelations of) strengths of preferences (without any comonotonicity restriction). The characterizing conditions could just as well have been formulated directly in terms of the primitive, the preference relation; it is hoped that the present formulations, in terms of derived strengths of preferences, are more transparent. The issue of how these derived strengths of preferences may be related to other notions of strengths of preferences, such as strengths of preferences under certainty, is not taken up in this paper. References on related discussions are Allais(1952, 1979), Krzysztofowicz(1983).

Nonadditivity of probabilities does not provide the only way to interpret Schmeidler's generalization of expected utility. In subsection 7.1 we shall expound that Schmeidler's approach can also be considered the generalization of expected utility which allows probabilities to be act-dependent through the ordering of states of nature according to 'favourableness'. Given an act, the state of nature giving the most preferred consequence is the most favourable one; and so on. (The act-dependent probabilities will then be required to satisfy a consistency condition (7.2).) Further section 7 gives examples and special cases of capacities, ranging from max-min-behaviour to fuzzy sets. This should show that Schmeidler's approach is general enough to entail many other approaches as special cases. Section 8 concludes the main body of the paper. In the Appendix the proof of the main theorem will be given.

Our results have been formulated for decision making under uncertainty, where the coordinates refer to states of nature. One may also let coordinates refer to individuals, and for instance reinterpret acts as allocations. In that case our theorems give results for welfare theory. Definitions 6.1 can then be interpreted to formalize a role of individual strengths of preferences in social welfare.

The formulations of the main Theorem 6.8 and the previously published, now reformulated, Theorem 6.6 are such that they can be understood without consultation of the remainder of the text, with the exception of the involved definitions of course; these definitions are listed directly above Theorem 6.8.

2. HISTORY OF THE SUBJECT

The main restrictive assumption in Savage's <u>derivation</u> of subjective expected utility maximization (giving a list of sufficient conditions) for decision making under uncertainty is his postulate 'P6', requiring the state space to be infinitely divisible. A second derivation of subjective expected utility maximization, important for this paper, has been given by Anscombe&Aumann(1963). They use a well-known result of von Neumann&Morgenstern (1947, 1953), to <u>characterize</u> subjective expected utility maximization (i.e. give a list of conditions, not only sufficient, but also necessary) for the case where lotteries on consequences are available and the utility function on the consequences is a 'von Neumann & Morgenstern' utility function, i.e. it is linear. Mathematically the result of Anscombe&Aumann is closely related to the result of de Finetti(1937; see also de Finetti,1972, 1974), who obtained his result for the case where consequences are amounts of money, and utility is linear. This restriction is not common in economic literature, hence de Finetti's result has not received much attention there, as opposed to statistical literature where his 'coherence condition' is well-known. The third derivation of subjective expected utility maximization considered in this paper is the author's result in Wakker(1984; 1986, Theorems IV.3.3 and V.6.1). There subjective expected utility maximization is derived under the restrictive assumption of continuity of the utility function. This has been done with the purpose to obtain a restriction, suited for economic applications.

The most important innovation in Savage's work may be the 'sure-thing principle' which adapts the 'independence condition' of von Neumann&Morgenstern to the context of decision making under uncertainty. It is mainly this principle which implies the additivity of the obtained subjective ('personalistic') probability measure. As the independence condition of von Neumann&Morgenstern (and Anscombe&Aumann), the sure-thing principle has been subjected to heavy criticisms. For one part these criticisms are based on the finding of systematic violations of the principle by decision makers in experiments and in actual decision situations. Secondly the criticisms are based on introspections of scientists, primarily Allais(1953, 1979), leading them to disagree with the normative appeal as was ascribed to the sure-thing principle, for instance in Savage(1954, first paragraph of section 2.7, in particular the last sentence), or Raiffa(1970, p. 82). The critics usually refer to the Allais paradox (see Example 7.5) and the Ellsberg Paradox (see Example 7.4). A well-contemplated exposition on this is McClennen(1983). The dissatisfaction with the sure-thing principle has impelled the introduction of new approaches.

Schmeidler's approach can best be considered to be the generalization to nonadditive probabilities of the approach of Anscombe&Aumann(1963). The 'nonadditive generalization' of de Finetti's linear-utility-result can be obtained in a way which mathematically is analogous. An analogue of Savage's result for the case of nonadditive probabilities has been obtained by Gilboa(1985a), who was inspired by an early version of Schmeidler's work. The main result of this paper, Theorem 6.8, will adapt Theorems IV.3.3 and V.6.1 of Wakker(1986) to the case of nonadditive probabilities. As compared to the results of Schmeidler(1984a) and the nonadditive version of de Finetti's result, our result does not need lotteries on consequences any more, or linearity of utility. As compared to Gilboa's adaptation of Savage(1954), no restrictions on the state space are needed anymore; the state space may for instance be finite, or have atoms. The restriction for our result is continuity of the utility function w.r.t. a connected[1] separable[2] topology. This restriction can be satisfied for instance in the case where consequences are amounts of money, or commodity bundles. The restriction is also applicable in contexts where no (physical) quantification for consequences is available yet.

In the proof of the main theorem we shall use Theorem VI.5.1 of Wakker(1986; see also Wakker,1987, Theorem 5.1); in this theorem the desired adaptation was already obtained for finite state spaces, without yet using the derived strength of preference relation.

3. THE EVALUATION OF MORE GENERAL APPROACHES

The prevailing view in the literature on decision making under uncertainty is nowadays that the expected-utility-approach excludes too many interesting ways of decision making under uncertainty. Such ways are revealed for instance in the Allais paradox and in the Ellsberg paradox. Hence there appears an increasing number of approaches to generalize expected utility. One such approach is Schmeidler's. Obviously the advantage of a more general approach, to allow for more interesting ways of decision making under uncertainty, should be weighed against the disadvantage to allow for more *un*interesting ways of decision making under uncertainty. By this disadvantage fewer theorems and predictions can be obtained, and methods to elicit, or help a decision maker choose, the parameters in such a more general approach will be more complicated (in the approach studied in this paper the parameters will be the utilities and capacities).

Another example of a generalization of expected utility maximization is the approach proposed by Machina(1982). His approach may be interpreted to permit the expected utility model to depend 'locally' on a considered act. As long as the preferences are smooth enough Machina's approach is less specified, i.e. more general, than expected utility. It allows for the ways of decision making under uncertainty such as usually exhibited in the Allais Paradox and the Ellsberg Paradox. An important step in Machina's propagation of his approach is to show that it still excludes sufficiently many uninteresting ways of decision making under uncertainty to allow the derivation of theorems on the measurement of risk aversion in the spirit of Arrow&Pratt. Besides Machina's approach, many other generalizations of expected utility have been considered. For the context of decision making under uncertainty we mention Arrow&Hurwicz(1972) and Cohen&Jaffray(1980) on complete ignorance, Skyrms(1980) and Goldsmith&Sahlin(1981) on higher-order-probabilities, Bell(1982) and Loomes&Sudgen(1982, 1986) on regret theory, Fishburn(1984) on the skew-symmetric bilinear theory, and Gilboa(1986) and Jaffray(1986) on a an approach dealing with the 'security factor'.

Like Machina's approach, and almost all generalizations of expected utility, Schmeidler's approach does not exclude the behaviour such as usually exhibited in the Allais paradox and in the Ellsberg paradox. Such behaviour can now be explained as 'uncertainty aversion', formalized as superadditivity of the nonadditive probability measure (see subsection 7.1, or Schmeidler,1984a, below Remark 3; see also Schmeidler,1984b,1984c).

4. STATES, ACTS, CONSEQUENCES, AND THE PREFERENCE RELATION

Let S be a set of states (of nature). Exactly one of them is the 'true state', the other states are untrue. By Γ we denote a set of consequences. An act f is a function from S to Γ. If a decision maker chooses an act f, this results in consequence f(s) for (him or) her, where s is the true state of nature. The decision maker is unsure about which state of nature is the true one, hence is unsure about what the consequence of an act is. By $\geq$ we denote the preference relation of the decision maker on the acts; it is a binary relation on Γ^S.

For any α in Γ, $\overline{\alpha}$ is the constant act which assigns consequence α to every state. For any set $A \subset S$, consequence α, and act f, $f_{-A}\alpha$ is the act which assigns f(s) to every $s \in A^c$, and α to every $s \in A$.

Next we introduce measure-theoretic structure. A reader not interested in that may simply assume that Σ, introduced below, is 2^S. Then all functions from S to Γ are 'measurable', and all measure-theoretical conditions in the sequel will be trivially satisfied, so can be ignored. By Σ we denote an algebra on S, i.e. $\Sigma \subset 2^S$ contains $\emptyset$, S, and is closed w.r.t. finite union and intersection taking and w.r.t. complement taking. Elements A,B of Σ are called events. Further we assume that an algebra Δ on Γ is given. By Φ we denote the set of acts f which are (Δ-Σ-) measurable, i.e. for every set $E \in \Delta$, $\{s \in S: f(s) \in E\}$ is in Σ. Note that $f_{-A}\alpha$ is in Φ whenever A is an event and f is in Φ.

We say that the binary relation $\geq$ is a weak order on a subset E of Γ^S, if it is complete on E (i.e. for all f,g in E: $f \geq g$ or $g \geq f$) and transitive on E (i.e. for all f,g,h in E: if $f \geq g$ and $g \geq h$ then $f \geq h$). As usual we write $f > g$ if $f \geq g$ and not $g \geq f$, $f \approx g$ if $f \geq g$ and $g \geq f$, further $f \leq g$ if $g \geq f$, and $f < g$ if $g > f$. We call $\geq$ trivial on $E \subset \Gamma^S$ if $f \approx g$ for all f,g in E. We call $\geq$ an ordering if it is a weak order with $f \approx g$ only if $f = g$. As usual, for $\geq$ on Γ^S we denote also by $\geq$ the binary relation on Γ defined by $\alpha \geq \beta$ if $\overline{\alpha} \geq \overline{\beta}$.

Let again $E \subset \Gamma^S$. A function $\varphi : E \rightarrow \mathbb{R}$ represents $\geq$ on E if, for all f,g in E, $f \geq g$ iff $\varphi(f) \geq \varphi(g)$. A function is said to be continuously ordinal (w.r.t. a list of properties) if the set of all functions possessing these properties is the set of all continuous increasing transforms of the concerned function. [We use the term 'increasing' instead of 'strictly increasing'.] A function is said to be cardinal (w.r.t. a list of properties) if the set of all functions possessing these properties is the set of all positive affine transforms of the concerned function. [A positive affine transform is obtained by adding a real number, and multiplying with a positive real number.]

Let $\pi = (A_j)_{j=1}^m$ be an ordered partition, or partition for short, of S, consisting of events. Note that in our notation the ordering of the events in the partition is relevant. By Σ^π we denote the algebra of the subsets of S which are a union of events from π. We write $\sum_{j=1}^m \alpha_j 1_{A_j}$ for the act in Φ which assigns α_j to every s in A_j, j = 1,...,m. Obviously this act is simple, i.e. it has a finite range. The notation for the simple act above is just suggestive. It does not designate any addition or multiplication operation, these not even being defined for general Γ. For a partition $\pi = (A_1,...,A_m)$ of S consisting of events, Φ^π is the set of acts of the form $\sum_{j=1}^m \alpha_j 1_{A_j}$, and Φ_c^π is the set of acts of the form $\sum_{j=1}^m \alpha_j 1_{A_j}$ with furthermore $\alpha_{j-1} \geq \alpha_j$ for all $j \geq 2$. Note that indeed the ordering of events in π is relevant here. Let $\Phi^s := \{f \in \Phi\text{: f is simple}\}$. Then $f_{-A}\alpha \in \Phi^s$ whenever $f \in \Phi^s$ and $A \in \Sigma$.

We shall assume throughout the sequel that Γ is a connected[1] separable[2] topological space, e.g. Γ is $\mathbb{R}$, or $(\mathbb{R}^+)^m$. The following topological condition is of a finite-dimensional character, thus is weaker than most of the other continuity conditions, used in literature.

DEFINITION 4.1. The preference relation $\geq$ is simple-continuous, or s-continuous, if for any partition $(A_j)_{j=1}^m$, consisting of events, and any act $f = \sum_{j=1}^m \beta_j 1_{A_j}$, the sets $\{(\alpha_1,...,\alpha_m) \in \Gamma^m\text{: } \sum_{j=1}^m \alpha_j 1_{A_j} \geq f\}$ and $\{(\alpha_1,...,\alpha_m) \in \Gamma^m\text{: } \sum_{j=1}^m \alpha_j 1_{A_j} \leq f\}$ are closed w.r.t. the product topology on Γ^m.

One may formulate s-continuity as: the binary relation $\geq'$ on Γ^m, defined by:

(4.1) $(\alpha_1,...,\alpha_m) \geq' (\beta_1,...,\beta_m)$ iff $\sum_{j=1}^m \alpha_j 1_{A_j} \geq \sum_{j=1}^m \beta_j 1_{A_j}$

is continuous w.r.t. the product topology on Γ^m. This continuity is weaker than continuity of $\geq$ w.r.t. the product topology on Γ^S, and also weaker than the sup-metric continuity as used in Koopmans(1972).

For the extension of the Choquet-integral to nonsimple acts, as derived in section A2 of the Appendix, the following definitions will be used.

DEFINITION 4.2. The preference relation $\geq$ is constant-continuous on $\Phi' \subset \Gamma^S$ if $\{\alpha \in \Gamma\text{: } \overline{\alpha} \geq f\}$ and $\{\alpha \in \Gamma\text{: } \overline{\alpha} \leq f\}$ are closed for all $f \in \Phi'$.

As s-continuity, constant-continuity is implied by the product-topology-continuity, and by Koopmans' sup-metric-continuity.

DEFINITION 4.3. The preference relation $\geq$ satisfies pointwise monotonicity if:

$[f(s) \geq g(s)$ for all $s \in S] \Rightarrow [f \geq g]$.

We shall obtain the desired representation only for nonsimple acts which are 'strongly bounded':

DEFINITION 4.4. An act f is strongly bounded if there exist consequences α,β such that $\alpha \geq f(s) \geq \beta$ for all states s.

The set of all strongly bounded acts is denoted as Φ^b. Obviously, if a best and worst consequence exist, then every act is strongly bounded. If $\geq$ is a weak order then every simple act is strongly bounded (by means of its 'maximal' and 'minimal' consequences).

5. THE CHOQUET INTEGRAL AND COMONOTONICITY

In the sequel the following kind of set functions will be taken to measure the decision makers degrees of belief in the truth of events.

DEFINITION 5.1. A function $v : \Sigma \rightarrow \mathbb{R}$ is a capacity (on S) if:

(5.1) $v(\emptyset) = 0$

(5.2) $v(S) = 1$

(5.3) $A \subset B \Rightarrow v(A) \leq v(B)$ (monotonicity)

Note that the range of v must be contained in [0,1]. A (finitely-additive) probability measure P is a capacity which satisfies additivity, i.e. $P(A \cup B) = P(A) + P(B)$ for all disjoint events A,B. The following generalization of the notion of an integral, originating from Choquet(1953-54, formula 48.1) will be used in this paper:

DEFINITION 5.2. Let $v : A \rightarrow \mathbb{R}$ be a capacity. Let $\varphi : S \rightarrow \mathbb{R}$ be such that $\varphi^{-1}[\tau,\infty)$ and $\varphi^{-1}(-\infty,\tau]$ are events for all real τ. Then the Choquet integral of φ w.r.t. v, denoted as $\int_S \varphi dv$, or as $\int \varphi dv$, is

(5.4) $$\int_0^{\infty} v(\{s \in S: \varphi(s) \geq \tau\})d\tau + \int_{-\infty}^{0} [v(\{s \in S: \varphi(s) \geq \tau\})-1]d\tau$$

In this paper we shall deal only with bounded functions φ. For these (5.4) is always defined and finite. Integration by parts shows that for additive capacities the Choquet integral coincides with the usual integral (i.e. expectation). It is well-known (see for instance Wakker,1986, formulas VI.2.8 to VI.2.10) that the Choquet integral satisfies positive homogeneity ($\int\lambda\varphi dv = \lambda\int\varphi dv$ for all $\lambda \geq 0$), translation invariance ($\int(\lambda+\varphi)dv = \lambda + \int\varphi dv$ for all real λ), and monotonicity (if $\varphi(s) \geq \zeta(s)$ for all s, then $\int\varphi dv \geq \int\zeta dv$). For a nonadditive capacity v, the Choquet integral is not additive, i.e. the equality [$\int(\varphi+\zeta)dv = \int\varphi dv + \int\zeta dv$] does not hold for some φ,ζ. The latter equality is known to hold for all φ,ζ which satisfy [$\varphi(s) > \varphi(t)$ => not $\zeta(t) > \zeta(s)$]. So we define analogously:

DEFINITION 5.3. Acts f and g are comonotonic if for all $s,t \in S$, not simultaneously

(5.5) $f(s) > f(t)$ and $g(t) > g(s)$.

Hence comonotonicity will be important in our work. A set of acts is called comonotonic if each pair of acts from the set is comonotonic. Note that Φ_c^π is comonotonic; this explains the index c. If consequences are real numbers, indicating money, and f and g are comonotonic, then receipt of g in addition to an already received f will not induce any 'hedging against riskiness'. Thus the condition in (5.5) already occurred in Yaari(1969, bottom of page 324 and top of page 328) in a study of risk aversion.

A set C of acts may suggest that an event can be ignored, e.g. for being impossible:

DEFINITIONS 5.4. Let $C \subset \Gamma^S$. Event A is inessential (w.r.t. $\geq$) on C if, for all f,g in C which coincide outside of A, we have $f \approx g$. The opposite of 'inessential' is essential. For a partition π of S, consisting of events, event A is π-inessential (respectively π-essential) if $A \in \Sigma^\pi$ and A is inessential (respectively essential) on Φ_c^π.

6. THE MAIN THEOREM

The plan in this paper will be to work as much as possible with finite-dimensional aspects, thus with simple functions. Only with simple functions will be dealt in the (intuitively-)central conditions, those given in Definitions 6.1 through 6.5 below. And first for simple functions the desired representation will be derived, in the proof in section A1 of the Appendix. Only after that the extension of the desired representation to nonsimple acts will be obtained, with proof provided in section A2 in the Appendix.

The first idea is to derive strengths of preferences w.r.t. consequences from the preference relation over the acts. This idea was mentioned in Wakker(1984, section 3).

DEFINITIONS 6.1. We write, for consequences $\alpha,\beta,\gamma,\delta$,

(6.1) $\alpha\beta >^* \gamma\delta$

if there exist an event A, and simple acts $f_{-A}\alpha$, $g_{-A}\beta$, $f_{-A}\gamma$, $g_{-A}\delta$ such that both

(6.2) $f_{-A}\alpha \geq g_{-A}\beta$ and
$f_{-A}\gamma < g_{-A}\delta$.

We write $\geq^*$ instead of $>^*$ in (6.1) if in (6.2) we have $\leq$ instead of $<$ and if furthermore A is essential on Φ^s.

□

The idea of (6.2) is that replacement of α,β by γ,δ, contingent on event A, apparently has made event A a less favourable argument (or a more unfavourable argument) for the left act against the right act, in view of the reversion of preference. The idea is further that this should be interpreted to reveal that the strength of preference of α over β is larger than the strength of preference of γ over δ. If A is inessential on Φ^s, then (6.2) with $\leq$ instead of $<$ can be arranged for all $\alpha,\beta,\gamma,\delta$, by the choice f = g. This obviously should not give information about strengths of preferences. Hence the essentiality condition at the end of the above definition of $\geq^*$. Note that we have not assumed that $>^*$ is the asymmetric part of $\geq^*$.

The following lemma is a preparation for the second part of Definitions 6.3.

LEMMA 6.2. *The four simple acts* $f_{-A}\alpha$, $g_{-A}\beta$, $f_{-A}\gamma$, *and* $g_{-A}\delta$, *are comonotonic if and only if there exists a partition* π *of* S, *consisting of events and containing* A, *such that* Φ^{π}_{c} *contains all four acts.*

PROOF. See Wakker(1986, Lemma VI.3.5.iv).

□

In this paper we shall not deal with additive contexts, in our work comonotonicity will be central. Hence:

DEFINITIONS 6.3. We write, for consequences $\alpha,\beta,\gamma,\delta$,

(6.3) $\alpha\beta >_c^* \gamma\delta$

if there exist an event A, and comonotonic simple acts $f_{-A}\alpha$, $g_{-A}\beta$, $f_{-A}\gamma$, and $g_{-A}\delta$, for which the two preferences in (6.2) hold.

We write $\geq_c^*$ instead of $>_c^*$ in (6.3) if in the involved (6.2) we have $\leq$ instead of $<$ and if furthermore A is π-essential for a π such that the four involved acts are in Φ_c^π (compare Lemma 6.2 above).

□

In the proof of Lemma 6.10 we shall show (by deriving (6.9) from the first part of (6.7)) that under the assumption of Choquet integral representation:

$\alpha\beta \geq_c^* \gamma\delta \Rightarrow U(\alpha)-U(\beta) \geq U(\gamma)-U(\delta)$.

Analogously $\alpha\beta >_c^* \gamma\delta \Rightarrow U(\alpha)-U(\beta) > U(\gamma)-U(\delta)$.

The next definition is mainly useful in additive contexts.

DEFINITION 6.4. The preference relation $\geq$ exhibits contradictory strengths of preferences (between consequences) if there exist consequences $\alpha,\beta,\gamma,\delta$, such that

[both $\alpha\beta \geq^* \gamma\delta$ and $\gamma\delta >^* \alpha\beta$] .

In the present nonadditive context we shall mainly use the following definition. Lemma 6.10 below has been added to illustrate its meaning.

DEFINITION 6.5. The preference relation $\geq$ exhibits comonotonic-contradictory strengths of preferences (between consequences) if there exist consequences $\alpha,\beta,\gamma,\delta$ such that

[both $\alpha\beta \geq_c^* \gamma\delta$ and $\gamma\delta >_c^* \alpha\beta$] .

In Wakker(1986, Theorem V.4.4) the following characterization of subjective expected utility maximization was given, using a 'cardinal coordinate independence' condition. The terminology in Definition 6.4 and statement (ii) below, by means of derived strengths of preferences, is straightforwardly seen to be equivalent, and is hoped to be more appealing.

THEOREM 6.6. *The following two statements are equivalent for the preference relation* $\geq$ *on* Γ^S*:*

(i) *There exist a finitely additive probability measure* P *on* S, *and a continuous 'utility function'* $U : \Gamma \to \mathbb{R}$, *such that* $f \mapsto \int(U \circ f)dP$ *represents* $\geq$ *on* Φ^b.

(ii) *The preference relation* $\geq$ *is a constant- and s-continuous pointwise monotone weak order on* Φ^b, *and does not exhibit contradictory strengths of preferences on consequences.*

□

Another characterization of the same representation, in terms of a 'mean groupoid operation' on the set of consequences, derived from the preference relation, is given in Grodal(1978). Next we will formulate the main new result of the present paper, the adaptation of the above theorem to the 'nonadditive' case. Before, we summarize the assumptions made so far:

ASSUMPTION 6.7 (Structural Assumption). S is a nonempty set, endowed with an algebra Σ. Γ is a connected[1] separable[2] topological space, endowed with an algebra Δ containing all open subsets of Γ.

Let us recall that capacities have been introduced in Definition 5.1, that Φ^b has been introduced below Definition 4.4, that simple-continuity, constant-continuity, and pointwise monotonicity have been introduced in Definitions 4.1, 4.2, and 4.3, and that the comonotonic-contradictory-condition occurring in (ii) below has been introduced through Definitions 6.3 and 6.5.

THEOREM 6.8 (Main Theorem). *Under the Structural Assumption 6.7, the following two statements are equivalent for the preference relation* $\geq$ *on* Γ^S*:*

(i) *There exist a capacity* v *on* S *and a continuous 'utility function'* $U : \Gamma \to \mathbb{R}$, *such that* $f \mapsto \int(U \circ f)dv$ *represents* $\geq$ *on* Φ^b.

(ii) *The preference relation* $\geq$ *is a constant- and simple-continuous pointwise monotone weak order on* Φ^b, *and does not exhibit comonotonic-contradictory strengths of preferences on consequences.*

□

The integral in (i) above is well-defined since for every act f in Φ^b, $U \circ f$ is bounded. For the case where S is finite and Σ equals 2^S a, complicated, proof of this theorem, and of the uniqueness results listed below, has been given in Wakker(1986, Chapter VI). The idea of that proof is to first restrict attention to comonotonic subsets of the set of acts. Within such subsets, the comonotonicity restrictions are trivially satisfied and can be ignored, and results for additive contexts from literature (e.g. Debreu,1960, Theorem 3, or Krantz et al.,1971, Theorem 6.14) can be used to obtain 'local' expected utility representations. Next these local expected utility representations have to be fit together to give a 'global' representation, with now probabilities depending on the favourableness ordering of the states. The latter orders the states according to which state receives the better consequence from the act under consideration. The proof of the extension to general (S,Σ), as given in Theorem 6.8, is presented in the appendix of this paper. In this proof the assumption of topological separability is necessary only for the (unimportant) case described in (6.5) below, as can be inferred from the reference Wakker(1986) used in the proof. For the sake of clarity of presentation we have assumed topological separability for all cases. Proof of the following theorem will also be provided in the Appendix.

THEOREM 6.9 (Uniqueness results for Theorem 6.8). *Let, under the conditions of Theorem 6.8, Statement (i) there hold. Then we have the following uniqueness results, where* π *denotes a partition of* S *consisting of events:*

(6.4) *If some* π *contains two or more* π*-essential events, then* U *is cardinal, and* v *is uniquely determined.*

(6.5) *If* $\geq$ *is not trivial, and no* π *contains more than one* π*-essential event, then* U *is continuously ordinal, and* v *is uniquely determined;* v *assigns* 1 *to every* (A,A^c)*-essential event* A, *and* 0 *to every* (A,A^c)*-inessential event* A.

(6.6) *If* $\geq$ *is trivial, then* U *is any constant function, and* v *is arbitrary.*

□

The following lemma gives part of the implication (i) => (ii) in Theorem 6.8, and may be clarifying.

LEMMA 6.10. *Let there exist a capacity* v *on* S, *and a 'utility function'* $U : \Gamma \to \mathbb{R}$, *such that* $f \mapsto \int(U \circ f)dv$ *represents* $\geq$ *on* Φ^b. *Then* $\geq$ *does not exhibit comonotonic-contradictory strengths of preferences.*

PROOF. Obviously $\geq$ is a weak order. Now suppose that $\geq$ exhibits comonotonic-contradictory strengths of preferences. Contradiction will follow. There exist $\alpha,\beta,\gamma,\delta \in \Gamma$, such that

(6.7) $[\text{both } \alpha\beta \geq_c^* \gamma\delta \text{ and } \gamma\delta >_c^* \alpha\beta]$.

First we concentrate on $\alpha\beta \geq_c^* \gamma\delta$. By Lemma 6.2 there exist a partition π of S consisting of events, an event $A \in \Sigma^\pi$, and acts $f_{-A}\alpha$, $g_{-A}\beta$, $f_{-A}\gamma$, and $g_{-A}\delta \in \Phi_c^\pi$ with

$f_{-A}\alpha \geq g_{-A}\beta$ and $f_{-A}\gamma \leq g_{-A}\delta$,

and, to have π-essentiality of A as required in Definitions 6.3, there also exist $\hat{f}_{-A}\sigma$, $\hat{f}_{-A}\tau \in \Phi_c^\pi$ with $\hat{f}_{-A}\sigma > \hat{f}_{-A}\tau$. Apparently

$$\int U(f_{-A}\alpha)dv \geq \int U(g_{-A}\beta)dv \text{ and } \int U(f_{-A}\gamma)dv \leq \int U(g_{-A}\delta)dv, \text{ so}$$

$$(6.8) \qquad \int U(f_{-A}\alpha)dv - \int U(f_{-A}\gamma)dv \geq \int U(g_{-A}\beta)dv - \int U(g_{-A}\delta)dv .$$

Because of comonotonicity of the involved acts, in the left-hand side of inequality (6.8) the contributions to the Choquet integral of the states outside of A can be seen to cancel, thus this left-hand side of (6.8) equals $P^\pi(A)[U(\alpha)-U(\gamma)]$, where

$$P^\pi(A) := \quad v(\{s \in S: s \text{ is an element of an event placed in } \pi \text{ before } A\} \cup A) - v(\{s \in S: s \text{ is an element of an event placed in } \pi \text{ before } A\}) .$$

Analogously the difference in the right-hand side of inequality (6.8) can be seen to equal $P^\pi(A)[U(\beta)-U(\delta)]$. And since $\hat{f}_{-A}\sigma > \hat{f}_{-A}\tau$ implies $P^\pi(A)[U(\sigma)-U(\tau)] > 0$, we conclude that $P^\pi(A) > 0$. Thus (6.8) must imply $[U(\alpha)-U(\gamma)] \geq [U(\beta)-U(\delta)]$, or

$$(6.9) \qquad [U(\alpha)-U(\beta)] \geq [U(\gamma)-U(\delta)] .$$

This is what we wanted to derive from $\alpha\beta \geq_c^* \gamma\delta$.

Analogously from $\gamma\delta >_c^* \alpha\beta$ it can be derived that $[U(\gamma)-U(\delta)] > [U(\alpha)-U(\beta)]$, in <u>contradiction</u> with (6.9).

7. ILLUSTRATIONS

7.1. A finite State Space.

In this subsection we give an interpretation of the Choquet-integral-approach as a deviation from the usual expected utility approach by allowing for act-dependence of the probabilities. For the sake of representation we shall assume that the state space is a finite set $\{s_1,...s_n\}$. Lemma VI.3.5(ii) in Wakker(1986) gives the main tool to show analogous things for infinite state spaces.

Suppose the decision maker has to choose an act from a set of available acts. The decision maker will exhibit behaviour as represented in (i) of Theorem 6.8 if (he or) she proceeds as follows:

Stage 1. The decision maker determines her utility function U.

Stage 2. For every ordering $\geq^{\pi}$ of the state space the decision maker determines 'probabilities' $P^{\pi}(s_1),...,P^{\pi}(s_n)$.

Stage 3. Every single act f is evaluated as follows:

Step 3.1. The decision maker chooses a favourableness ordering $\geq^{\pi}$ compatible with f, i.e. one of the (possibly several) orderings $\geq^{\pi}$ of S such that for no states s_i, s_j simultaneously $s_i >^{\pi} s_j$ and $f(s_i) < f(s_j)$.

Step 3.2. Now the act is valued by its '(generalized) expected utility':

(7.1) $$\sum_{j=1}^{n} P^{\pi}(s_j)U(f(s_j)) \ .$$

Stage 4. Finally the decision maker chooses the available act with highest value in (7.1).

In Step 3.1 there was some arbitrariness in the choice of the favourableness ordering $\geq^{\pi}$. Hence we must guarantee that this arbitrariness is immaterial. For instance the valuation of an act $\alpha 1_A + \beta 1_{A^c}$ (say $\alpha > \beta$), writable as $[\sum_{s \in A} P^{\pi}(s)] \times [U(\alpha)-U(\beta)] + U(\beta)$, should be unambiguous. This leads to the following consistency requirement:

(7.2) CONSISTENCY: $\sum_{s \in A} P^{\pi}(s) = \sum_{s \in A} P^{\rho}(s)$
whenever, both w.r.t. $\geq^{\pi}$ and w.r.t. $\geq^{\rho}$,
A contains the $\|A\|$ most favourable states.

This consistency is necessary and sufficient for the possibility to define a capacity

(7.3) $v : A \mapsto \sum_{s \in A} P^{\pi}(s)$,

where the ordering $\geq^{\pi}$ is chosen such that A contains the $\|A\|$ most $\geq^{\pi}$-favourable states.

Under (7.2) and (7.3) indeed for any act f, and favourableness ordering $\geq^{\pi}$ compatible with f,

$$\int (U \circ f) dv = \sum_{j=1}^{n} P^{\pi}(s_j)U(f(s_j)),$$

as one derives from the definition of the Choquet integral (see Wakker, 1986, formula VI.2.7). Every $P^{\pi}(s)$ can now be recovered from v as the 'marginal capacity contribution' of s to the set of all states, more favourable than s:

(7.4) $P^{\pi}(s) = v[\{s': s' >^{\pi} s\} \cup \{s\}] - v[\{s': s' >^{\pi} s\}]$.

Formula (7.4), and formula (7.3) in case consistency (7.2) applies, give the interconnection between the capacity v, and the 'act-dependent probabilities' $P^{\pi}(s)$.

A topic for future research is the question what are good procedures to elicit, or help a decision maker determine, the utility function as in Stage 1, and the act-dependent 'probabilities' as in Stage 2.

The capacities studied most in literature are the superadditive capacities, i.e. capacities satisfying $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$ for all events A,B. Schmeidler(1984a, the Proposition; or Schmeidler,1984c, insertion at page 10), in the context where lotteries on consequences are available, characterizes this property. It is interpreted as 'uncertainty aversion'. In the present context, without lotteries available, Wakker(1986, Theorem VI.11.4) characterizes the property by means of a 'pessimism'-property of the preference relation, which reflects the idea that a state is given less weight as it becomes more favourable. It is well-known (see Schmeidler,1972) that superadditive capacities v assign to each event

$A \mapsto: \min\{P(A): P \text{ dominates } A\}$,

and that this equals

$\min\{P^{\pi}(A): P^{\pi} \text{ is obtained as above}\}$.

Thus it can be seen that for superadditive capacities the decision making represented by the Choquet integral as in (7.1) may also result in the following way:

A 'Bayesian-like' decision maker has decided that (his or) her prior probability shall be an element from a set of P^{π}'s as above, but she has not decided which element. Then, if the decision maker is a pessimistic one (there seem to be many such), and if she is willing to leave the Bayesian path, she may come to value any act f by an integral which assigns to every favourable event $A = \{s \in S: f(s) \geq \tau\}$ the most pessimistic, so the smallest possible, 'probability' $\min \{P^{\pi}(A)\}$, thus assigns to every unfavourable event $B = \{s \in S: f(s) < \tau\}$ the largest possible 'probability' $\max \{P^{\pi}(B)\}$. This indeed can be seen to assign to every act its Choquet integral. Further it is known that for superadditive capacities the Choquet integral, obtained this way, equals the minimum of the expected utilities $\int_S U(f)dP^{\pi}$, with the P^{π}'s as above; see for finite state spaces Huber(1981, Propositions 10.2.5, and 10.2.1 applied to $v^* : A \mapsto 1-v(A^c)$, or for arbitrary state spaces Schmeidler(1984b, Proposition 3); see also Anger(1977). The capacities introduced in Examples 7.1, 7.4 and 7.5 below can be seen to be super-additive.

7.2. Examples.

In the examples below always $\Gamma = \mathbb{R}^+$, indicating money; U is always assumed to be identity; further $v(\emptyset) = 0$, $v(S) = 1$.

EXAMPLE 7.1 (Maximin behaviour). Let $v(A) = 0$ whenever $A \neq S$. Then the Choquet integral assigns to every act the infimum utility of the consequences of that act. The decision maker chooses the act with highest infimum of payment. This behaviour is exhibited by the pessimistic decision maker who chooses her subjective probabilities dependent upon the involved act, and behaves as if being sure that the state (or the 'ultrafilter' of states) will be true which is the most unfavourable given the involved act.

EXAMPLE 7.2 (Maximax behaviour). Let $v(A) = 1$ whenever $A \neq \emptyset$. Now the Choquet integral assigns to every act the supremum utility of the consequence of that act and the decision maker chooses the act with highest supremum of payment. The decision maker is optimistic and behaves as if being sure that the state (or the ultrafilter of states). will be true which is the most favourable given the involved act.

EXAMPLE 7.3 (The α-Hurwicz criterion). Let $0 \leq \alpha \leq 1$, and let $v(A) = \alpha$ for all $\emptyset \neq A \neq S$. Then the Choquet integral of every act is $\alpha \times \sup(f(s)) + (1-\alpha) \times \inf(f(s))$. This approach is a mixture of the two approaches in the above examples.

EXAMPLE 7.4 (The Ellsberg paradox). First an arbitrary ball will be drawn from an urn, containing one blue and one green ball. Next a ball is drawn from a second urn containing either two blue balls, or two green balls. The decision maker does not know if the balls in the second urn are blue, or green, and has no information to consider one more likely than the other. Let $S = \{s_1,...,s_4\}$, with s_1 = bb the state of nature where two blue balls will be drawn, analogously s_2 = bg, s_3 = gb, s_4 = gg. We shall consider four acts f^1, g^1, f^2, g^2, given by

INSERT TABLE 7.1 ABOUT HERE

Act f^1 will yield \$1 if the ball drawn first is blue, \$0 otherwise; etc. So the consequences of the f-acts depend upon the first drawing, the consequences of the g-acts upon the second drawing.

The most commonly exhibited behaviour in this example is to prefer f^1 to g^1, and f^2 to g^2. Expected utility cannot explain this behaviour because the first preference would imply P(bb,bg) > P(bb,gb), the second P(gb,gg) > P(bg,gg). These two inequalities cannot hold simultaneously for an additive probability P. The behaviour can be explained by statement (i) in the Main Theorem 6.8: e.g. the decision maker may let a new state (e.g. bb) contribute in capacity, to an event, 2/6 if the other state referring to the same first ball (e.g. bg, if the new state is bb) as the new state was already present, and 1/6 otherwise. The capacity v determined this way (see (7.3)) is given in Table 7.2.

INSERT TABLE 7.2 ABOUT HERE

The acts f^1 and f^2 have Choquet integral 3/6, the Choquet integral of g^1 and g^2 is 2/6. The capacity v can be seen to be superadditive.

An alternative explanation of the behaviour usually exhibited in the Ellsberg paradox, and of uncertainty about uncertainty, is by means of 'second-order probabilities', see Skyrms(1980), or Goldsmith&Sahlin(1981).

EXAMPLE 7.5 (The Allais paradox). The Allais paradox usually involves given 'objective' probabilities. For our set-up it will be rephrased. A point (= state) will be chosen arbitrarily from the unit interval (0,1] =: S. We shall consider the four acts indicated in Table 7.3.

INSERT TABLE 7.3 ABOUT HERE

Usually people prefer f^1 to g^1, and g^2 to f^2. Again expected utility cannot explain this since the difference in expected utility of f^1 and g^1, after cancellation of the contributions of the interval (0.11 , 1], consists of exactly the same terms as the difference in expected utility of f^2 and g^2 after cancellation of the contributions of the interval (0.11 , 1]. Again the behaviour can be explained by (i) in the Main Theorem 6.8: Let λ be Lebesgue measure. Let $v(A) = \lambda(A)/2$ whenever $A \neq S$. Then the Choquet integral of f^1 is 500,000, that of g^1 is 347,500 , that of f^2 is 27,500, and the Choquet integral of g^2 is 125,000.

From the above preferences contradictory strengths of preference can be derived. To this end let $\alpha = 500{,}000$, $\beta = 0$, and let event A = (o.11 , 1] exhibit all strengths of preferences. The above two preferences, according to the Definitions 6.1, give the implausible $\alpha\alpha >^* \beta\beta$, whereas $f^2 \geq f^2$ together with $f^1 \leq f^1$ gives $\beta\beta \geq^* \alpha\alpha$. By Theorem 6.8 no comonotonic-contradictory strengths of preferences can result from the above preferences; indeed g^1 and f^2 are not comonotonic.

EXAMPLE 7.6 (The rank-order approach). One way to obtain a capacity is to take an additive probability measure P on Σ, a nondecreasing function $\varphi : [0,1] \rightarrow [0,1]$ with $\varphi(0) = 0$, $\varphi(1) = 1$, and to take as capacity the 'distorted probability' $v := \varphi \circ P$. We shall call this approach the rank-order approach. (Yaari indicated this way to consider the rank-order approach as a special case of the Choquet-integral-approach, see for instance Gilboa,1985b, footnote at page 3, or Yaari,1987, page 114. It is also apparent from Chateauneuf,1986.) Obviously the probability measure P may simply be a mathematical device, without any interpretation associated with it. We shall in the sequel consider two other cases, with two extreme interpretations associated with P.

The first case is the case of 'risk', where the probability measure P is an 'objective' probability measure, completely known and given in advance. For monetary consequences from the interval [0,1], linear utility, a coninuous transformation φ, and, finally, a probability space (S,Σ,P) which is 'rich enough' to generate all probability distributions, an elegant characterization of the rank-order approach has been provided in Yaari(1987). Let us now see what conditions are required on top of those in statement (ii) in Theorem 6.8 to obtain Yaari's approach. Yaari's axiom A1 of 'neutrality' states that acts which induce the same probability distributions over outcomes are equivalent. This can be seen to be necessary and sufficient for the condition that any pair of events A,B with the same probability $P(A) = P(B)$ shall also have the same capacity $v(A) = v(B)$ (for consequences α,β with $\alpha > \beta$, the acts $\alpha 1_A + \beta 1_{A^c}$ and $\alpha 1_B + \beta 1_{B^c}$ must be equivalent). Thus it is a necessary and sufficient condition for the capacity v to be a transform $\varphi \circ P$ of P. Yaari's axiom A4 of monotonicity w.r.t. 'first-order-stochastic dominance' (i.e. $P[f(s) \geq \gamma] \geq P[g(s) \geq \gamma]$ for all consequences γ) is necessary and sufficient for nondecreasingness of φ (again by comparing acts $\alpha 1_A + \beta 1_{A^c}$ and $\alpha 1_B + \beta 1_{B^c}$). If $\geq$ is nontrivial, then $\varphi(0) = 0$ and $\varphi(1) = 1$ can be arranged by applying a proper positive affine transformation on φ. So we have:

OBSERVATION 7.6.1. *The rank-order approach is equivalent to the Choquet-integral-approach together with the neutrality axiom and the condition of monotonicity w.r.t. first-order-stochastic dominance.*

Since both the neutrality axiom and the condition of monotonicity with respect to first-order stochastic dominance are uncontroversial in decision making under risk, in a loose-hand way it may be said that the rank-order approach is the risk-analogue of the Choquet-integral-approach.

Yaari also characterizes and uses continuity of the transformation φ, by means of a L_1-continuity condition for distribution functions. In our alternative set-up, without a richness-of-space assumption and without boundedness-of-outcomes, continuity of φ will require substantial adaptation, or a different characterizing condition; we do not take up this point. For the special case where consequences are monetary, linearity of utility can be defined, and can be characterized by the condition $[\alpha\beta \approx_c^* \gamma\delta$ whenever $\alpha-\beta = \gamma-\delta]$. Concavity of the utility function can be characterized by the condition $[\alpha\beta \geq_c^* \gamma\delta$ whenever $\alpha-\beta = \gamma-\delta$ and $\alpha \leq \gamma]$, a condition which reflects the idea of decreasing marginal utility. Yaari's work under linear utility makes very clear the main purpose of the rank-order approach, to explain risk aversion as a phenomenon related to probability, rather than to utility as is the case under expected utility. Also linearity of utility makes possible the presentation of the rank-order approach as a 'dual' of expected utility.

The second case considered in this example deals with subjective probability measures P which are not known and given in advance. In that case, when obtaining the representation of statement (ii) of Theorem 6.8, one may wonder whether *there exists* a probability measure P such that $v = \varphi \circ P$, with φ nondecreasing. This is the case if and only if there exists a probability measure P such that, for $\geq$ on Σ defined by $A \geq B$ iff $v(A) \geq v(B)$, we have $P(A) \geq P(B) \Rightarrow A \geq B$. Questions of this kind are studied in 'comparative probability theory', see Wakker(1981, supplying Savage,1954), Gilboa(1985b, explicitely indicating the significance for the rank-order approach), or Fishburn(1986, giving a survey). The five-point example in Kraft,Pratt&Seidenberg(1959) can be used to show that there exist, even for a finite state space, capacities which cannot be obtained as nondecreasing transforms of an additive probability, so which will never be obtained through the rank-order appproach.

EXAMPLE 7.7 (Belief functions). Let S be finite. Shafer(1976), following earlier work of Dempster(1967), considers belief functions v, i.e. functions $v : \Sigma \to \mathbb{R}$, with $v(\emptyset) = 0$, $v(S) = 1$ which furthermore can be characterized by nonnegativity of the (uniquely determined) function $m : \Sigma \to \mathbb{R}$ which satisfies $v : A \mapsto \sum_{B \subset A} m(B)$. Belief functions satisfy (5.3), thus are capacities. Wallsten&Forsyth(1985) refer to Krantz(1982) for consideration of the existence of an axiomatic base for Shafer's belief functions.

EXAMPLE 7.8 (Fuzzy sets). We start with some arbitrary finite set X of 'points', and a finite collection Σ, the elements of which are called fuzzy sets. For every point x and fuzzy set A there is a real number $m(x,A) \in [0,1]$, indicating the 'grade of membership' of x in A. Zadeh(1965) modeled this by means of 'membership functions' f_A, such that $f_A(x) = m(x,A)$ for all x,A. For every pair of fuzzy sets A,B the 'union' $A \cup B$ has a membership function defined by $f_{A \cup B} : x \mapsto \max\{f_A(x), f_B(x)\}$, and the 'intersection' $A \cap B$ has a membership function defined by $f_{A \cap B} : x \mapsto \min\{f_A(x), f_B(x)\}$. We write $A \supset B$ if $f_A \geq f_B$, and $A \subset B$ if $f_A \leq f_B$. We may assume that Σ is closed under union and intersection taking, and that Σ contains the constant-zero-function denoted as $\emptyset$, and the constant-1-function, denoted as f_S (if not, then we add all required fuzzy sets). By Sikorski(1969, section 8) there exists an algebra Σ' of subsets of a set S' that is isomorphic to Σ as an algebra (this also applies to infinite Σ). We shall identify Σ with Σ'.

The situation described above may also be modeled in a 'dual way', by means of a family $\{v_x\}_{x \in X}$ of capacities on Σ, instead of a family $\{f_A\}_{A \in \Sigma}$ of membership functions on X. To this end, for every $x \in X$ we define the function v_x on Σ by $v_x : A \mapsto m(x,A)$ $(= f_A(x))$. This function is easily seen to be a capacity, assigning maximums to unions and minimums to intersections. Thus $v(A) = \max\{v(\{s\}): s \in S\}$ for all $A \in \Sigma$. Reversedly, every family of capacities assigning maximums to unions and minimums to intersections can be transferred into a family of fuzzy sets. One interpretation for the set X could be that this is a group of decision makers.

A topic for future study is the question what the intersection of the Choquet-integral-approach is with other, independent, approaches. The intersection with the complete-ignorance-models of Arrow&Hurwicz(1972) and Cohen&Jaffray(1980) is given by Example 7.3, the intersection with the security-factor-models of Jaffray(1986) and Gilboa(1986) is the representation $f \mapsto \lambda EU(f) + (1-\lambda)\inf(U(f))$, where EU is (additive) expected utility (see Gilboa,1986, section 1.5). The intersection with the skew-symmetric theory of Fishburn(1984), by transitivity, is expected utility.

On many places in literature (nonadditive) capacities play a role, and are required to satisfy conditions suited for the particular situation. Wallsten&Forsyth(1985), for the context of risk assessment, give many references to studies casting doubts on additivity of (probability) measures to assess human probability judgments. Huber(1981, section 10.2) and Huber&Strassen(1973) use capacities in the study of robustness in statistics. In Sugeno&Murofushi(1987) the term 'fuzzy measure' is used for capacities which satisfy certain continuity conditions w.r.t. decreasing and increasing sequences of events; a way to integrate w.r.t. these is given, very different from the Choquet integral; it does not extend the (additive) Lebesgue integral. Capacities are called 'characteristic functions', or 'games', in cooperative game theory with side-payments, see Shapley(1972) and Driessen(1985). From pure mathematics we mention Adams(1981) for L^p-Potential theory, Dellacherie(1970), and in particular Anger(1977); Theorem 3 in the latter, for the case where the domain of the capacity is finite, can be considered a predecessor of the characterization given in section 2 in Schmeidler(1984c) of functionals which are Choquet integrals. Papamarcou&Fine(1986) consider 'undominated lower probabilities' (again a special class of capacities) and give references to physical applications.

8. CONCLUSION.

Schmeidler's approach to decision making under uncertainty by means of ('nonadditive probabilities' =) capacities in the Choquet integral is one of the new approaches to decision making under uncertainty, aiming to avoid the exclusion of several interesting ways of decision making under uncertainty which are excluded by expected utility. Schmeidler's approach can be considered to allow for act-dependence of probabilities through the 'favourableness'-ordering of the states as induced by an act under consideration. In contexts where Schmeidler's approach, in its full generality, is considered to include too many *un*interesting ways of behaviour to give interesting results, still his characterization theorems (e.g. the Theorem in section 3 in Schmeidler,1984a), or our Main Theorem 6.8, may be useful. These theorems then may still serve as a convenient starting point for the characterization of more specified approaches, such as those mentioned in section 7. A topic for future research is the question which strategic properties for preference relations must be added in (i) of Theorem 6.8, to give characterizations of these more specified approaches.

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APPENDIX. DERIVATION OF THE MAIN THEOREM 6.8.

In this appendix the Main Theorem 6.8 will be derived in full generality; also the uniqueness results in Theorem 6.9 will be established. We shall take as point of departure the following observation, proved in Wakker(1986, Theorem VI.5.1; 1987, Theorem 5.1).

OBSERVATION A1. *The Main Theorem 6.8 holds if* S *is finite and* $\Sigma = 2^S$.

□

Throughout this appendix we shall, without further mention, assume that any partition π is a partition of S, consisting of events. In the first paragraph A1 we treat simple acts.

A1. The main theorem for simple acts

The next lemma treats the 'degenerate' case where the decision maker does not exhibit uncertainty, but behaves as if knowing for sure, for every act, what consequence (or ultrafilter of consequences) will result from that act.

LEMMA A1.2. *Let no partition* π *contain more than one* π*-essential event. Under the Structural Assumption 6.7 the following two statements are equivalent:*

(i) *There exists a capacity* v *on* S, *and a continuous* $U : \Gamma \to \mathbb{R}$, *such that* $f \mapsto \int(U \circ f)dv$ *represents* $\geq$ *on* Φ^s.

(ii) $\geq$ *is an s-continuous weak order on* Φ^s.

Furthermore, if (i) holds, then $\geq$ *does not exhibit comonotonic-contradictory strengths of preferences.*

Uniqueness results are as in (6.5) and (6.6).

PROOF. First suppose (i) holds. Let $\pi = \{A_1,...,A_m\}$ be a partition. The map $(\alpha_1,...,\alpha_m) \mapsto (U(\alpha_1),...,U(\alpha_m))$ from Γ^m (endowed with the product topology) to $\mathbb{R}^m$ is continuous; so is, by Wakker(1986, Proposition VI.2.4), the function $(\tau_1,...,\tau_m) \mapsto \int_S \varphi dv$ ($\varphi := \sum_{j=1}^m \tau_j 1_{A_j}$) from $(U(\Gamma))^m$ to $\mathbb{R}$. The, consequently continuous, composition of these two represents $\geq'$ as defined in (4.1). So this $\geq'$ is continuous, and s-continuity of $\geq$ follows. The remainder of (ii) is obvious. The "Furthermore"-statement is by Lemma 6.10.

Next we suppose that (ii) holds, and derive (i) and the uniqueness results. If $\geq$ is trivial on Φ^s, then obviously no event A and partition π exist with A π-essential. Also the reversed implication holds. (For any two simple measurable acts take a partition π so fine that both acts are in Φ^π. Then transform one simple act, step by step, on one element of the partition after the other, into the other simple act; always staying within the same $\approx$-equivalence class.) For this case everything is straightforward.

So from now on we assume that $\geq$ is nontrivial on Φ^s. By s-continuity the binary relation $\geq$ on Γ as defined in section 4 is a continuous weak order. Hence by Debreu(1954,1964) we can find a continuous, continuously ordinal, function U on Γ which represents this $\geq$ on Γ. Note that the function U to be found in (i) must indeed represent this $\geq$. So all we can hope is that the presently obtained U will be as required in (i), further that *every* possible U representing this $\geq$ on Γ will be as required in (i), the latter to obtain continuous ordinality of U.

We define, for all events A, v(A) = 1 if A is (A,A^c)-essential, v(A) = 0 otherwise. First we show that v is monotone. So let, for events A and B, $A \subset B$ and v(A) = 1. To derive is that v(B) = 1, i.e. that B is ρ-essential for $\rho := (B,B^c)$. Since A is π-essential for $\pi := (A,A^c)$, there exist consequences α,β,θ such that not $\alpha 1_A + \theta 1_{A^c} \approx \beta 1_A + \theta 1_{A^c}$, and (to be in Φ^{π}_{c}) such that $\alpha \geq \theta$, $\beta \geq \theta$. Say $\alpha 1_A + \theta 1_{A^c} > \beta 1_A + \theta 1_{A^c}$. Let $A_1 := A$, $A_2 := B\backslash A$, $A_3 := B^c$, and let $\pi' := (A_1,A_2,A_3)$. Then obviously A is π'-essential. By the first sentence of the Lemma, A_2 is π'-inessential. So $\alpha 1_{A_1} + \alpha 1_{A_2} + \theta 1_{A_3} \approx \alpha 1_A + \theta 1_{A^c} > \beta 1_A + \theta 1_{A^c} \approx \beta 1_{A_1} + \beta 1_{A_2} + \theta 1_{A_3}$. This shows that B is ρ-essential. So v(B) = 1, and v is monotone.

Obviously $v(\emptyset) = 0$. To show that v(S) = 1, we first recall from the exposition for the case where $\geq$ was assumed trivial on Φ^s, that the presently-supposed nontriviality of $\geq$ on Φ^s implies existence of an event A, and a partition π, such that A is π-essential. So $f_{-A}\alpha > f_{-A}\beta$ for some $f_{-A}\alpha, f_{-A}\beta$ in Φ^{π}_{c}. By the first sentence of the Lemma, A^c is π-inessential, and $\overline{\alpha}_j \approx f_{-A}\alpha > f_{-A}\beta \approx \overline{\beta}$ follows. This implies that v(S) = 1.

Also $\alpha > \beta$ implies that for each partition $\pi = (A_1,...,A_m)$ there is at least (so exactly) one event A_j in π which is π-essential. Then any act $f = \sum \alpha_i 1_{A_i}$ in Φ^{π}_{c} is equivalent to $\overline{\alpha}_j$, thus the function assigning $U(A_j)$ to this f represents $\geq$ on Φ^s. Note that $U(\alpha_j)$ is equal to $\int_S (U \circ f)dv$. Also note that v must be as we defined it, to obtain the results of the last sentences, and thus to have (i) satisfied. So indeed v is uniquely determined.

□

LEMMA A1.3. *Let the Structural Assumption 6.7 hold. Let $\geq$ be an s- continuous weak order on Φ^s. Let $\geq$ exhibit no comonotonic-contradictory strengths of preferences. Let $\pi^1 = (A_1,...,A_s)$ be a partition containing at least two π^1-essential events. Then there exists a capacity v^1 on Σ^{π^1}, and a continuous $U^1 : \Gamma \to \mathbb{R}$, such that the function $f \mapsto \int (U^1 \circ f)dv^1$ represents $\geq$ on Φ^{π^1}.*

This v^1 is uniquely determined, and U^1 is cardinal.

PROOF. Define $\geq'$ on Φ^s as in (4.1), and apply Observation A1.

□

LEMMA A1.4. *Let, under the assumptions and notations of Lemma A1.3,* $\pi^2 = (B_1,...,B_t)$ *be another partition containing at least two* π^2*-essential events. Let application of Lemma A1.3 to* π^2 *give* v^2 *and* U^2*. Then* $U^2 = \varphi \circ U^1$ *for a positive affine* φ*, and if some event* A *is both in* $\Sigma\pi^1$ *and* $\Sigma\pi^2$*, then* $v^1(A) = v^2(A)$.

PROOF. Define

$$\pi^3 := (A_1 \cap B_1\ , \ ...\ , A_1 \cap B_t\ , A_2 \cap B_1\ , ...A_2 \cap B_t\ , \ ...\ , A_s \cap B_1\ , \ ...\ , A_s \cap B_t).$$

Let $A \in \Sigma\pi^1$ be π^1-essential, i.e. $f > g$ for some $f,g \in \Phi\pi^1_c$ which coincide outside of A. Let $A = C_1 \cup ... \cup C_k$ for disjoint C_j's from π^3, such that, for $j = 2,...,k$, C_{j-1} is placed before C_j in π^3. Let, for $j = 0,...,k$, f^j be such that it coincides with f and g outside A, with f on $C_1,...,C_j$, and with g on $C_{j+1},...,C_k$. Since $f^k = f > g = f^0$, there is i with $f^i > f^{i-1}$. And since all f^j are in $\Phi\pi^3_c$ (note here that f must assign to A a better consequence than g does), π^3-essentiality of C_i follows. So every π^1-essential event in π^1 contains at least one π^3-essential event from π^3. We conclude that π^3 contains at least two disjoint π^3-essential events. So we obtain the v^3,U^3 for π^3, as resulting from Lemma A1.3. If we let v^{31} be the restriction of v^3 to $\Sigma\pi^1$, v^{31} and U^3 satisfy all requirements for v^1 and U^1 of Lemma A1.3, for π^1. So $v^{31} = v^1$, and $U^1 = \varphi^1 \circ U^3$ for a positive affine φ^3. Analogously the restriction of v^3 to $\Sigma\pi^2$ is equal to v_2, and $U^2 = \varphi^2 \circ U^3$. From this everything follows.

□

The following theorem is essentially Theorem 6.8 restricted to Φ^S.

THEOREM A1.5. *Under the Structural Assumption 6.7, for the preference relation* $\geq$ *on* Γ^S *the following two statements are equivalent:*

(i) *There exists a capacity* v *on* S, *and a continuous function* $U : \Gamma \to \mathbb{R}$, *such that* $f \mapsto \int (U \circ f) dv$ *represents* $\geq$ *on* Φ^s.

(ii) $\geq$ *is an s-continuous weak order on* Φ^s*, and* $\geq$ *does not exhibit comonotonic-contradictory strengths of preferences.*

Uniqueness results are as in Theorem 6.9.

PROOF. The implication (i)=>(ii) follows straightforwardly from application of Observation A1 to every Φ^π_c. So in the sequel we assume (ii), and derive (i) and the uniqueness results. For the case described in (6.6) (i.e. $\geq$ is trivial) everything is straightforward. The case described in (6.5) has been treated in Lemma A1.2. So let there be a partition π containing more than one π-essential event. By Lemmas A1.3 and A1.4 for every partition ρ containing at least two ρ-essential events, v^ρ and U^ρ can be taken, independent of ρ. That we do, and we leave out indexes ρ. It is straightforwardly checked that v and U satisfy all requirements.

□

A2. The extension to strongly bounded acts

This section completes the proof of Theorem 6.8 by treating nonsimple acts. The extension of the implication (i)⇒(ii) to nonsimple acts is straightforward, thus omitted. So we assume (ii), and derive (i) and the uniqueness results. We can apply Theorem A1.5 to obtain U and v to represent $\geq$ on Φ^s. This also gives the uniqueness results. Remains to be proved that these U and v also work for nonsimple acts.

It is elementarily verified (see Wakker,1986, Lemma V.4.3) that, for a constant-continuous pointwise monotone weak order on Φ^b, every act f in Φ^b has a <u>certainty equivalent</u> $\bar{\alpha}$, i.e. :

(A2.1) $\quad f \approx \bar{\alpha}$.

Obviously we can assign $U(\alpha)$ to any f in Φ^b as above, thus obtain a function which represents $\geq$ on Φ^b. This we do. Remains to be shown that this function equals the Choquet integral. That it does so for simple acts is implied by Theorem A1.5. So let f in Φ^b be arbitrary, $f \approx \bar{\alpha}$. Let $\mu,\nu \in \Gamma$ be such that $\mu \geq f(s) \geq \nu$ for all states s.

If $\bar{\mu} \approx \bar{\nu}$ then by pointwise monotonicity $f \approx \bar{\mu}$, so $\bar{\alpha} \approx \bar{\mu}$, $U \circ f$ is constant, and indeed $U(\alpha) = \int(U \circ f)dv$. So from now on we assume that $\bar{\mu} > \bar{\nu}$.

For notational convenience we shall assume that $U(\mu) = 1$, $U(\nu) = 0$. We shall construct a sequence of pairs of simple functions (f^m,g^m) such that, for all s,m:

(A2.2) $\quad U(f(s))-1/m \leq U(f^m(s)) \leq U(f(s)) \leq U(g^m(s)) \leq U(f(s)) + 1/m.$

For any m, and $0 \leq k \leq m-1$,

$$A_k := \{s \in S : k/m \leq U(f(s)) < (k + 1)/m\}$$

is an event. Since $U(\Gamma)$ is an interval, there exists, for any $0 \leq k \leq m$, an α_k such that $U(\alpha_k) = k/m$. Let

$$f^m := \sum_{k=0}^{m-1} \alpha_k 1_{A_k} + \alpha_{m-1} 1_{\{s:\ U(f(s))=1\}}$$

and

$$g^m := \sum_{k=0}^{m-1} \alpha_{k+1} 1_{A_k} + \alpha_m 1_{\{s:\ U(f(s))=1\}} .$$

We have $U(g^m(s)) \geq U(f(s)) \geq U(f^m(s))$ so $g^m(s) \geq f(s) \geq f^m(s)$ for all s. By pointwise monotonicity $g^m \geq f \geq f^m$. Hence by Theorem A1.5,

$$\int(U \circ g^m)dv \geq U(\alpha) \geq \int(U \circ f^m)dv.$$

Further, by monotonicity of the Choquet integral:

$$\int(U \circ g^m)dv \geq \int(U \circ f)dv \geq \int(U \circ f^m)dv.$$

And we have

$$\int(U \circ g^m)dv - \int(U \circ f^m)dv = 1/m.$$

By letting m go to infinity, we find that indeed $U(\alpha)$ equals the Choquet integral of $U \circ f$. This completes the proof of the main theorem.

□

FOOTNOTES

[1]: A topological space is connected if there is no nontrivial subset of Γ which is simultaneously open and closed.

[2]: A topological space is separable if it has a countable dense subset.

TABLE 7.1. The acts in the Ellsberg Paradox

	bb	bg	gb	gg
f^1	1	1	0	0
g^1	1	0	1	0
f^2	0	0	1	1
g^2	0	1	0	1

TABLE 7.2.

A contains one state	:	v(A)=1/6
A contains two states	:	v(A)=3/6 if A is {bb,bg} or {gg,gb}, v(A)=2/6 otherwise
A contains three states	:	v(A)=4/6
further	:	v(∅)=0, v(S)=1

TABLE 7.3. The acts involved in the Allais Paradox

	(0 , 0.01]	(0.01 , 0.11]	(0.11 , 1]
f^1	\$500,000	\$500,000	\$500,000
g^1	\$0	\$2,500,000	\$500,000
f^2	\$500,000	\$500,000	\$0
g^2	\$0	\$2,500,000	\$0