4. THE WEIGHTED UNIFORM EMPIRICAL PROCESS

The fact that the empirical process U_n is 0 at the point 0 suggests that we might weight the process by dividing by a non-random function that decreases to 0 as the argument decreases to 0. After deriving a probability inequality for the global fluctuations of the thus obtained process, we again, as in chapter 3, consider strong and weak convergence properties.

To be more precise, we will study the weighted empirical process

$$U_n(t)/q(t), t \in (0,1],\ U_n(0)/q(0) = 0\ , \tag{4.1}$$

where the weight function q will always be either an element of the class

$$\mathcal{Q}^* = \{q : [0,1] \to [0,\infty) : q \text{ continuous and nondecreasing},\ q > 0 \text{ on } (0,1]\} \tag{4.2}$$

or an element of the more restricted class

$$\mathcal{Q} = \{q \in \mathcal{Q}^* : q/I^{1/2} \text{ nonincreasing on } (0,1]\}\ . \tag{4.3}$$

(Of course, one may also consider $U_n(t)/q(1-t), t \in [0,1)$, since $U_n(1)$ is also equal to 0. By symmetry considerations, however, we may and will restrict attention to the growth properties of U_n near 0.)

First we will derive a global probability inequality for the weighted empirical process in (4.1), which is based on the fundamental inequality (2.19). For its proof we need a special countably infinite partition of $(0,1]$, that becomes arbitrarily fine near 0. For a fixed $\theta \in (0,1)$ this partition is the collection of half-open intervals

$$\mathcal{I}(\theta) = \{(\theta^k, \theta^{k-1}] :\ k \in I\!N\}\ .$$

For any $0 < \alpha \le \beta \le 1$ introduce

$$\mathcal{J}(\theta;\alpha,\beta) = \{k \in I\!N : \theta^{k-1} \ge \alpha,\ \theta^k < \beta\}\ .$$

Observe that $\{(\theta^{k-1}, \theta^k] : k \in \mathcal{J}(\theta;\alpha,\beta)\}$ is the set of all intervals in $\mathcal{I}(\theta)$ having a non-empty intersection with $[\alpha,\beta]$. The inclusions

$$[\alpha,\beta] \subset \bigcup_{k\in\mathcal{J}(\theta;\alpha,\beta)} (\theta^k, \theta^{k-1}] \subset [\theta\alpha, \beta/\theta]\ . \tag{4.4}$$

are immediate.

THEOREM 4.1. Let $q \in \mathcal{Q}$. Then we have for any $\varepsilon \in (0,1)$ and $0 < \alpha \le \beta \le \frac{1}{2}(1-\varepsilon)$

$$I\!P\Big(\sup_{\alpha\le t\le\beta} |U_n(t)|/q(t) \ge \lambda\Big) \le C \int_{(1-\varepsilon)\alpha}^{\beta/(1-\varepsilon)} \frac{1}{t}\exp\left(\frac{-(1-\varepsilon)\lambda^2 q^2(t)}{2t}\psi\left(\frac{\lambda q(\alpha)}{n^{1/2}\alpha}\right)\right) dt,\ \lambda \ge 0, \tag{4.5}$$

where $C \in (0, \infty)$ only depends on ε.

PROOF. For $\theta \in (0,1)$ it follows from the monotonicity of q and from Theorem 2.5 that

$$(4.6) \qquad \begin{aligned} & I\!P\Big(\sup_{\alpha \le t \le \beta} |U_n(t)|/q(t) \ge \lambda \Big) \\ & \le I\!P\Big(\max_{k \in \mathcal{J}(\theta,\alpha,\beta)} \sup_{\theta^k < t \le \theta^{k-1}} |U_n(t)|/q(\theta^k) \ge \lambda \Big) \\ & \le \sum_{k \in \mathcal{J}(\theta;\alpha,\beta)} I\!P\Big(\sup_{0 < t \le \theta^{k-1}} |U_n(t)| \ge \lambda q(\theta^k) \Big) \\ & \le C \sum_{k \in \mathcal{J}(\theta;\alpha\beta)} \exp\left(\frac{-(1-\varepsilon)\lambda^2 q^2(\theta^k)}{2\theta^{k-1}} \psi\left(\frac{\lambda q(\theta^k)}{n^{1/2}\theta^{k-1}} \right) \right) . \end{aligned}$$

Because $q \in \mathcal{Q}$ we have

$$(4.7) \qquad \frac{(1-\varepsilon)\lambda^2 q^2(\theta^k)}{2\theta^{k-1}} = \frac{(1-\varepsilon)\lambda^2 \theta q^2(\theta^k)}{2\theta^k} \ge \frac{(1-\varepsilon)\lambda^2 \theta q^2(t)}{2t}, \text{ for } \theta^k < t \le \theta^{k-1} \ ,$$

and jointly with the monotonicity of ψ

$$(4.8) \qquad \psi\left(\frac{\lambda q(\theta^k)}{n^{1/2}\theta^{k-1}} \right) \ge \psi\left(\frac{\lambda q(\theta^{k-1})}{n^{1/2}\theta^{(k-1)/2}\alpha^{1/2}} \right) \ge \psi\left(\frac{\lambda q(\alpha)}{n^{1/2}\alpha} \right), \text{ for } k \in \mathcal{J}(\theta;\alpha,\beta) \ .$$

Let us next observe that

$$(4.9) \qquad 1 = \frac{1}{1-\theta}\theta^{1-k} \int_{\theta^k}^{\theta^{k-1}} 1 dt \le \frac{1}{1-\theta} \int_{\theta^k}^{\theta^{k-1}} \frac{1}{t} dt \ .$$

Combining (4.9) with (4.4) and (4.6)-(4.8) we find

$$\begin{aligned} & I\!P\Big(\sup_{\alpha \le t \le \beta} |U_n(t)|/q(t) \ge \lambda \Big) \\ & \le \frac{C}{1-\theta} \int_{\theta\alpha}^{\beta/\theta} \frac{1}{t} \exp\left(\frac{-\theta(1-\varepsilon)\lambda^2 q^2(t)}{2t} \psi\left(\frac{\lambda q(\alpha)}{n^{1/2}\alpha} \right) \right) dt \ . \end{aligned}$$

The expression on the right in (4.5) is found by choosing $\theta = 1 - \varepsilon$ and by some relabeling. □

We now consider strong limit theorems for U_n/q. Our first result is the extension of Theorem 3.3 to the weighted empirical process. Write

$$\mathcal{Q}_0 = \{q \in \mathcal{Q} : \int_0^{1/e} \frac{1}{q^2(t)\log\log(1/t)} dt < \infty\} ,$$

and, for $q \in \mathcal{Q}$,

$$F_q = \{f/q : f \in F\} .$$

THEOREM 4.2. Let $q \in \mathcal{Q}$. If $q \in \mathcal{Q}_0$, then there exists an $\Omega_0 \in \mathcal{F}$ with $I\!P\Omega_0 = 1$, such that for each $\omega \in \Omega_0$ the sequence $\{U_{n,\omega}/((2\log\log n)^{1/2}q)\}_{n=2}^{\infty}$ is relatively compact in D with respect to ρ and the set of limit points is equal to F_q. Conversely, if $q \notin \mathcal{Q}_0$, then

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} \frac{|U_n(t)|}{(2\log\log n)^{1/2}q(t)} = \infty \text{ a.s.} \tag{4.10}$$

PROOF. We will only sketch a proof of the first part of this theorem. Let $q \in \mathcal{Q}_0$ and define, for $\delta \in (0,1)$,

$$q_\delta(t) = \begin{cases} q(\delta) , & 0 \le t < \delta , \\ q(t) , & \delta \le t \le 1 . \end{cases}$$

Since q_δ is bounded away from 0 it easily follows from Theorem 3.3 that the first part of Theorem 4.2 holds with q replaced by q_δ everywhere.

Suppose that we can show that for every $\varepsilon > 0$ there exists a $\delta \in (0,1)$ such that

$$\limsup_{n\to\infty} \sup_{0\le t\le\delta} \frac{|U_n(t)|}{(2\log\log n)^{1/2}q(t)} < \varepsilon \text{ a.s.} \tag{4.11}$$

Note that this implies that

$$\sup_{0\le t\le\delta} |f(t)|/q(t) < \varepsilon , \tag{4.12}$$

for $f \in F$. Combining the first part of Theorem 4.2 for $q = q_\delta$ with (4.11) and (4.12) it is not hard to see that for $f \in F$ and $q \in \mathcal{Q}_0$ almost surely, infinitely often,

$$\sup_{t\in[0,1]} \left| \frac{U_n(t)}{(2\log\log n)^{1/2}q(t)} - \frac{f(t)}{q(t)} \right| < 3\varepsilon . \qquad \square$$

It is clear from the above that the main ingredient for the proof of Theorem 4.2 is (4.11). It will turn out that the proof of (4.11) can be given with the help of properties of certain suprema of the empirical process weighted by the function $q = I^{1/2}$. Note that near zero weighting by $I^{1/2}$ is essentially the same as weighting by $(I(1-I))^{1/2}$, which is the standard deviation function of U_n. In this way we are naturally led into questions about the almost sure behavior of the supremum of the weighted empirical process. We will now consider some of those questions in more detail and in passing prove (4.11). We will restrict ourselves to weight functions of the form I^ν, for some $0 \leq \nu \leq 1$. Let us first consider the supremum over the entire unit interval. Because $I^\nu \in \mathcal{Q}_0$ provided that $0 \leq \nu < 1/2$, Theorem 4.2 guarantees a decent behavior of this supremum. For $1/2 \leq \nu \leq 1$, however, the picture is quite different.

THEOREM 4.3. Let $0 \leq \nu < 1/2$. Then we have

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} \frac{|U_n(t)|}{(2\log\log n)^{1/2} t^\nu} = \sup_{f\in F} \sup_{t\in[0,1]} \frac{|f(t)|}{t^\nu} \text{ a.s.} \tag{4.13}$$

PROOF. In order to see that $I^\nu, 0 \leq \nu < 1/2$, is an element of $\mathcal{Q}_0$, first note that trivially $I^\nu \in \mathcal{Q}$. As far as the integral condition in the definition of $\mathcal{Q}_0$ is concerned we may focus on th integral over a neighbourhood of 0 and note that for som $\eta > 0$

$$\int_0^\eta \frac{1}{t^{2\nu}\log\log(1/t)} dt < \int_0^\eta t^{-2\nu} dt < \infty \ ,$$

where the condition $\nu < \frac{1}{2}$ is crucial. Now apply Theorem 4.2 which yields (4.13) in the same way as (3.6) follows from Theorem 3.3. □

THEOREM 4.4. Let $\frac{1}{2} \leq \nu \leq 1$ and let $\{a_n\}_{n=1}^\infty$ be a sequence of positive real numbers.

(I) If $\sum_{n=1}^\infty a_n = \infty$, then

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} n^{1/2} a_n^\nu |U_n(t)|/t^\nu = \infty \text{ a.s.} \tag{4.14}$$

(II) If $\sum_{n=1}^\infty a_n < \infty$ and $na_n \downarrow$, then

$$\lim_{n\to\infty} \sup_{t\in[0,1]} n^{1/2} a_n^\nu |U_n(t)|/t^\nu = 0 \text{ a.s.} \tag{4.15}$$

COROLLARY 4.1. Let $\frac{1}{2} \leq \nu \leq 1$. There exists no sequence of positive real numbers $\{a_n\}_{n=1}^\infty$ such that $na_n \downarrow$ and

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} n^{1/2} a_n^\nu |U_n(t)|/t^\nu = 1 \text{ a.s.}$$

For convenience we prove Theorem 4.4 only for the case $\nu = \frac{1}{2}$, which may be seen as the "hardest case". For this proof we need two additional results. Recall that $\xi_1, \xi_2, ...$ is a sequence of independent $UN(0,1)$ random variables.

THEOREM 4.5. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive real numbers. If $\sum_{n=1}^\infty a_n < \infty$ and $a_n \downarrow$, then $\mathbb{P}(\xi_{1:n} \leq a_n \text{ i.o.}) = 0$.

PROOF. Note that $a_n \downarrow 0$. Hence $\{\xi_n \leq a_n \text{ i.o.}\} = \{\xi_{1:n} \leq a_n \text{ i.o.}\}$. The Borel-Cantelli lemma yields that $\mathbb{P}(\xi_n \leq a_n \text{ i.o.}) = 0$ and hence that $\mathbb{P}(\xi_{1:n} \leq a_n \text{ i.o.}) = 0$. □

The next theorem, stated without proof, is an analogue of Theorem 3.2 for $U_n/I^{1/2}$.

THEOREM 4.6. Let $0 < \alpha \le \beta \le 1, \varepsilon \in (0,1)$ and write $n_k = [(1+\varepsilon/12)^k]$, $k \in \mathbb{N}$. Then we have for all $k \in \mathbb{N}$ and for $\lambda > (2/\varepsilon)^{1/2}$

$$\mathbb{P}(\max_{n_k < n \le n_{k+1}} \sup_{\alpha \le t \le \beta} |U_n(t)|/t^{1/2} \ge \lambda) \le 2\mathbb{P}(\sup_{\alpha \le t \le \beta} |U_{n_{k+1}}(t)|/t^{1/2} \ge (1-\varepsilon)\lambda) \ .$$

PROOF OF THEOREM 4.4. Part I. Assume $\sum_{n=1}^{\infty} a_n = \infty$, then for all $\varepsilon > 0$ we have $\sum_{n=1}^{\infty} \varepsilon a_n = \infty$ and hence $\sum_{n=1}^{\infty} \mathbb{P}(\xi_n \le \varepsilon a_n) = \infty$. The Borel-Cantelli lemma now implies $\mathbb{P}(\xi_n \le \varepsilon a_n \text{ i.o.}) = 1$. But

$$(na_n)^{1/2} \sup_{0<t\le 1} n^{1/2}|\Gamma_n(t) - t|/t^{1/2} \ge (na_n)^{1/2} n^{1/2} \frac{1}{2n}/\xi_n^{1/2} = \tfrac{1}{2}(a_n/\xi_n)^{1/2} \ .$$

Combination of both results now implies

$$\limsup_{n\to\infty} (na_n)^{1/2} \sup_{0<t\le 1} n^{1/2}|\Gamma_n(t) - t|/t^{1/2} \ge \frac{1}{2\varepsilon^{1/2}} \text{ a.s.}$$

Letting $\varepsilon \downarrow 0$ proves this part of the theorem.

Part II. We prove this part under the weak additional regularity assumption that $n^c a_n \uparrow$ for some $c > 1$. We have to show that

$$\text{(4.16)} \qquad \lim_{n\to\infty} na_n^{1/2} \sup_{0<t\le 1} |\Gamma_n(t) - t|/t^{1/2} = 0 \text{ a.s.}$$

The observation that $\{Ma_n\}_{n=1}^{\infty}$ ($1 < M < \infty$ arbitrary) also satisfies the conditions on $\{a_n\}_{n=1}^{\infty}$, gives that we only have to prove that for some $L \in (0,\infty)$

$$\text{(4.17)} \qquad \limsup_{n\to\infty} na_n^{1/2} \sup_{0<t\le 1} |\Gamma_n(t) - t|/t^{1/2} \le L \text{ a.s.}$$

From the conditions on $\{a_n\}_{n=1}^{\infty}$ we find that $na_n \log n \to 0$ $(n \to \infty)$, hence because of Theorem 3.1 it suffices to prove (4.17) when the supremum is restricted to e.g. $(0, \frac{1}{4}]$. Theorem 4.5 implies that

$$\limsup_{n\to\infty} na_n^{1/2} \sup_{0<t\le a_n} |\Gamma_n(t) - t|/t^{1/2}$$

$$= \limsup_{n\to\infty} na_n^{1/2} \sup_{0<t\le a_n} t/t^{1/2} = \limsup_{n\to\infty} na_n = 0 \text{ a.s.}$$

Hence it finally remains to show that

$$\text{(4.18)} \qquad \limsup_{n\to\infty} na_n^{1/2} \sup_{a_n\le t\le 1/4} |\Gamma_n(t) - t|/t^{1/2} \le L \text{ a.s.}$$

Applying Theorem 4.6 (with $\varepsilon = \frac{1}{2}$) we see by the Borel-Cantelli lemma that it suffices to show that

$$(4.19) \qquad \sum_{k=1}^{\infty} I\!P\Big(\sup_{a_{n_{k+1}} \le t \le 1/4} |U_{n_{k+1}}(t)|/t^{1/2} \ge L/(2(n_k a_{n_k})^{1/2})\Big) < \infty \ .$$

Next we use Theorem 4.1 (with $\varepsilon = \frac{1}{2}$) to obtain sufficiently sharp upper bounds for these probabilities, assuming that k is large enough. We have, if $L \ge 2$,

$$(4.20) \qquad I\!P\Big(\sup_{a_{n_{k+1}} \le t \le 1/4} |U_{n_{k+1}}(t)|/t^{1/2} \ge L/(2(n_k a_{n_k})^{1/2})\Big)$$

$$\le C \int_{a_{n_{k+1}}/2}^{1/2} \frac{1}{t} dt \exp\left(-\frac{L^2}{16 n_k a_{n_k}} \psi\left(\frac{L}{2(n_k a_{n_k})^{1/2} (n_{k+1} a_{n_{k+1}})^{1/2}} \right)\right)$$

$$\le C \log \frac{1}{a_{n_{k+1}}} \exp\left(\frac{-L^2}{16 n_k a_{n_k}} \frac{2(n_k a_{n_k})^{1/2} (n_{k+1} a_{n_{k+1}})^{1/2}}{L} \log\left(\frac{L}{2 n_k a_{n_k}} \right)\right)$$

$$= C \log \frac{1}{a_{n_{k+1}}} \exp\left(\frac{-L}{8} \left(\frac{n_{k+1} a_{n_{k+1}}}{n_k a_{n_k}} \right)^{1/2} \log\left(\frac{L}{2 n_k a_{n_k}} \right)\right)$$

$$\le C \log \frac{1}{a_{n_{k+1}}} (n_k a_{n_k})^{\frac{L}{8}\left(\frac{n_{k+1} a_{n_{k+1}}}{n_k a_{n_k}} \right)^{1/2}} \ .$$

Observe that $n^c a_n \uparrow$ implies that $(n_{k+1} a_{n_{k+1}})^{1/2} / (n_k a_{n_k})^{1/2} \ge (23/25)^{(c-1)/2}$ for large k. This yields, if $(L/8)(23/25)^{(c-1)/2} \ge 2$, that the number on the right in (4.20) is bounded from above by $C(\log(1/a_{n_{k+1}})(n_k a_{n_k})^2$.

So it remains to show that

$$(4.21) \qquad \sum_{k=1}^{\infty} \log \frac{1}{a_{n_{k+1}}} (n_k a_{n_k})^2 < \infty \ .$$

First note that $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \downarrow$ imply

$$(4.22) \qquad \sum_{k=1}^{\infty} n_k a_{n_k} < \infty \ .$$

From $n^c a_n \uparrow$ we have $a_n \ge a_1 / n^c$. Hence writing $\tilde{c} = 1 \wedge \frac{1}{a_1}$, we see that

$$(4.23) \qquad \log \frac{1}{a_{n_{k+1}}} (n_k a_{n_k})^2 \le (\log(\tilde{c} n_{k+1}^c))(n_k a_{n_k})^2$$

$$= (\log \tilde{c})(n_k a_{n_k})^2 + c \log n_{k+1} (n_k a_{n_k})^2 \ .$$

Since (4.22) implies $\sum_{k=1}^{\infty}(n_k a_{n_k})^2 < \infty$, the first term in the right hand side of (4.23) is summable in k. Hence it suffices to prove that

$$(4.24) \qquad \sum_{k=1}^{\infty} k(n_k a_{n_k})^2 < \infty \ .$$

Observe that (4.22) and $n_k a_{n_k} \downarrow$ imply $k n_k a_{n_k} \to 0$ $(k \to \infty)$, yielding that $k(n_k a_{n_k})^2 \le n_k a_{n_k}$ for large k. This observation and again (4.22) yield (4.24). $\square$

Corollary 4.1 states that no proper standardization is possible for $\nu \in [\frac{1}{2}, 1]$, when the supremum is taken over the entire unit interval. However, if we restrict the supremum to $[a_n, 1]$ for some $a_n \in (0,1)$ with $a_n \to 0$, a suitable standardization might still be possible for such values of ν.

THEOREM 4.7. Let $\frac{1}{2} \le \nu \le 1$ and suppose that the $a_n \in (0,1)$ satisfy

$$(4.25) \qquad \lim_{n\to\infty} \frac{n a_n}{\log\log n} = c \in (0, \infty) \ ,$$

then we have

$$(4.26) \qquad \limsup_{n\to\infty} \sup_{a_n \le t \le 1} \frac{n^{1/2-\nu}|U_n(t)|}{(\log\log n)^{1-\nu} t^{\nu}} = c^{1-\nu}(\beta_c - 1) \vee 2 1_{\{1/2\}}(\nu) \text{ a.s.},$$

where β_c is defined by $\beta_c > 1$ and $\beta_c(\log \beta_c - 1) + 1 = 1/c$.

The proof of this result will not be given in detail, since it is again rather technical. It might be interesting, however, to point out one of the major tools for the proof that $c^{1-\nu}(\beta_c - 1) \vee 2 1_{\{1/2\}}(\nu)$ is almost surely an upper bound for the expression on the left in (4.26). It is convenient to consider the cases $\nu = \frac{1}{2}$ and $\frac{1}{2} < \nu \le 1$ separately. Restricting ourselves to the latter case we "split" $[a_n, 1]$ into the three subintervals $[a_n, n^{-1}\log n], [n^{-1}\log n, \frac{1}{4}], [\frac{1}{4}, 1]$, and first consider

$$(4.27) \qquad \limsup_{n\to\infty} \sup_{a_n \le t \le n^{-1}\log n} \frac{n^{1/2-\nu}|U_n(t)|}{(\log\log n)^{1-\nu} t^{\nu}} \le c^{1-\nu}(\beta_c - 1) \text{ a.s.}$$

One of the main tools to prove (4.27) is the inequality

$$(4.28) \qquad I\!P\Big(\sup_{\alpha \le t \le \beta} \frac{n^{1/2-\nu}|U_n(t)|}{t^{\nu}} \ge \lambda \Big)$$

$$\le C \int_{(1-\varepsilon)\alpha}^{\beta/(1-\varepsilon)} \frac{1}{t} dt \exp\left(\frac{-(1-\varepsilon)\lambda^2 (n\alpha)^{2\nu-1}}{2} \psi(\lambda (n\alpha)^{\nu-1}) \right) \ .$$

This inequality can be derived from (4.5); note that we cannot apply (4.5) directly because the present weight function $I^{\nu}, \frac{1}{2} < \nu \le 1$, is not in the class $\mathcal{Q}$. We do have, however,

$$I\!\!P\left(\sup_{\alpha\le t\le\beta}\frac{n^{1/2-\nu}|U_n(t)|}{t^\nu}\ge\lambda\right)=I\!\!P\left(\sup_{\alpha\le t\le\beta}\frac{|U_n(t)|}{t^{1/2}t^{\nu-1/2}}\ge\lambda n^{\nu-1/2}\right)$$

$$\le I\!\!P\left(\sup_{\alpha\le t\le\beta}\frac{|U_n(t)|}{t^{1/2}}\ge\lambda(n\alpha)^{\nu-1/2}\right)\ ,$$

and (4.5) may be applied to this last expression. This yields (4.28).

The inequality in (4.28) is also used in showing that

$$\lim_{n\to\infty}\ \sup_{n^{-1}\log n\le t\le 1/4}\frac{n^{1/2-\nu}|U_n(t)|}{(\log\log n)^{1-\nu}t^\nu}=0 \text{ a.s.}$$

Finally we have

$$\lim_{n\to\infty}\ \sup_{1/4\le t\le 1}\frac{n^{1/2-\nu}|U_n(t)|}{(\log\log n)^{1-\nu}t^\nu}=0 \text{ a.s.,}$$

as is immediate from (3.1) because $\nu>1/2$.

Now we can finish the sketch of the proof of Theorem 4.2 by outlining a proof of the crucial relation (4.11). First it will be shown that

$$\lim_{n\to\infty}\ \sup_{0<t\le n^{-1/2}}\frac{|U_n(t)|}{(\log\log n)^{1/2}q(t)}=0 \text{ a.s.}$$

Writing $a_n=\{n^{3/2}\log\log n q^2(n^{-1/2})\}^{-1}$ it clearly suffices to prove that

$$\lim_{n\to\infty}\ \sup_{0<t\le 1}\frac{(na_n)^{1/2}|U_n(t)|}{t^{1/2}}=0 \text{ a.s.}$$

This follows from Theorem 4.4(II), with $\nu=\frac{1}{2}$, because $na_n\downarrow$ $(n\ge 2)$ and $\sum_{n=k}^{\infty}a_n<\infty$ for some $k\in I\!\!N$. To see this last statement if suffices to show that for some $a\in(0,\infty)$

$$\int_a^\infty\frac{1}{t^{3/2}(\log\log t)q^2(t^{-1/2})}dt<\infty\ . \tag{4.29}$$

Making the substitution $s=t^{-1/2}$ we see that the left hand side of (4.29) is equal to

$$\int_0^{a^{-1/2}}\frac{2}{q^2(t)\log\log(1/t^2)}dt\le 2\int_0^{a^{-1/2}}\frac{1}{q^2(t)\log\log(1/t)}dt<\infty\ ,$$

since $q\in\mathcal{Q}_0$.

It remains to show that for every $\varepsilon>0$ there exists a $\delta\in(0,1)$ such that

$$(4.30)\qquad \limsup_{n\to\infty} \sup_{n^{-1/2}\le t\le\delta} \frac{|U_n(t)|}{(2\log\log n)^{1/2}q(t)} < \varepsilon \text{ a.s.}$$

Theorem 4.7, with $\nu = \frac{1}{2}$, entails, since $\lim_{c\to\infty} c^{1/2}(\beta_c - 1) = 2^{1/2}$,

$$(4.31)\qquad \limsup_{n\to\infty} \sup_{n^{-1/2}\le t\le 1} \frac{|U_n(t)|}{(\log\log n)^{1/2}t^{1/2}} \le 2 \text{ a.s.}$$

Using $q \in \mathcal{Q}_0$ again, we see that $t^{1/2}/q(t) \downarrow 0$, as $t \downarrow 0$, so that (4.30) is immediate from (4.31).

We now address the problem of weak convergence of the weighted empirical processes U_n/q. For convenience we actually consider the convergence in probability to zero of the suprema of the weighted differences of $\tilde{U}_n$ and $\tilde{U}$, where $\tilde{U}_n$ and $\tilde{U}$ are obtained from the Skorohod construction (3.23). Note that for each $n \in \mathbb{N}$ there exist independent and $UN(0,1)$ distributed random variables $\xi_1^{(n)}, ..., \xi_n^{(n)}$, such that $\tilde{U}_n$ is the uniform empirical process pertaining to these random variables. We will, however, suppress both the tilde in the notation for the processs and the upper index in the notation for the random variables. In particular we will write

$$(4.32)\qquad \sup_{t\in[0,1]} |U_n(t) - U(t)| \to 0 \text{ a.s. } (n\to\infty)\ .$$

Set

$$\mathcal{Q}_0^* = \{q \in \mathcal{Q}^* : \int_0^1 \frac{1}{t}\exp\left(\frac{-\lambda q^2(t)}{t}\right) dt < \infty, \text{ for all } \lambda > 0\}\ .$$

THEOREM 4.8. Let $q \in \mathcal{Q}^*$. Then we have

$$(4.33)\qquad \sup_{t\in[0,1]} \frac{|U_n(t) - U(t)|}{q(t)} \underset{\mathbb{P}}{\to} 0, \text{ as } n\to\infty\ ,$$

if and only if $q \in \mathcal{Q}_0^*$.

PROOF. Let us restrict ourselves to a sketch of the proof of the "if" part. To avoid too many technical details we will assume that $q \in \mathcal{Q}$, meaning that in addition $q/I^{1/2}$ is non-increasing. It can be shown that $q \in \mathcal{Q}_0^* \cap \mathcal{Q}$ implies that $g(t) = q(t)/(t\log\log(1/t))^{1/2} \to \infty$ as $t \downarrow 0$. We may and will assume without loss of generality that also $g \le (\log\log(1/I))^{1/2}$ in a neighbourhood of zero.

For any $0 < \delta \le \frac{1}{4}$ we have

$$(4.34)\qquad \sup_{t\in[0,1]} \frac{|U_n(t) - U(t)|}{q(t)} \le \sum_{k=1}^{5} Y_{nk}$$

where, with $\alpha_n = 1/(nc), c \in (0,\infty), \beta_n = q^2(\frac{1}{n})$ the random variables Y_{nk} are given by

$$(4.35)\qquad Y_{n1} = \sup_{0\le t\le\alpha_n} |U_n(t)|/q(t)\ ,$$

$$(4.36)\qquad Y_{n2} = \sup_{\alpha_n\le t\le\beta_n} |U_n(t)|/q(t)\ ,$$

$$(4.37)\qquad Y_{n3} = \sup_{\beta_n\le t\le\delta} |U_n(t)|/q(t)\ ,$$

$$(4.38)\qquad Y_{n4} = \sup_{0\le t\le\delta} |U(t)|/q(t)\ ,$$

$$(4.39)\qquad Y_{n5} = \sup_{t\in[0,1]} |U_n(t)-U(t)|/q(\delta)\ .$$

It will be shown that for any $\varepsilon > 0$ and each $k \in \{1,2,3,4,5\}$ there exists a $c = c(\varepsilon)$, $\delta = \delta(\varepsilon)$ and $N = N(\varepsilon) \in \mathbb{N}$ such that

$$(4.40)\qquad \mathbb{P}(Y_{nk} \ge \varepsilon) \le \varepsilon \text{ for } n \ge N\ .$$

Let us first consider Y_{n1}. Observe that $\mathbb{P}(\xi_{1:n} > \alpha_n) = (1-1/(nc))^n$. Hence $\lim_{n\to\infty} \mathbb{P}(\xi_{1:n} > \alpha_n) = e^{-1/c} > 1-\varepsilon$ for large c. If $\xi_{1:n} > \alpha_n$ we have $|U_n(t)| = n^{1/2}t$ for $0 \le t \le \alpha_n$, implying that for n sufficiently large, with probability larger than $1-\varepsilon$,

$$Y_{n1} \le n^{1/2} \sup_{0\le t\le\alpha_n} t/q(t) = n^{1/2}/((cn\log\log(1/nc)))^{1/2} g(1/(nc)) < \varepsilon\ .$$

Hence (4.40) for $k = 1$ follows.

An indirect application of Theorem 4.1 with $q = I^{1/2}$ and $\varepsilon = \frac{1}{2}$ yields for large n

$$\mathbb{P}(Y_{n2} \ge \varepsilon) = \mathbb{P}(\sup_{\alpha_n\le t\le\beta_n} |U_n(t)|/t^{1/2} \ge \varepsilon g(\beta_n)(\log\log(1/\beta_n))^{1/2})$$

$$\le \mathbb{P}(\sup_{\alpha_n\le t\le\beta_n} |U_n(t)|/t^{1/2} \ge (\log\log n)^{1/2})$$

$$\le C \int_{\alpha_n/2}^{\beta_n} \frac{1}{t} dt \exp\left(\frac{-\log\log n}{4} \psi\left(\frac{(\log\log n)^{1/2}(nc)^{1/2}}{n^{1/2}}\right)\right)$$

$$\le 3C \log\log\log n \exp(-(\log\log n)^{1/2}) \le \varepsilon\ ,$$

where for the last but one inequality the second property in (2.15) is used.

In order to prove (4.40) for Y_{n3} we apply Theorem 4.1 (with $\varepsilon = \frac{1}{2}$) directly, yielding

$$(4.41)\qquad \mathbb{P}(Y_{n3} \ge \varepsilon) \le C \int_{\beta_n/2}^{2\delta} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 q^2(t)}{4t} \psi\left(\frac{\varepsilon q(\beta_n)}{n^{1/2}\beta_n}\right)\right) dt$$

$$\leq C \int_0^{2\delta} \frac{1}{t} \exp\left(\frac{-\varepsilon^2 \psi(\varepsilon) q^2(t)}{4t} \right) dt \ .$$

Since $q \in \mathcal{Q}_0^*$ we can choose δ so small that the expression on the right in (4.41) is bounded from above by ε (for all $n \in I\!N$).

Note that Y_{n4} does not depend on n. It is a well-known result that indeed (4.40) holds for δ small enough. Finally the validity of (4.40) for $k = 5$ is immediate from (4.32). $\square$

For the weighted empirical process the following approximation result exists (cf. Theorem 3.13); its proof will be omitted.

THEOREM 4.9. There exists a sequence of Brownian bridges $U^{(1)}, U^{(2)}, ...,$ such that for all $0 < \nu \leq \frac{1}{2}$ and all $0 < \lambda \leq 1$

$$\sup_{\frac{\lambda}{n} \leq t \leq 1 - \frac{\lambda}{n}} \frac{n^{1/2-\nu} |U_n(t) - U^{(n)}(t)|}{(t(1-t))^\nu} = O_{I\!P}(1), \text{ as } n \to \infty \ . \tag{4.42}$$

Usually we conclude the chapters with one or more applications. At the end of this chapter, however, we will only roughly indicate possible applications due to their complexity. More detailed applications of weighted empirical processes can be found at the end of chapter 5; here we will only focus on some general ideas.

Let $X_1, ..., X_n$ be independent and identically distributed random variables with common continuous and strictly increasing distribution function F and denote their order statistics as in (1.16) with

$$-\infty = X_{0:n} < X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n} < X_{n+1:n} = \infty \ .$$

Now consider the statistically important trimmed sum $\sum_{i=k}^{m} X_{i:n}$, for $1 \leq k \leq m \leq n$. Of course choosing $k = 1$ and $m = n$ gives the whole sum $\sum_{i=1}^{n} X_{i:n} = \sum_{i=1}^{n} X_i = n\bar{X}_n$, where $\bar{X}_n$ is the mean of the X_i, $1 \leq i \leq n$. However, if $k \neq 1$ or $m \neq n$, it might be difficult to deal with the trimmed sum above because of the dependence between the $X_{i:n}$. Therefore it is useful to note that, almost surely,

$$\sum_{i=k}^{m} X_{i:n} = \int_{(X_{k-1:n}+X_{k:n})/2}^{(X_{m:n}+X_{m+1:n})/2} t dn F_n(t) \ , \tag{4.43}$$

where, as before, F_n denotes the empirical distribution function of the X_i. From now on we do not write the integration limits anymore. Centering and dividing by $n^{1/2}$ in the right hand side of (4.43) leads to the expression

$$\int t dU_{n,F}(t) \ ,$$

where $U_{n,F} = n^{1/2}(F_n - F)$ is the empirical process. Now integrating by parts yields as one of the terms

$$-\int U_{n,F}(t)dt = -\int U_n(F(t))dt = -\int U_n(s)dF^{-1}(s) ,$$

where F^{-1} is defined as in (1.18). Here the first equality follows as in the proof of Theorem 1.4 and the second one from the change of variables $F(t) = s$.

This may lead, depending on F, to an expression which has as one of the main terms

$$-\int_0^1 \frac{U_n(t)}{t^{1+\alpha}}dt, \text{ with } 0 \leq \alpha < \tfrac{1}{2} .$$

We now can, when required, easily bound this expression in probability. We have, choosing $0 < \varepsilon < \frac{1}{2} - \alpha$ and noting that $I^{\alpha+\varepsilon} \in \mathcal{Q}_0^*$,

$$\left| -\int_0^1 \frac{U_n(t)}{t^{1+\alpha}}dt \right| \leq \sup_{t\in[0,1]} \frac{|U_n(t)|}{t^{\alpha+\varepsilon}} \int_0^1 \frac{1}{t^{1-\varepsilon}}dt \tag{4.44}$$

$$= \frac{1}{\varepsilon} \sup_{t\in[0,1]} \frac{|U_n(t)|}{t^{\alpha+\varepsilon}} = O_{\mathbb{P}}(1) ,$$

which readily follows from Theorem 4.8 and the fact that for $q \in \mathcal{Q}_0^*$, $\sup_{t\in[0,1]} |U(t)|/q(t) < \infty$ a.s. Obviously this technique fails if only the unweighted process U_n is taken out of the integral, since in that case the resulting integral is infinite.

A technique like the one in (4.44) is e.g. also very useful in establishing asymptotic normality for (certain) linear combinations of a function of the order statistics: expressions of the form

$$\sum_{i=1}^n c_{in}\Psi(X_{i:n}) ,$$

where $c_{1n}, ..., c_{nn}$ are real numbers and where $\Psi : \mathbb{R} \to \mathbb{R}$ is a measurable function. These general statistics contain as special cases the mean, the trimmed mean, and sums of extreme values.

PROBLEMS

4.1 Show that $I^{1/2} \notin \mathcal{Q}_0$. Now prove, using Theorem 4.4, that (4.10) holds for $q = I^{1/2}$.

4.2 Prove that (4.11) implies (4.12).

4.3 Show that (4.17) is indeed sufficient for (4.16).

4.4 Prove that $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \downarrow$ imply $\sum_{k=1}^{\infty} n_k a_{n_k} < \infty$ (see (4.22) and above it).

4.5 Use Theorem 4.9 to show that for all $0 < \nu \leq \frac{1}{2}$

$$\sup_{0<t<1} \frac{n^{1/2-\nu}|U_n(t) - \bar{U}^{(n)}(t)|}{(t(1-t))^{\nu}} = O_{\mathbb{P}}(1), \text{ as } n \to \infty ,$$

where

$$\bar{U}^{(n)}(t) = \begin{cases} U^{(n)}(t) & \text{for } \dfrac{1}{n} \leq t \leq 1 - \dfrac{1}{n} ; \\ 0 & \text{elsewhere} . \end{cases}$$

5. THE TAIL EMPIRICAL PROCESS

Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of numbers such that

$$(5.1) \qquad 0 < k_n < n,\ k_n/n \to 0 \text{ and } k_n \to \infty\ (n \to \infty)\ .$$

Then the tail empirical process is defined by

$$(5.2) \qquad w_n(t) = \left(\frac{n}{k_n}\right)^{1/2} U_n\left(t\frac{k_n}{n}\right),\ t \in [0,1]\ .$$

This is a rescaled and properly standardized version of the uniform empirical process on $[0, k_n/n]$. This process is not only an interesting object in itself, but its properties are also key tools to derive theorems about small or large (via symmetry) order statistics (needed in, e.g., extreme value theory). We will consider the functional central limit theorem (using a Skorohod construction as in Theorem 4.8) and the functional law of the iterated logarithm for the (weighted) tail empirical process. Note that our probability inequalities remain applicable to this process.

THEOREM 5.1. Let $\{k_n\}_{n=1}^{\infty}$ be as in (5.1).

(I) There exist $\tilde{w}_n, n \in I\!N$, such that $\tilde{w}_n \stackrel{d}{=} w_n$ and a Wiener process $\tilde{W}$ such that

$$(5.3) \qquad \sup_{t\in[0,1]} |\tilde{w}_n(t) - \tilde{W}(t)| \to 0 \text{ a.s. } (n \to \infty)\ .$$

(II) Let $\tilde{w}_n, n \in I\!N$, be as in part I and let $q \in \mathcal{Q}^*$. Then

$$(5.4) \qquad \sup_{t\in[0,1]} \frac{|\tilde{w}_n(t) - \tilde{W}(t)|}{q(t)} \to 0 \text{ a.s. } (n \to \infty)$$

if and only if $q \in \mathcal{Q}_0^*$.

PROOF. We will give a proof of part I and the "if" part of part II at one time; the proof of the "only if" part of part II will be omitted.

Set $K_n = n\Gamma_n(k_n/n)$ and note that K_n is binomially distributed with parameters n and k_n/n. Hence by the central limit theorem $Z_n = (K_n - k_n)/k_n^{1/2}$ converges weakly to a standard normal random variable. We have

$$(5.5) \qquad w_n(t) = \left(\frac{K_n}{k_n}\right)^{1/2} K_n^{1/2}\left(\frac{n}{K_n}\Gamma_n\left(t\frac{k_n}{n}\right) - t\right) + Z_n t,\ t \in [0,1]\ .$$

By two separate applications of the Skorohod construction (3.22), there exist a sequence of uniform empirical processes $\{\tilde{U}_n\}_{n=1}^{\infty}$, a Brownian bridge $\tilde{U}$, a sequence of random variables $\{\tilde{Z}_n\}_{n=1}^{\infty}$ with $\tilde{Z}_n \stackrel{d}{=} Z_n$, and a standard normal random variable $\tilde{Z}$, such that

$$(5.6) \qquad \sup_{t\in[0,1]} |\tilde{U}_n(t) - \tilde{U}(t)|/q(t) \to 0 \text{ a.s. and } |\tilde{Z}_n - \tilde{Z}| \to 0 \text{ a.s. } (n \to \infty)\ .$$

Moreover, we have that the sequences $\{\tilde{U}_n\}_{n=1}^\infty$ and $\{\tilde{Z}_n\}_{n=1}^\infty$ are independent and that $\tilde{U}$ and $\tilde{Z}$ are independent. Now the following observation is crucial:

$$(5.7)\qquad w_n \stackrel{d}{=} \left(\frac{\tilde{K}_n}{k_n}\right)^{1/2} \tilde{U}_{\tilde{K}_n} + \tilde{Z}_n I = \tilde{w}_n\ ,$$

cf. (5.5). Write

$$(5.8)\qquad \tilde{W} = \tilde{U} + \tilde{Z} I\ .$$

It is easy to see that $\tilde{W}$ is a Wiener process.

For $q \in \mathcal{Q}^*$ we have

$$(5.9)\qquad \sup_{t\in[0,1]} \frac{|\tilde{w}_n(t) - \tilde{W}(t)|}{q(t)}$$

$$\leq \left(\frac{\tilde{K}_n}{k_n}\right)^{1/2} \sup_{t\in[0,1]} \frac{|\tilde{U}_{\tilde{K}_n}(t) - \tilde{U}(t)|}{q(t)} + \left|\left(\frac{\tilde{K}_n}{k_n}\right)^{1/2} - 1\right| \sup_{t\in[0,1]} \frac{|\tilde{U}(t)|}{q(t)}$$

$$+|\tilde{Z}_n - \tilde{Z}| \sup_{t\in[0,1]} \frac{t}{q(t)}\ .$$

Now we show that the right hand side of (5.9) tends to zero almost surely ($n \to \infty$). The first term can be dealt with using $\tilde{K}_n/k_n \to 1$ a.s., $\tilde{K}_n \to \infty$ a.s. ($n \to \infty$), and (5.6); for the second term use again $\tilde{K}_n/k_n \to 1$ a.s. and the fact that $\sup_{t\in[0,1]} |\tilde{U}(t)|/q(t) < \infty$ a.s. for $q \in \mathcal{Q}_0^*$. Finally the almost sure convergence to zero of the third term follows from (5.6) and the fact that $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$ for $q \in \mathcal{Q}_0^*$. □

Next we consider the functional law of the iterated logarithm for w_n and w_n/q. On $\{k_n\}_{n=1}^\infty$ we need, in addition, the following assumption:

$$(5.10)\qquad \frac{k_n}{n} \downarrow,\ k_n \uparrow \text{ and } k_n/\log\log n \to \infty\ (n \uparrow \infty)\ .$$

THEOREM 5.2. Assume $\{k_n\}_{n=1}^\infty$ satisfies (5.1) and (5.10). Then there exists an $\Omega_0 \in \mathcal{F}$ with $I\!P\Omega_0 = 1$, such that for each $\omega \in \Omega_0$ the sequence $\{w_{n,\omega}/(2\log\log n)^{1/2}\}_{n=2}^\infty$ is relatively compact in D with respect to ρ and the set of limit points is equal to S. (The set of functions S is defined as F in Theorem 3.3 but without the restriction $f(1) = 0$.)

The proof of Theorem 5.2 and its "weighted" analogue Theorem 5.3 will be omitted. In the latter result we restrict ourselves to weight functions of the form $I^\nu, 0 < \nu \leq 1$.

THEOREM 5.3. Assume $\{k_n\}_{n=1}^\infty$ satisfies (5.1) and (5.10) and let $0 < \nu \leq 1$.

(I) If $\displaystyle\sum_{n=2}^\infty \frac{k_n}{n} \frac{1}{(k_n \log\log n)^{1/(2\nu)}} < \infty$, then there exists an $\Omega_0 \in \mathcal{F}$ with $I\!P\Omega_0 = 1$, such that for each $\omega \in \Omega_0$ the sequence $\{w_{n,\omega}/((2\log\log n)^{1/2}I^\nu)\}_{n=2}^\infty$ is relatively compact in D with respect to ρ and the set of limit points is equal to $\{f/I^\nu : f \in S\}$.

(II) If $\sum_{n=2}^\infty \frac{k_n}{n}\ \frac{1}{(k_n \log\log n)^{1/(2\nu)}} = \infty$, then

$$\limsup_{n\to\infty}\ \sup_{t\in[0,1]} \frac{|w_n(t)|}{(2\log\log n)^{1/2}t^\nu} = \infty \text{ a.s.} \tag{5.11}$$

Note that for $\frac{1}{2} \le \nu \le 1$, we always have $\sum_{n=2}^\infty \frac{k_n}{n}\frac{1}{(k_n \log\log n)^{1/(2\nu)}} = \infty$; also for any $0 < \nu < \frac{1}{2}$, there are both sequences $\{k_n\}_{n=1}^\infty$, satisfying (5.1) and (5.10), such that the above sum is infinite or finite respectively. It is striking that in Theorem 5.3 the class of weight functions for which there is a functional law of the iterated logarithm depends on the sequence $\{k_n\}_{n=1}^\infty$, whereas in its weak analogue (Theorem 5.1, part II), the class of weight functions for which the functional central limit theorem holds does not depend on $\{k_n\}_{n=1}^\infty$. Theorem 5.3 entails, under the conditions of part I,

$$\limsup_{n\to\infty}\ \sup_{t\in[0,1]} \frac{|w_n(t)|}{(2\log\log n)^{1/2}t^\nu} = \sup_{f\in S}\ \sup_{t\in[0,1]} \frac{f(t)}{t^\nu} = 1 \text{ a.s.}; \tag{5.12}$$

cf. (3.6) and (4.13).

We conclude this chapter with three applications. We begin with estimating the (right) tail of a one dimensional distribution function. Denote as usual this distribution function with F and its generalized inverse F^{-1} (as defined in (1.18)) with Q. Write $\bar F = 1 - F$ and let F_n be the empirical distribution function based on a sample of size n from F; also $\bar F_n = 1 - F_n$. Now assume $\{k_n\}_{n=1}^\infty$ satisfies (5.1), then we can show that there exists a Skorohod construction such that, as $n \to \infty$,

$$\sup_{t \ge Q(1-\frac{k_n}{n})} \left| \frac{n}{k_n^{1/2}} \left(\bar F_n^*(t) - \bar F(t)\right) - W^*\left(\frac{n}{k_n}\bar F(t)\right)\right| \to 0 \text{ a.s. }, \tag{5.13}$$

where $F_n^* \stackrel{d}{=} F_n, n \in I\!N$, and W^* is a Wiener process. This readily implies, for a sequence of positive numbers $\{m_n\}_{n=1}^\infty$ with $m_n/k_n^{1/2} \to \infty$, that

$$\sup_{t \ge Q(1-\frac{k_n}{n})} \frac{n}{m_n}|\bar F_n(t) - \bar F(t)| \underset{I\!P}{\to} 0 \ (n\to\infty)\ . \tag{5.14}$$

Note that (5.14) is not a trivial statement. Although both terms inside the absolute value tend to zero uniformly on the stated region, it is not true in general that $\sup_{t\ge Q(1-k_n/n)} (n/m_n)\bar F(t) \to 0$. E.g., for continuous F and $m_n = k_n^{3/4}$, this expression is equal to $k_n/m_n = k_n^{1/4} \to \infty$ $(n \to \infty)$.

We now prove (5.13). Recall (5.3) and write for $0 \le s \le k_n/n$,

$$\tilde\Gamma_n(s) = \frac{k_n^{1/2}}{n}\tilde w_n\left(s\frac{n}{k_n}\right) - s\ .$$

Then (5.3) can be rewritten as

$$\sup_{t\in[0,1]} \left| \frac{n}{k_n^{1/2}} \left(\tilde{\Gamma}_n \left(t\frac{k_n}{n} \right) - t\frac{k_n}{n} \right) - \tilde{W}(t) \right| \to 0 \text{ a.s. } (n \to \infty) \ .$$

Obviously, a similar statement holds true for the right tail empirical process, i.e. we have

$$\sup_{t\in[0,1]} \left| \frac{n}{k_n^{1/2}} \left(1 - \Gamma_n^* \left(1 - t\frac{k_n}{n} \right) - t\frac{k_n}{n} \right) - W^*(t) \right| \to 0 \text{ a.s. } (n \to \infty) \ ,$$

where Γ_n^* is the uniform empirical distribution function on $[1 - k_n/n, 1]$ pertaining to this Skorohod construction and W^* another Wiener process. Writing $tk_n/n = 1 - s$ yields, as $n \to \infty$,

$$\sup_{1-\frac{k_n}{n}\le s\le 1} \left| \frac{n}{k_n^{1/2}} \left(1 - \Gamma_n^*(s) - (1-s) \right) - W^* \left(\frac{n}{k_n}(1-s) \right) \right| \to 0 \text{ a.s. } (n \to \infty) \ .$$

Now setting $s = F(t)$ (if possible) leads to

$$\sup_{F(t)\ge 1-\frac{k_n}{n}} \left| \frac{n}{k_n^{1/2}} \left(1 - \Gamma_n^*(F(t)) - \bar{F}(t) \right) - W^* \left(\frac{n}{k_n}\bar{F}(t) \right) \right| \to 0 \text{ a.s. } (n \to \infty) \ .$$

Finally, defining $F_n^*(t) = \Gamma_n^*(F(t))$, and observing that F_n^* behaves on $[Q(1 - k_n/n), \infty)$ as the empirical distribution function of a random sample of size n from F yields (5.13).

The next application is about asymptotic confidence bands for so-called intermediate quantiles. Define the empirical quantile function Q_n on $[0, 1]$ as the generalized inverse of the one dimensional empirical distribution function F_n, see (1.18). Note that

$$(5.15) \qquad Q_n(t) = X_{i:n}, \ (i-1)/n < t \le i/n, \ i = 1, ..., n \ .$$

(Also, for $t < 0$ we set $Q_n(t) = -\infty$.) For $0 < \alpha < 1$ and $q \in \mathcal{Q}_0^*$ define $c = c(\alpha, q)$ by

$$I\!P(\sup_{0<t\le 1} |W(t)|/q(t) \ge c) = \alpha \ ,$$

where again W is a Wiener process. Then we have for continuous F, $q \in \mathcal{Q}_0^*$ and $\{k_n\}_{n=1}^{\infty}$ satisfying (5.1) the following asymptotic confidence band statement for quantiles in the region $(0, k_n/n]$:

$$(5.16) \qquad I\!P\left(Q_n \left(\frac{tk_n}{n} \left(1 - \frac{cq(t)}{tk_n^{1/2}} \right) \right) \le Q\left(\frac{tk_n}{n} \right) < Q_n \left(\frac{tk_n}{n} \left(1 + \frac{cq(t)}{tk_n^{1/2}} \right) \right), 0 < t \le 1 \right)$$

$$\to 1 - \alpha \ (n \to \infty) \ ,$$

where Q is the (theoretical) quantile function, defined as in the previous application. It should be emphasized that continuity is the only condition on F that is needed for (5.16). Usually results of this type are established under additional (mostly extreme value type) conditions. Even for the definition of the standardized quantile process (see chapter 6), which is often used to generate confidence bands for quantiles, one has to assume that F has a density.

Now we prove (5.16). From (5.4) and (5.2) it follows that with probability tending to $1-\alpha$ $(n \to \infty)$

$$-c \le \frac{n}{k_n^{1/2}} \left(\Gamma_n \left(\frac{tk_n}{n} \right) - \frac{tk_n}{n} \right) / q(t) < c, \;\; 0 < t \le 1 \;.$$

Rewriting this yields

$$\text{(5.17)} \qquad \frac{tk_n}{n} \left(1 - \frac{cq(t)}{tk_n^{1/2}} \right) \le \Gamma_n \left(\frac{tk_n}{n} \right) < \frac{tk_n}{n} \left(1 + \frac{cq(t)}{tk_n^{1/2}} \right), \;\; 0 < t \le 1 \;.$$

Let Γ_n^{-1} be the uniform quantile function (set $\Gamma_n^{-1}(s) = 0$ for $s \le 0$) and observe that for $0 < s, t < 1$ we have

$$\{\Gamma_n(t) < s\} = \{t < \Gamma_n^{-1}(s)\} \;.$$

Hence (5.17) can be written as

$$\text{(5.18)} \qquad \Gamma_n^{-1} \left(\frac{tk_n}{n} \left(1 - \frac{cq(t)}{tk_n^{1/2}} \right) \right) \le \frac{tk_n}{n} < \Gamma_n^{-1} \left(\frac{tk_n}{n} \left(1 + \frac{cq(t)}{tk_n^{1/2}} \right) \right), \;\; 0 < t \le 1 \;.$$

Now assuming without loss of generality that $X_{i:n} = Q(\xi_{i:n})$ and noting that $Q_n = Q \circ \Gamma_n^{-1}$ and that Q is strictly increasing on $(0,1)$ since F is continuous, we see that (5.18) is equivalent to

$$Q_n \left(\frac{tk_n}{n} \left(1 - \frac{cq(t)}{tk_n^{1/2}} \right) \right) \le Q \left(\frac{tk_n}{n} \right) < Q_n \left(\frac{tk_n}{n} \left(1 + \frac{cq(t)}{tk_n^{1/2}} \right) \right), \;\; 0 < t \le 1 \;,$$

which completes the proof.

The last application is in the area of extreme value theory. To be more precise, we consider the estimation of the extreme value index of a distribution in the domain of max-attraction of an extreme value distribution. When doing statistics of extreme values, one often wants to extrapolate far beyond the largest observation, which can not be done properly without conditions on F. We want this conditions on F to be weak and natural.

Let $X_1, X_2, ...$ be a sequence of independent random variables with common distribution function F. We now assume that $X_{n:n}$ (the maximum of $X_1, ..., X_n$), properly standardized, converges weakly to some limiting distribution, i.e. we have

$$\text{(5.19)} \qquad I\!P \left(\frac{X_{n:n} - b_n}{a_n} \le t \right) \to G(t) \quad (n \to \infty) \;,$$

for all continuity points of G, a distribution function. Here $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of numbers, with $a_n > 0, n \in \mathbb{N}$. Just as for weak convergence of means, not every distribution function can arise in the limit. To be more specific, it can be shown that apart from affine transformations

$$G(t) = G_\gamma(t) = \exp(-(1+\gamma t)^{-1/\gamma}),\ \gamma \in \mathbb{R}, t \in \mathbb{R}\ ,$$

where $(1+\gamma t)^{-1/\gamma}$ has to be interpreted as e^{-t} for $\gamma = 0$ and t is such that $1+\gamma t > 0$. Hence G is known apart from the real valued parameter γ. This parameter gives important information about the right tail of F.

The problem we will address here is to estimate γ from the sample $X_1, ..., X_n$. Let k_n be a positive integer less than n. Define for $j = 1, 2$

$$M_n^{(j)} = \frac{1}{k_n} \sum_{i=0}^{k_n-1} (\log X_{n-i:n} - \log X_{n-k_n:n})^j$$

and define the estimator of γ by

$$\hat{\gamma}_n = M_n^{(1)} + 1 - \tfrac{1}{2}\left(1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}}\right)^{-1} .$$

We confine ourselves to strong consistency of $\hat{\gamma}_n$. Assume that (5.19) holds and that $\sup\{t : F(t) < 1\} > 0$. Also assume $\{k_n\}_{n=1}^{\infty}$ satisfies $k_n/n \to 0$ and $k_n/(\log n)^\delta \to \infty$ $(n \to \infty)$ for some $\delta > 0$. Then we have

$$\lim_{n\to\infty} \hat{\gamma}_n = \gamma \text{ a.s.} \tag{5.20}$$

We will not prove (5.20), but only note that the following result is crucial for its proof. Let $0 < k_n < n$ be such that $k_n/(\log n)^\delta \to \infty$ for some $\delta > 0$. Let $F(t) = 1 - t^{-\alpha}$ $(t \geq 1)$ and $\alpha > 2(1+\delta)/\delta$. Then

$$\lim_{n\to\infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \frac{X_{n-i:n}}{X_{n-k_n:n}} = \frac{\alpha}{\alpha - 1} \quad \text{a.s.} \tag{5.21}$$

We will present a sketch of the proof of (5.21), which is based on the following result. Let $\{k_n\}_{n=1}^{\infty}$ be as above (5.21) and $0 \leq a < \delta/(2(1+\delta))$, then

$$\lim_{n\to\infty} \left(\frac{n}{k_n}\right)^{1-a} \int_0^{k_n/n} \frac{\Gamma_n(t) - t}{t^{1+a}} dt = 0 \text{ a.s.} \tag{5.22}$$

We first prove (5.22); for convenience we will use, in addition, $k_n \uparrow, k_n/n \downarrow 0$ $(n \uparrow \infty)$. For $0 < \eta < \delta/(2(1+\delta)) - a$ and with $\nu = a + \eta$ we have

$$(5.23)\qquad \left| \left(\frac{n}{k_n}\right)^{1-a} \int_0^{k_n/n} \frac{\Gamma_n(t)-t}{t^{1+a}} dt \right| \le \left(\frac{n}{k_n}\right)^{1-a} \sup_{0\le t\le k_n/n} \frac{|\Gamma_n(t)-t|}{t^\nu} \int_0^{k_n/n} t^{\eta-1} dt$$

$$= \frac{1}{\eta}\left(\frac{2\log\log n}{k_n}\right)^{1/2} \sup_{t\in[0,1]} \frac{|w_n(t)|}{(2\log\log n)^{1/2} t^\nu}\,.$$

Now this last expression tends to 0 almost surely $(n\to\infty)$, because of (5.12) (note that the sum in Theorem 5.3, part I, is indeed finite) and since $k_n/\log\log n \to \infty$ $(n\to\infty)$.

Now we are prepared to turn to (5.21). Let F_n be the empirical distribution function based on a sample of size n from $F = 1 - I^{-\alpha}, \alpha > 2(1+\delta)/\delta$. Then (5.22), with $a = 1/\alpha$, yields

$$(5.24)\qquad \lim_{n\to\infty} \left(\frac{n}{k_n}\right)^{1-1/\alpha} \int_{(n/k_n)^{1/\alpha}}^{\infty} (1-F_n(t))dt = \frac{1}{\alpha-1} \quad \text{a.s.}$$

It is known that

$$(5.25)\qquad \lim_{n\to\infty} \left(\frac{k_n}{n}\right)^{1/\alpha} X_{n-k_n:n} = 1 \text{ a.s.}$$

Combination of (5.24) and (5.25) yields, for $0 < \varepsilon < 1$,

$$\limsup_{n\to\infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \frac{X_{n-i:n}}{X_{n-k_n:n}} = \limsup_{n\to\infty} \left(\frac{n}{k_n}\right)^{-1/\alpha} \frac{1}{k_n} \sum_{i=0}^{k_n-1} X_{n-i:n}$$

$$= \limsup_{n\to\infty} \left(\frac{n}{k_n}\right)^{-1/\alpha} \frac{n}{k_n} \int_{X_{n-k_n+1:n}}^{\infty} t dF_n(t)$$

$$= \limsup_{n\to\infty} \left(\left(\frac{n}{k_n}\right)^{-1/\alpha} X_{n-k_n+1:n} + \left(\frac{n}{k_n}\right)^{1-1/\alpha} \int_{X_{n-k_n+1:n}}^{\infty} (1-F_n(t))dt \right)$$

$$\le 1 + \limsup_{n\to\infty} \left(\frac{n}{k_n}\right)^{1-1/\alpha} \int_{(n(1-\varepsilon)/k_n)^{1/\alpha}}^{\infty} (1-F_n(t))dt$$

$$= 1 + \frac{(1-\varepsilon)^{-1+1/\alpha}}{\alpha-1} \text{ a.s.}$$

This with a similar lower bound gives (5.21).

Extreme value theory can e.g. be used to determine the heights of sea dikes; the heights have to be such that the probability of a flood is extremely small.

PROBLEMS

5.1 Prove that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}(\rho(w_n, A_m(w_n)) \geq \varepsilon) = 0 \ .$$

(See the proof of Theorem 3.9, where a similar statement for U_n is proved.)

5.2 Prove in two ways that $w_n \Rightarrow W$, as $n \to \infty$, in $(D, \mathcal{D}_\rho, \rho)$:

(i) by using (5.3);

(ii) by using problem 5.1.

5.3 Prove that $q \in \mathcal{Q}_0^*$ implies $\lim_{t\downarrow 0} q(t)/t^{1/2} = \infty$. (See the end of the proof of Theorem 5.1.)

5.4 Show that (5.22) indeed yields (5.24).

6. THE QUANTILE PROCESS AND THE BAHADUR-KIEFER PROCESS

From a statistical point of view the empirical distribution function F_n is an estimator of F. Sometimes it is more important to estimate the quantile function $Q = F^{-1}$, see (1.18); note that e.g. $Q(\frac{1}{2})$ is (roughly) the median of the distribution. This leads us to the theory of quantile processes, a theory that is substantially different from the theory of empirical processes, but very much intertwined with it. In this chapter we will briefly consider these quantile processes and the related Bahadur-Kiefer processes.

We begin with $UN(0,1)$ random variables. In this case we have on $[0,1]$ that $Q = I = F$. This leads in the usual setup to the uniform quantile process defined by

$$(6.1)\qquad V_n(t) = n^{1/2}(\Gamma_n^{-1}(t) - t), \quad t \in [0,1]\ ,$$

where Γ_n^{-1} (see below (5.17)) is the generalized inverse of the uniform empirical distribution function Γ_n. Observe that

$$(6.2)\qquad \Gamma_n^{-1}(t) = \xi_{k:n},\ \text{for}\ t \in \left(\frac{k-1}{n}, \frac{k}{n}\right], \quad k = 1, ..., n\ .$$

This shows once more that Γ_n^{-1} is left-continuous. Establishing weak convergence of V_n is relatively easy.

THEOREM 6.1. We have, with U a Brownian bridge,

$$(6.3)\qquad V_n \Rightarrow U,\ \text{as}\ n \to \infty,\ \text{in}\ (D_L, \mathcal{D}_\rho, \rho)\ ,$$

where D_L is the "left-continuous and right-hand limits" analogue of D.

For the proof of this theorem the following result from analysis is very useful; its proof will not be given.

THEOREM 6.2. For each $n \in I\!N$, let x_n be a nondecreasing function on [0,1] with $x_n(0) = 0$ and $x_n(1) = 1$. Moreover, let y be a continuous function on [0,1] and $\{m_n\}_{n=1}^{\infty}$ a sequence of positive numbers converging to ∞. If as $n \to \infty$,

$$\sup_{t\in[0,1]} |m_n(x_n(t) - t) - y(t)| \to 0\ ,$$

then

$$\sup_{t\in[0,1]} |m_n(x_n^{-1}(t) - t) + y(t)| \to 0\ .$$

PROOF OF THEOREM 6.1. Let $\tilde{U}_n$ and $\tilde{U}$ be as in (3.23) and let $\tilde{\omega} \in \tilde{\Omega}$ such that $\rho(\tilde{U}_n, \tilde{U}) \to 0$. Now applying Theorem 6.2 yields that for the same $\tilde{\omega}$ we have $\rho(\tilde{V}_n, -\tilde{U}) \to 0$, where $\tilde{V}_n = n^{1/2}(\tilde{\Gamma}_n^{-1} - I)$ and $\tilde{\Gamma}_n$ is the uniform empirical distribution function pertaining to $\tilde{U}_n$. Hence

$$\tilde{V}_n \Rightarrow -\tilde{U}, \text{ as } n \to \infty, \text{ in } (D_L, \mathcal{D}_\rho, \rho) \ .$$

Since $V_n \stackrel{d}{=} \tilde{V}_n$ $(n \in I\!N)$ and $U \stackrel{d}{=} -\tilde{U}$, this implies (6.3). □

Observe that we easily obtain from the proof of Theorem 6.1 that

$$\text{(6.4)} \qquad \sup_{t\in[0,1]} |U_n(t) + V_n(t)| \underset{I\!P}{\to} 0, \text{ as } n \to \infty \ .$$

A related useful and easy to prove statement is

$$\text{(6.5)} \qquad V_n = -U_n \circ \Gamma_n^{-1} + n^{1/2}(\Gamma_n \circ \Gamma_n^{-1} - I) \ .$$

We now turn to the general quantile process. Let Q_n be the empirical quantile function as defined above (5.15) and assume that F has a derivative f on $I\!R$. Then the quantile process is defined by

$$\text{(6.6)} \qquad V_{n,F}(t) = n^{1/2} f(Q(t))(Q_n(t) - Q(t)), \ 0 < t < 1 \ .$$

We restrict ourselves to weak convergence of $V_{n,F}$ on a subinterval of $[0,1]$; extension to the whole interval $(0,1)$ is possible, but much more complicated.

THEOREM 6.3. Let $0 \leq a < b \leq 1$. If $f \circ Q$ is positive and continuous on an open subinterval of [0,1] containing $[a,b]$, then there exist $\tilde{V}_{n,F}$, $n \in I\!N$, such that $\tilde{V}_{n,F} \stackrel{d}{=} V_{n,F}$ (on $D_L[a,b]$) and a Brownian bridge $\tilde{U}$ such that

$$\text{(6.7)} \qquad \sup_{a<t<b} |\tilde{V}_{n,F}(t) - \tilde{U}(t)| \to 0 \text{ a.s. } (n \to \infty) \ .$$

PROOF. Let $\tilde{V}_n$ be as in the proof of Theorem 6.1 and denote the $UN(0,1)$ random variables on which $\tilde{V}_n$ is based with $\xi_i, 1 \leq i \leq n$. Set $X_i = Q(\xi_i)$ and note that $\tilde{Q}_n = Q \circ \tilde{\Gamma}_n^{-1}$, cf. below (5.18). Applying the mean-value theorem it follows from (6.6) that

$$\tilde{V}_{n,F}(t) = n^{1/2} f(Q(t))(\tilde{Q}_n(t) - Q(t))$$

$$= n^{1/2} \frac{f(Q(t))}{f(Q(t_n^*))}(\tilde{\Gamma}_n^{-1}(t) - t) = \frac{f(Q(t))}{f(Q(t_n^*))}\tilde{V}_n(t), \ a < t < b \ ,$$

for some t_n^* between t and $\tilde{\Gamma}_n^{-1}(t)$. But

$$\sup_{a<t<b} |t_n^* - t| \leq \sup_{t\in[0,1]} |\tilde{\Gamma}_n^{-1}(t) - t| \to 0 \text{ a.s. } (n \to \infty) \ ,$$

and hence

$$\text{(6.8)} \qquad \sup_{a<t<b} \left| \frac{f(Q(t))}{f(Q(t_n^*))} - 1 \right| \to 0 \text{ a.s. } (n \to \infty) \ .$$

But

$$\sup_{a<t<b} |\tilde{V}_{n,F}(t) + \tilde{U}(t)| \leq \sup_{a<t<b} |\tilde{V}_{n,F}(t) - \tilde{V}_n(t)| + \sup_{a<t<b} |\tilde{V}_n(t) + \tilde{U}(t)|$$

$$\leq \sup_{a<t<b} |\tilde{V}_n(t)| \sup_{a<t<b} \left| \frac{f(Q(t))}{f(Q(t_n^*))} - 1 \right| + \sup_{a<t<b} |\tilde{V}_n(t) + \tilde{U}(t)| \ .$$

From (6.8), the proof of Theorem 6.1 and relabeling $\tilde{U}$ by $-\tilde{U}$, (6.7) now easily follows. □

As far as strong convergence is concerned, let us confine ourselves to considering the uniform case.

THEOREM 6.4. Theorem 3.3 and hence also Theorem 3.1 remain true if U_n is replaced by V_n.

PROOF. Theorem 6.4 follows immediately from an application of Theorem 6.2. □

Let us also briefly consider the quantile analogue of the tail empirical process: the tail quantile process is defined by

$$v_n(t) = \left(\frac{n}{k_n}\right)^{1/2} V_n(t\frac{k_n}{n}), t \in [0,1] \ ,$$

where $\{k_n\}_{n=1}^{\infty}$ is as in (5.1).

THEOREM 6.5. We have, with W a Wiener process,

$$v_n \Rightarrow W, \text{ as } n \to \infty, \text{ in } (D_L, \mathcal{D}_\rho, \rho) \ .$$

PROOF. The proof follows from a modification of Theorem 6.2. (Especially, some care is needed near $t = 1$, since it is not true that $\frac{n}{k_n}\Gamma_n(\frac{k_n}{n}) = 1$ a.s.) □

We now consider the aforementioned Bahadur-Kiefer process, which is simply the sum of the empirical process and the quantile process; again we will restrict ourselves to $UN(0,1)$ distributed random variables. In (6.4) it was observed that

$$\sup_{t\in[0,1]} |U_n(t) + V_n(t)| \underset{\mathbb{P}}{\to} 0, \text{ as } n \to \infty \ .$$

This statement easily carries over, e.g., weak convergence of U_n to weak convergence of V_n. However, from a mathematical point of view "convergence to zero" is an unsatisfactory statement. In the next part of this chapter we will investigate the exact rate of convergence in (6.4). To this end define the uniform Bahadur-Kiefer process by

$$(6.9)\qquad \begin{aligned} R_n(t) &= U_n(t) + V_n(t) \\ & n^{1/2}(\Gamma_n(t) - t) + n^{1/2}(\Gamma_n^{-1}(t) - t) \\ &= n^{1/2}(\Gamma_n(t) + \Gamma_n^{-1}(t) - 2t),\ t \in [0,1]\ . \end{aligned}$$

First the behaviour of $\sup_{t\in[0,1]}|R_n(t)|$ will be considered.

THEOREM 6.6. We have

$$(6.10)\qquad \limsup_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}(\log\log n)^{1/4}} \sup_{t\in[0,1]} |R_n(t)| = 2^{-1/4} \text{ a.s.}$$

THEOREM 6.7. We have

$$(6.11)\qquad \lim_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}} \frac{\sup_{t\in[0,1]}|R_n(t)|}{(\sup_{t\in[0,1]}|U_n(t)|)^{1/2}} = 1 \text{ a.s.}$$

We will omit the long and difficult proof of Theorem 6.7. Observe that U_n may be replaced by V_n in (6.11), since $\sup_{t\in[0,1]}|U_n(t)| = \sup_{t\in[0,1]}|V_n(t)|$; observe also that (6.10) is a consequence of (6.11). Another consequence of Theorem 6.7 is that

$$(6.12)\qquad \frac{n^{1/4}}{(\log n)^{1/2}} \sup_{t\in[0,1]} |R_n(t)| \Rightarrow (\sup_{t\in[0,1]} |U(t)|)^{1/2},\ \text{as } n\to\infty,\ \text{in } (I\!R, \mathcal{B}, \delta_1)\ ,$$

where as usual U denotes a Brownian bridge.

PROOF OF THEOREM 6.6. We will only prove that $2^{-1/4}$ is almost surely an upper bound for the left hand side of (6.10).

From (6.5) we have that

$$R_n - (U_n - U_n \circ \Gamma_n^{-1}) = n^{1/2}(\Gamma_n \circ \Gamma_n^{-1} - I)\ .$$

Analysing the behaviour of the function $\Gamma_n \circ \Gamma_n^{-1}$ yields that

$$\sup_{t\in[0,1]} |R_n(t) - (U_n(t) - U_n(\Gamma_n^{-1}(t)))| = n^{-1/2} \text{ a.s.}$$

Hence it is sufficient to show that

$$(6.13)\qquad \limsup_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}(\log\log n)^{1/4}} \sup_{t\in[0,1]} |U_n(t) - U_n(\Gamma_n^{-1}(t))| \le 2^{-1/4} \text{ a.s.}$$

Let $\varepsilon > 0$ and set $a_n = (1+\varepsilon)\frac{1}{2}(2\log\log n)^{1/2} n^{-1/2}$, then by Theorem 6.4 it follows that the left hand side of (6.13) is almost surely bounded from above by

$$\limsup_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}(\log\log n)^{1/4}}\omega_n(a_n) ,$$

with ω_n as defined in (3.7). Now we can apply Theorem 3.6. Note that

$$(a_n \log(1/a_n))^{1/2} = (1+o(1))(1+\varepsilon)^{1/2}2^{-3/4}(\log\log n)^{1/4}(\log n)^{1/2}n^{-1/4} .$$

Hence

$$\limsup_{n\to\infty} \frac{n^{1/4}}{(\log n)^{1/2}(\log\log n)^{1/4}}\omega_n(a_n) \le \limsup_{n\to\infty} \frac{(1+\varepsilon)^{1/2}2^{-3/4}\omega_n(a_n)}{(a_n\log(1/a_n))^{1/2}}$$

$$\le (1+\varepsilon)^{1/2}2^{-3/4}2^{1/2} = (1+\varepsilon)^{1/2}2^{-1/4} \text{ a.s.}$$

Since $\varepsilon > 0$ is arbitrary, this proves (6.13). □

Finally we briefly consider weak convergence of $R_n(t)$.

THEOREM 6.8. Let $t \in (0,1)$ be fixed, then

$$n^{1/4}R_n(t) \Rightarrow Z|U(t)|^{1/2}, \text{ as } n\to\infty, \text{ in } (I\!R, \mathcal{B}, \delta_1) ,$$

where Z and U are independent, $\mathcal{L}(Z) = \mathcal{N}(0,1)$ and U is a Brownian bridge (hence $\mathcal{L}(U(t)) = \mathcal{N}(0, t(1-t))$).

We will not prove this result, but only use it, in conjunction with (6.12), to note that there is no sequence of positive real numbers $\{b_n\}_{n=1}^{\infty}$, such that b_nR_n converges weakly in $(D, \mathcal{D}_\rho, \rho)$ to a nondegenerate limiting process. (To achieve that $b_nR_n \in D$, we take the right-continuous version of V_n.)

We now discuss some applications of the results in this chapter. We begin with a brief and simple application to location estimation. Define (using slightly different definitions than usual) $Q(\frac{1}{2})$ and $Q_n(\frac{1}{2})$ to be the median of the probability distribution and its estimator the sample median respectively. It is well known that

$$n^{1/2}(Q_n(\tfrac{1}{2}) - Q(\tfrac{1}{2})) \Rightarrow Z/(2f(Q(\tfrac{1}{2}))), \text{ as } n\to\infty ,$$

where Z is a standard normal random variable. Note that this is nothing else than

$$(6.14) \qquad V_{n,F}(\tfrac{1}{2}) \Rightarrow U(\tfrac{1}{2}), \text{ as } n\to\infty .$$

Hence (6.14) easily follows from Theorem 6.3. Moreover, that theorem shows that in (6.14), $\frac{1}{2}$ may be replaced by any $t \in (a,b)$ and that the weak convergence even holds uniformly in $t \in [c,d] \subset (a,b)$.

Let us turn to a longer and less straightforward application to the two-sample situation, see also the last application in chapter 1. Assume for convenience that F_1 and F_2 (and hence Q_1 and Q_2) are continuous and strictly increasing. Observe that $F_1 = F_2$ implies that $Q_2 \circ F_1 = I$. Deviations of $F_1 = F_2$ will be measured by the shift function $\Delta = Q_2 \circ F_1 - I$. (The function Δ is also of interest when $F_2 = F_1(I-\mu)$ or $F_2 = F_1((I-\mu)/\sigma)$, $\mu \in I\!R, \sigma > 0$.) We will estimate Δ by (in the obvious notation) $\Delta_{n_1,n_2} = Q_{2,n_2} \circ F_{1,n_1} - I$, where $Q_{2,n_2} \circ F_{1,n_1}$ is the so-called quantile-quantile plot or, for short, $Q-Q$ plot. The $Q-Q$ (plot) process is now defined by

$$D_{n_1,n_2}(t) = \left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2} f_2(Q_2(F_1(t)))(Q_{2,n_2}(F_{1,n_1}(t)) - Q_2(F_1(t)))$$

$$= \left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2} f_2(Q_2(F_1(t)))(\Delta_{n_1,n_2}(t) - \Delta(t)),\ t \in I\!R\ ,$$

where $f_2 = F_2'$. We have, similar to Theorem 6.3, the following result (here we suppress the tilde in the notation). Let $0 \le a < b \le 1$. If $f_2 \circ Q_2$ is positive and continuous on an open subinterval of $[0,1]$ containing $[a,b]$, then

$$\sup_{Q_1(a)<t<Q_1(b)} |D_{n_1,n_2}(t) - U_{(n_1,n_2)}(F_1(t))| \to 0 \text{ a.s. } (n_1 \to \infty \text{ and } n_2 \to \infty)\ , \tag{6.15}$$

where $U_{(n_1,n_2)}$ denotes a Brownian bridge for all $n_1, n_2 \in I\!N$. Statement (6.15) can, of course, be rewritten as

$$\sup_{a<t<b} |D_{n_1,n_2}(Q_1(t)) - U_{(n_1,n_2)}(t)| \to 0 \text{ a.s. } (n_1 \to \infty \text{ and } n_2 \to \infty)\ .$$

We now prove (6.15). In the obvious notation we have

$$\begin{aligned} D_{n_1,n_2}(t) &= \left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2} f_2(Q_2(F_1(t)))(Q_{2,n_2}(F_{1,n_1}(t)) - Q_2(F_{1,n_1}(t))) \\ &+ \left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2} f_2(Q_2(F_1(t)))\frac{Q_2(F_{1,n_1}(t)) - Q_2(F_1(t))}{F_{1,n_1}(t) - F_1(t)}(F_{1,n_1}(t) - F_1(t)) \\ &= \left(\frac{n_1}{n_1+n_2}\right)^{1/2} \frac{f_2(Q_2(F_1(t)))}{f_2(Q_2(s_n^*))} V_{2,n_2}(F_{1,n_1}(t)) \\ &+ \left(\frac{n_2}{n_1+n_2}\right)^{1/2} f_2(Q_2(F_1(t)))\frac{Q_2(F_{1,n_1}(t)) - Q_2(F_1(t))}{F_{1,n_1}(t) - F_1(t)} U_{1,n_1}(F_1(t)), Q_1(a) < t < Q_1(b)\ , \end{aligned} \tag{6.16}$$

where s_n^* lies between $F_{1,n_1}(t)$ and $\Gamma_{2,n_2}^{-1}(F_{1,n_1}(t))$. Assume that we have Shorohod constructions such that

$$\sup_{t\in[0,1]} |U_{1,n_1}(t) - U_1(t)| \to 0 \text{ a.s. } (n_1 \to \infty)\ , \tag{6.17}$$

$$\sup_{t\in[0,1]} |V_{2,n_2}(t) - U_2(t)| \to 0 \text{ a.s. } (n_2 \to \infty)\ , \tag{6.18}$$

with U_1 and U_2 independent Brownian bridges. Then it follows from (6.18), similar as in the proof of Theorem 6.3 that

$$\sup_{Q_1(a)<t<Q_1(b)} \left| \frac{f_2(Q_2(F_1(t)))}{f_2(Q_2(s_n^*))} V_{2,n_2}(F_{1,n_1}(t)) - U_2(F_{1,n_1}(t)) \right| \to 0 \text{ a.s. } (n_2 \to \infty) ,$$

and from (6.17) and the mean-value theorem we see that

$$\sup_{Q_1(a)<t<Q_1(b)} |f_2(Q_2(F_1(t))) \frac{Q_2(F_{1,n_1}(t)) - Q_2(F_1(t))}{F_{1,n_1}(t) - F_1(t)} U_{1,n_1}(F_1(t))$$

$$-U_1(F_1(t))| \to 0 \text{ a.s. } (n_1 \to \infty) .$$

This, in combination with (6.16), yields

$$\sup_{Q_1(a)<t<Q_1(b)} \left| D_{n_1,n_2}(t) - \left\{ \left(\frac{n_1}{n_1+n_2}\right)^{1/2} U_2(F_{1,n_1}(t)) + \left(\frac{n_2}{n_1+n_2}\right)^{1/2} U_1(F_1(t)) \right\} \right| \tag{6.19}$$

$$\to 0 \text{ a.s. } (n_1 \to \infty \text{ and } n_2 \to \infty) .$$

From Theorem 1.1 and the uniform continuity of U_2 it follows that (6.19) remains true with F_{1,n_1} replaced by F_1. The proof of (6.15) is complete by noting that for $\lambda \in (0,1)$, the process $\lambda^{1/2}U_1 + (1-\lambda)^{1/2}U_2$ is a Brownian bridge.

Finally, we consider briefly a different statistic in the two-sample situation, that is based on probability-probability plots ($P-P$ plots) and can be seen as a two-sample analogue of $\sup_{t\in[0,1]}|R_n(t)|$ in (6.10). Write

$$R_{n_1,n_2} = \sup_{0<t<1} |F_{1,n_1}(Q_{2,n_2}(t)) + F_{2,n_2}(Q_{1,n_1}(t)) - 2t| .$$

If F_1, F_2 are continuous and $F_1 = F_2$ then, with U a Brownian bridge,

$$\left(\frac{n_1 n_2}{n_1+n_2}\right)^{3/4} \left(\log\left(\frac{n_1 n_2}{n_1+n_2}\right)\right)^{-1/2} R_{n_1,n_2} \Rightarrow (\sup_{t\in[0,1]} |U(t)|)^{1/2}, \text{ as } n_1 \to \infty \text{ and } n_2 \to \infty ,$$

cf. (6.12). We do not prove this result, but just mention that it can be used to test the null hypothesis $F_1 = F_2$.

PROBLEMS

6.1 Provide the details for the proof of Theorem 6.4.

6.2 Show that (6.10) is indeed a consequence of (6.11).

6.3 Prove (6.12), using (6.11).

6.4 Let $t \in (0,1)$ be fixed. Show, using Theorem 6.8, that for all $x \in \mathbb{R}$

$$\mathbb{P}(n^{1/4} R_n(t) \leq x) \to \frac{2}{(t(1-t))^{1/2}} \int_0^\infty \Phi\left(\frac{x}{y^{1/2}}\right) \Phi'\left(\frac{y}{(t(1-t))^{1/2}}\right) dy, \text{ as } n \to \infty .$$

6.5 Let the $P-P$ process be defined by

$$E_{n_1,n_2}(t) = \left(\frac{n_1 n_2}{n_1 + n_2}\right)^{1/2} (F_{2,n_2}(Q_{1,n_1}(t)) - F_2(Q_1(t))), t \in [0,1] ;$$

see for notation the application to the $Q-Q$ process above. Assume that F_1, F_2 are continuous and that $F_1 = F_2$. Prove that

$$E_{n_1,n_2} \Rightarrow U, \text{ as } n_1 \to \infty \text{ and } n_2 \to \infty, \text{ in } (D_L, \mathcal{D}_\rho, \rho) ,$$

where U is a Brownian bridge.

7. GENERALIZATIONS

In this last chapter we briefly introduce some processes, that generalize the empirical and quantile processes that we have considered in great detail before. We will not give much more than definitions; for results and proofs the reader is referred to the literature (see the selected references.)

The first situation that we will consider is the random censorship from the right model. Let $X_1, ..., X_n$ be a random sample from a univariate continuous distribution function F and assume that we are interested in estimating F. However, we are not able to observe the X_i, $1 \le i \le n$, but only Z_i and δ_i, $1 \le i \le n$, where $Z_i = X_i \wedge Y_i$, $\delta_i = 1_{(-\infty,0]}(X_i - Y_i)$ and $Y_1, ..., Y_n$ is a random sample, independent of the X_i, from a distribution function G. This model is, e.g., very useful for a variety of problems in biostatistics. It is obvious that the empirical distribution function can not be constructed from the data, hence estimating F is not obvious. Instead of trying to estimate F, we confine ourselves for convenience to estimating $\Lambda = -\log(1-F)$. Besides F the function Λ is another important characteristic of a probability distribution on $\mathbb{R}$, since $\Lambda' = F'/(1-F)$ is the hazard rate, a name explained by the fact that $\Lambda'(t) = \lim_{h \downarrow 0} \mathbb{P}(t < X_1 \le t+h | X_1 > x), t \in \mathbb{R}$. We need some more definitions. Write H for the distribution function of the Z_i and note that $H = 1-(1-F)(1-G)$. The empirical distribution function of the $Z_i, 1 \le i \le n$, is denoted by H_n. Define the corresponding empirical process by

$$U_{n,H}(t) = n^{1/2}(H_n(t) - H(t)),\ t \in \mathbb{R}\ .$$

We write $H^1(t) = \mathbb{P}(Z_1 \le t, \delta_1 = 1) = \int_{-\infty}^{t} 1 - G(s-)dF(s)$. Note that in general $\lim_{t\to\infty} H^1(t) < 1$. The empirical-distribution-type function pertaining to H^1 is defined by

$$H_n^1(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_i 1_{(-\infty,t]}(Z_i),\ t \in \mathbb{R}\ , \tag{7.1}$$

and the empirical process by

$$U_{n,H^1}(t) = n^{1/2}(H_n^1(t) - H^1(t)),\ t \in \mathbb{R}\ .$$

Observe that

$$\Lambda(t) = -\log(1-F(t)) = \int_{-\infty}^{t} \frac{1}{1-F(s)} dF(s)$$

$$= \int_{-\infty}^{t} \frac{1}{1-F(s)} \frac{1-G(s-)}{1-G(s-)} dF(s) = \int_{-\infty}^{t} \frac{1}{1-H(s-)} dH^1(s), t < G^{-1}(1)\ .$$

Hence $\Lambda(t)$ can be estimated by

$$\Lambda_n(t) = \int_{-\infty}^{t} \frac{1}{1-H_n(s-)} dH_n^1(s),\ t \in \mathbb{R}\ ,$$

since, in contrast to F_n, the functions H_n and H_n^1 can be constructed from the data. This leads to the empirical cumulative hazard process

$$T_n(t) = n^{1/2}(\Lambda_n(t) - \Lambda(t)),\ t \in \mathbb{R}\ .$$

It is now easy to show that

$$(7.2) \qquad T_n(t) = \int_{-\infty}^{t} \frac{1}{1 - H_n(s-)} dU_{n,H^1}(s) + \int_{-\infty}^{t} \frac{U_{n,H}(s)}{1 - H_n(s-)} d\Lambda(s),\ t \leq Z_{n:n}\ .$$

Without going into details, we conclude the exposition on this model by mentioning that (7.2) can be used, in conjunction with the joint weak convergence of $U_{n,H}$ and U_{n,H^1}, to derive weak convergence of T_n. With a bit more effort, properties of the process $n^{1/2}(\hat{F}_n - F)$ can be derived along these lines; here $\hat{F}_n$ denotes the so-called product-limit estimator, a natural generalization of the empirical distribution function to this situation.

Up till now we mainly considered the (uniform) empirical process on $\mathbb{R}$ (or [0,1]), but in chapters 2 and 1 we indexed the multivariate (or even more general) empirical probability measure by quadrants or rectangles and the more general Vapnik-Cervonenkis classes, respectively. Indexing the multivariate empirical process by quadrants is often felt as somewhat unnatural. Therefore we mention that we may extend our theory to multivariate empirical processes indexed by closed rectangles with sides parallel to the coordinate axes. The basic idea to achieve this is a modification of the "point representation of rectangles" as presented in the proof of Theorem 2.2. Occasionally proofs are more complicated or results appear to depend on the dimension, when $d \geq 2$. Note that when $d \geq 2$, we do not have results that allow us (if F is continuous) to restrict ourselves to the uniform distribution (cf., e.g., Theorem 1.4).

For more general indexing sets, like, e.g., the Vapnik-Cervonenkis classes of chapter 1 (and for more general spaces than $\mathbb{R}^d$), we have to resort to different techniques like, e.g., symmetrization; see the second proof of Theorem 1.1. In this case, the empirical process is, of course, defined by

$$(7.3) \qquad n^{1/2}(P_n(A) - P(A)),\ A \in \mathcal{A}\ .$$

Since $P_n(A)$ is the average of the $1_A(X_i)$, $1 \leq i \leq n$, one can also index the empirical measure by functions by simply replacing the indicator functions 1_A by a more general function g. This leads to the empirical process, indexed by functions

$$(7.4) \qquad n^{1/2}(P_n(g) - P(g)),\ g \in \mathcal{G}$$

where $P_n(g) = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$, $P(g) = \int_{\$} g dP$ and $\mathcal{G}$ is a certain class of functions. The theory of the empirical processes in (7.3) and (7.4) is well developed, e.g. weak convergence can be defined properly and proved under a variety of natural conditions (like VC for the process in (7.3)) on the classes $\mathcal{A}$ and $\mathcal{G}$.

Another possible generalization of the empirical process is to relax the assumption that the random variables $X_1, ..., X_n$ are independent and identically distributed. One can consider the empirical process of independent, but non-identically distributed random variables.

Denote in that case the distribution function of X_i by $F_{(i)}, 1 \leq i \leq n$. Define F_n as usual and set $\bar{F}_{(n)} = \frac{1}{n}\sum_{i=1}^{n} F_{(i)}$, then the empirical process is defined by

$$n^{1/2}(F_n(t) - \bar{F}_{(n)}(t)),\ t \in I\!R\ .$$

When the X_i are identically distributed, but not independent the definition of the empirical process is obvious, but the results are not. They depend on the dependence structure of the X_i. We mention as examples the empirical process based on (uniform) 1-spacings or k-spacings (see the last application of chapter 3), the empirical process of U-statistic structure or the empirical process of $X_i = \sum_{k \in Z\!\!Z} a_k Z_{i-k}, 1 \leq i \leq n$, where the $Z_j, j \in Z\!\!Z$, are independent and identically distributed random variables and the a_k are certain non-negative numbers. (The X_i form a so-called general linear process.) Apart from the above specific dependence structures, one can also study the empirical process under various mixing conditions.

Next, we consider the generalization to the compound empirical process, a process which can be dealt with easily, using the techniques we presented to deal with the ordinary empirical process. Let $\langle X_1, Y_1\rangle, ..., \langle X_n, Y_n\rangle$ be independent and identically distributed random vectors in $I\!R^2$, where typically X_i and Y_i are dependent. Now set

$$F_{n,c}(t) = \frac{1}{n}\sum_{i=1}^{n} Y_i 1_{(-\infty,t]}(X_i),\ t \in I\!R\ ,$$

and

$$F_c(t) = I\!E Y_i 1_{(-\infty,t]}(X_i),\ t \in I\!R\ .$$

Define the compound empirical process by

$$n^{1/2}(F_{n,c}(t) - F_c(t)),\ t \in I\!R\ . \tag{7.5}$$

Note that H_n^1 defined in (7.1) is a special case of $F_{n,c}$ above. However, one can deal more easily with H_n^1 since the δ_i are 0-1 random variables. The process in (7.5) is useful for studying nonparametric regression. Writing

$$m(t) = I\!E(Y_i|X_i = t),\ t \in I\!R\ ,$$

for the regression function, we have that

$$F_c(t) = \int_{-\infty}^{t} m(s) dF_X(s),\ t \in I\!R\ ,$$

where F_X is the distribution function of the X_i. This relation leads to nonparametric estimators $\hat{m}_n$ of m. Properties of the process in (7.5) can then be used to derive results, like e.g. strong uniform consistency, for $\hat{m}_n$.

We conclude by defining a generalization of the quantile process in (6.6). In the setup above (1.13), we write

$$Q(t) = \inf\{\lambda(A) : P(A) \geq t,\ A \in \mathcal{A}\},\ 0 < t < 1\ ;$$

$$Q_n(t) = \inf\{\lambda(A) : P_n(A) \geq t,\ A \in \mathcal{A}\},\ 0 < t < 1\ ,$$

where λ is a real-valued function on the class of sets $\mathcal{A} \subset \mathcal{E}$. Writing $F = Q^{-1}$, the generalized quantile process is defined by

$$n^{1/2}F'(Q(t))(Q_n(t) - Q(t)),\ 0 < t < 1\ .$$

Note that the definitions of Q_n and Q offer a flexible technique to summarize properties of multivariate (or more general) data and probability distributions, respectively.

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