# Portfolio Diversification Effects of Downside Risk

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ABSTRACT
Risk managers use portfolios to diversify away the unpriced risk of individual securities. In this article we compare the benefits of portfolio diversification for downside risk in case returns are normally distributed with the case of fat-tailed distributed returns. The downside risk of a security is decomposed into a part which is attributable to the market risk, an idiosyncratic part, and a second independent factor. We show that the fat-tailed-based downside risk, measured as value-at-risk (VaR), should decline more rapidly than the normal-based VaR. This result is confirmed empirically.

KEYWORDS: diversification, portfolio decomposition, value-at-risk

Risk managers use portfolios to diversify away the unpriced risk of individual securities. This topic has been well studied for global risk measures like the variance [see, e.g., the textbook by Elton and Gruber (1995, chap.4)]. In this article we study the benefits of portfolio diversification with respect to an extreme downside risk measure known as the zeroth lower partial moment and its inverse; where the inverse of the zeroth lower partial moment is better known as the value-at-risk VaR measure. Choice theoretic considerations for this risk measure are offered in Arzac and Bawa's (1977) analysis of the safety-first criterion. In Gourieroux, Laurent, and Scaillet (2000), the implications under the assumption of normally distributed returns are investigated, while Jansen, Koedijk, and de Vries (2000) implement the safety-first criterion for heavy-tailed distributed returns. There is some concern in the literature that the VaR measure lacks subadditivity as a global risk measure. As a measure for the downside risk,

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however, the VaR exhibits subadditivity if one evaluates this criterion sufficiently deep in the tail area.[1]

The portfolio diversification effects for the downside risk are evaluated in terms of the diversification speed. The diversification speed is measured in two different ways. Let the "VaR diversification speed" be the rate at which the VaR changes as the number of assets $k$ included in the portfolio increases. Usually the safety-first criterion and the VaR criterion are evaluated at a fixed probability level. It is also possible to do the converse analysis by fixing the VaR level and letting the probability level change as the number of assets $k$ increases. This gives what we term the "diversification speed of the risk level." We will study both concepts.

Much of the theoretical literature in finance presumes that the returns are normally distributed. For a host of questions, this is a reasonable assumption to make. Empirically it is well known that the return distributions have fatter tails than the normal [see, e.g., Jansen and De Vries (1991)]. For the downside risk measures, this data feature turns out to make a crucial difference. The diversification speeds are shown to be quite different for the cases of the normal and the fat-tailed distributions. The VaR diversification speed is higher for the class of (finite variance) fat-tailed distributions in comparison to the normal distribution, but is lower with respect to the diversification speed of the risk level. The intuition for this result is as follows. Start with the latter result. The tails of the normal density go down exponentially fast, while the tails of fat-tailed distributions decline at a power rate (this is the defining characteristic of these distributions). Since an exponential function eventually beats any power, it stands to reason that the diversification speed of the risk level under normality is larger. The VaR diversification speed measures the speed in terms of quantiles, which are the inverse of the probabilities. Taking the inverse reverses the diversification speed.

Consider, for example, the case of the normal versus Student's $t$-distributed returns with $v$ degrees of freedom. It is well known that the VaR diversification speed for the normal distribution follows the square root rule. In contrast, the Student's $t$ VaR diversification speed is $1 - 1/v$. This is greater than $1/2$ if $v > 2$ (guaranteeing a finite variance). This intuition is made rigorous below by means of the celebrated Feller convolution theorem for heavy-tailed (i.e., regularly varying) distributions.

For the empirical counterpart of this analysis, we briefly review the semiparametric approach to estimating the (extreme) downside risk. The heavy-tail feature is captured by a Pareto distribution-like term, of which one needs to estimate the tail index (the equivalent of the degrees of freedom $v$ in case of Student's law) and a scale coefficient. We consider estimation by means of a pooled dataset on the basis of the assumption that the tail indexes of the different securities and risk components are equal. We do allow for heterogeneity of the scale coefficients, though. Most securities' distributions display equal hyperbolic tail coefficients, but do differ considerably in terms of their scale coefficients [see Hyung and de Vries (2002)].

[1] At least this holds for the normal distribution and the class of fat-tailed distributions investigated in this article.

Within this framework it is possible to calculate the diversification effects beyond the sample range and for hypothetically larger portfolios if we make some assumptions regarding the market model betas and scale coefficients of the orthogonal risk factors. The diversification speeds are analyzed graphically.

We start our essay by reviewing the Feller's convolution theorem for distributions with heavy tails. Subsequently we study the diversification problem in more detail by adding the market factor. The relevance of the theoretical results for the downside risk portfolio diversification question is demonstrated by an application to Standard & Poor's (S&P) stock returns.

## 1 DIVERSIFICATION EFFECTS AND THE FELLER CONVOLUTION THEOREM

In this section we only consider securities that are independently distributed. In the next section this counterfactual assumption, as least as far as equities are concerned, is relaxed by allowing for common factors. Let $R_i$ denote the logarithmic return of the $i$th security. Suppose the $\{R_i\}$ are generated by a distribution with heavy tails in the sense of regular variation at infinity. Thus, far from the origin, the Pareto term dominates:

$$\Pr\{R_i \leq -x\} = A_i x^{-\alpha}[1 + o(1)],\ \alpha > 0,\ A_i > 0 \tag{1}$$

as $x \to \infty$. The Pareto term implies that only moments up to $\alpha$ are bounded and hence the informal terminology of heavy tails. In contrast, the normal distribution has all moments bounded thanks to the exponential tail shape. Distributions like the Student's $t$, Pareto, and nonnormal sum-stable distributions all have regularly varying tails. Downside risk measures like the VaR, that is, at the desired probability level $\delta$: $\Pr\{R_i \leq -\mathrm{VaR}\} = \delta$, directly pick up differences in tail behavior.

An implication of the regular variation property is the simplicity of the tail probabilities for convoluted data. Suppose the $\{R_i\}$ are generated by a heavy-tailed distribution that satisfies Equation (1). From Feller's theorem (1971, VIII.8), the distribution of the $k$ sum satisfies[2]

$$\Pr\left\{\sum_{i=1}^{k} R_i \leq -x\right\} = kAx^{-\alpha}[1 + o(1)],\ \text{as } x \to \infty.$$

From this, one can derive the diversification effect for the equally weighted portfolio $\overline{R} = \frac{1}{k}\sum_{i=1}^{k} R_i$ [see Dacorogna et al. (2001)]. The following first-order approximation for the equally weighted portfolio diversification effect regarding the downside risk obtains[3]

[2] Note that in this analysis $x \to \infty$, while $k$ is a fixed number.

[3] Note that this diversification result only holds as $x \to \infty$. Garcia, Renault, and Tsafack (2003) show that for symmetric stable distributions, the diversification result applies anywhere below the median. This has to do with the fact that the sum stable distributions are self-additive throughout their support, while this only applies in the tail region for the class of fat-tailed distributions.

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{1-\alpha} A x^{-\alpha}. \tag{2}$$

Under the heterogeneity of the scale coefficients $A_i$, the equivalent of Equation (2) reads

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha} \left(\sum_{i=1}^{k} A_i\right) x^{-\alpha}. \tag{3}$$

To summarize, if at a constant VaR level $x$, one increases the number $k$ of securities included in the portfolio, this decreases the probability of loss by $k^{1-\alpha}$ [see Equation (2)].

The other case is where the $R_i$ are independent standard normally distributed

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \sim N(0, \frac{1}{k}).$$

The following is the equivalent of Equation (1) for the normal distribution

$$\Pr\{R_i \leq -x\} = \frac{1}{x}\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)[1 + o(1)] \text{ as } x \to \infty.$$

For the equally weighted portfolio it thus holds

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} = \Pr\left\{\frac{1}{\sqrt{k}} R_i \leq -x\right\} \simeq \frac{1}{x\sqrt{k}}\frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}kx^2). \tag{4}$$

It follows that under normality

$$\frac{d \ln \Pr}{d \ln k} \simeq -\frac{1}{2} - \frac{1}{2}x^2 k, \tag{5}$$

while under the fat-tail model from Equation (2),

$$\frac{d \ln \Pr}{d \ln k} \simeq 1 - \alpha. \tag{6}$$

Hence, for sufficiently high but fixed $k$, the normal distribution implies a higher diversification speed of the risk level.

Next, consider holding the probability constant but letting the VaR level change, which is the typical case considered under the safety-first criterion, to determine the VaR diversification speed. Thus, in the case of the normal model, we are interested in comparing VaR levels $t$ and $s$ such that

$$\Pr\{R_i \leq -t\} = \Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -s\right\} = \Pr\left\{\frac{1}{\sqrt{k}} R_i \leq -s\right\}. \tag{7}$$

Using the additivity properties of the normal distribution, or equivalently using Equation (4) on both sides of Equation (7), it is immediate that

$$s = \frac{t}{\sqrt{k}}$$

so that the normal-based VaR diversification speed reads

$$\frac{d \ln s}{d \ln k} = -\frac{1}{2}. \tag{8}$$

For the fat-tailed model, the equivalent of Equation (7) is

$$A_i t^{-\alpha} = \Pr\{R_i \leq -t\} = \Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -s\right\} = k^{-\alpha}\left(\sum_{i=1}^{k} A_i\right) s^{-\alpha}.$$

Solving for $s$ gives

$$s = \frac{t}{k}\left(\frac{\sum_{i=1}^{k} A_i}{A_i}\right)^{1/\alpha}.$$

Furthermore, if the scale coefficients are identical, this simplifies to

$$s = \frac{t}{k^{1-1/\alpha}}.$$

So that if $\alpha > 2$, that is, when the variance exists,

$$\frac{d \ln s}{d \ln k} = -(1 - \frac{1}{\alpha}) < -\frac{1}{2}. \tag{9}$$

Compare Equation (9) to Equation (8). If $\alpha > 2$, then the VaR diversification speed is higher for fat-tailed distributed returns than if the returns were normally distributed.

## 2 DIVERSIFICATION EFFECTS IN FACTOR MODELS

We relax the assumption of independence between security returns and allow for nondiversifiable market risk. The market risk reduces the benefits from diversification to the elimination of the idiosyncratic component of the risk. First consider a single index model in which all idiosyncratic risk is assumed independent from the market risk $R$,

$$R_i = \beta_i R + Q_i, \tag{10}$$

and where $R$ is the (excess) return on the market portfolio, $\beta_i$ is the amount of market risk, and $Q_i$ is the idiosyncratic risk of the return on asset $i$. The idiosyncratic risk may be diversified away fully in arbitrarily large portfolios and hence is not priced. But the cross-sectional dependence induced by common market risk factors has to be held in any portfolio.

We apply Feller's theorem again for deriving the benefits from cross-sectional portfolio diversification in this single index model. Consider an equally weighted portfolio of $k$ assets. Let $\bar{\beta} = \frac{1}{k}\sum_{i=1}^{k}\beta_i$. The case of unequally weighted portfolios is but a minor extension left to the reader for consideration of space. In this single index model, the $Q_i$ are cross-sectionally independent and, moreover, are independent from the market risk factor $R$. Suppose, in addition, that the $Q_i$ satisfy $\Pr\{Q_i \leq -x\} \approx A_i x^{-\alpha}$ for all $i$, and that $\Pr\{R \leq -x\} \approx A_r x^{-\alpha}$. The diversification benefits from the equally weighted portfolio regarding the downside risk measure for the case of homogeneous scale coefficients $A_i = A$ then follow as

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} = k^{1-\alpha} A x^{-\alpha}[1 + o(1)] + \overline{\beta}^{\alpha} A_r x^{-\alpha}[1 + o(1)], \tag{11}$$

as $x \to \infty$. If the scale coefficients are heterogeneous, the equivalent of Equation (11) reads

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha}\left(\sum_{i=1}^{k} A_i\right)x^{-\alpha} + \overline{\beta}^{\alpha} A_r x^{-\alpha}. \tag{12}$$

In large portfolios one should see that almost all downside risk is driven by the market factor, if $\alpha > 1$,

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx \overline{\beta}^{\alpha} A_r x^{-\alpha}$$

for large, but finite $k$.

In general, one finds the single index model does not hold exactly due to the fact that $\mathrm{cov}[Q_i, Q_j]$ is typically nonzero for off-diagonal elements as well. Thus, though the $Q_i$ may be independent from the market risk factor $R$ (they are uncorrelated with $R$ by construction), they are typically not cross-sectionally independent from each other. This case is usually referred to as the market model. For example, let there be one other common factor $F$. This factor is assumed independent from $R$, but the $\mathrm{cov}[Q_i, F]/\mathrm{cov}[F, F] = \tau_i$. Let $\overline{\tau} = \frac{1}{k}\sum_{i=1}^{k}\tau_i$, and assume that $\Pr\{F \leq -x\} \approx A_f x^{-\alpha}$. Then, by analogy with the foregoing results,

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha}\left(\sum_{i=1}^{k} A_i\right)x^{-\alpha} + \overline{\beta}^{\alpha} A_r x^{-\alpha} + \overline{\tau}^{\alpha} A_f x^{-\alpha}. \tag{13}$$

To study the case of nonidentical $\alpha$ in Equation (12), one has to consider two cases:

**Case 1**: $\alpha_r = \alpha_1 = \ldots = \alpha_j < \alpha_{j+1} \leq \alpha_{j+2} \leq \ldots \leq \alpha_k$.

**Case 2**: $\alpha_1 = \ldots = \alpha_j < \alpha_{j+1} \leq \alpha_{j+2} \leq \ldots \leq \alpha_k$ and $\alpha_r > \alpha_1$.

Here, $\alpha_r$ stands for the tail index of the market portfolio return, and the $\alpha_\mathrm{i}$ are the indices of the idiosyncratic parts of the security $i$ return. Then corresponding expressions to Equation (12) are for Case 1,

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha_r}\left(\sum_{i=1}^{j} A_i\right) x^{-\alpha_r} + \overline{\beta}^{\alpha_r} A_r x^{-\alpha_r},$$

and for Case 2,

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha_1}\left(\sum_{i=1}^{j} A_i\right) x^{-\alpha_1}.$$

Next, consider holding the probability constant, but let the VaR level change in Equation (12) as the number of assets $k$ increases. From Equation (12), we have

$$\Pr\left\{\frac{1}{k}\sum_{i=1}^{k} R_i \leq -x\right\} \approx k^{-\alpha}\left[\sum_{i=1}^{k} A_i + \left(\sum_{i=1}^{k} \beta_i\right)^{\alpha} A_r\right] x^{-\alpha}.$$

By first-order inversion [cf. De Bruijn's theorem in Bingham, Goldie, and Teugels (1987)] one obtains

$$\mathrm{VaR} = x = \frac{1}{k}\left[\sum_{i=1}^{k} A_i + \left(\sum_{i=1}^{k} \beta_i\right)^{\alpha} A_r\right]^{1/\alpha} \bar{p}^{-1/\alpha}, \tag{14}$$

and where $\bar{p}$ is the fixed probability level. With homogeneous scale coefficients, we may simplify this to

$$\mathrm{VaR} = \frac{1}{k^{1-1/\alpha}}\left[A + \frac{\left(\sum_{i=1}^{k} \beta_i\right)^{\alpha}}{k} A_r\right]^{1/\alpha} \bar{p}^{-1/\alpha}.$$

This should be compared with the results from the previous section on the VaR diversification speed, where the part stemming from the market factor was absent. In particular, we find

$$\frac{d \ln \mathrm{VaR}}{d \ln k} = -1 + \frac{1}{\alpha}\frac{A}{A + \frac{\left(\sum_{i=1}^{k} \beta_i\right)^{\alpha}}{k} A_r},$$

which is smaller, that is, gives a higher speed, than the simple $-1 + 1/\alpha$ from before.

## 3 ESTIMATION BY POOLING

To investigate the relevance of the above downside risk diversification theory, we need to estimate the various downside risk components. To explain the details of the estimation procedure, consider again the simple setup in Equation (3). To be able to calculate the downside risk measure, one needs estimates of the tail index $\alpha$ and the scale coefficients $A_i$. A popular estimator for the inverse of the tail index is Hill's (1975) estimator. If the only sources of heterogeneity are the scale coefficients, one can pool all return series. Let $\{R_{11}, \ldots, R_{1n}, \ldots, R_{k1}, \ldots, R_{kn}\}$ be the sample of returns. Denote by $Z_{(i)}$ the $i$th descending order statistic from $\{R_{11}, \ldots, R_{1n}, \ldots, R_{k1}, \ldots, R_{kn}\}$. If we estimate the left tail of the distribution, it is understood that we take the losses (reverse signs). The Hill estimator reads

$$\widehat{1/\alpha} = \frac{1}{m}\sum_{i=1}^{m} \ln\left(Z_{(i)}\right) - \ln\left(Z_{(m+1)}\right). \tag{15}$$

This estimator requires a choice of the number of the highest-order statistics $m$ to be included, that is, one needs to locate the start of the tail area. We implemented the subsample bootstrap method proposed by Danielsson et al. (2000) to determine $m$. The estimator for the scale $A$ when $A_i = A$ for all $i$ is

$$\hat{A} = \frac{m}{kn}\left(Z_{(m+1)}\right)^{\hat{\alpha}}.$$

Note that $m/nk$ is the empirical probability associated with $Z_{(m+1)}$, and the estimator $\hat{A}$ follows intuitively from Equation (1). Under the heterogeneity of $A_i$ one takes

$$\hat{A}_i = \frac{m_i}{n}\left(Z_{(m+1)}\right)^{\hat{\alpha}},$$

where $m_i$ is such that

$$R_{i(1)} \geq \ldots \geq R_{i(m_i)} \geq Z_{(m+1)} \geq R_{i(m_i+1)} \geq \ldots \geq R_{i(n)}.$$

Note that $\sum_{i=1}^{k} m_i = m$. This implies that by the pooling method we obtain exactly the same portfolio probabilities whether or not one assumes (counterfactually incorrect) identical or heterogeneous scale coefficients, since

$$\begin{aligned} k^{-\hat{\alpha}}\left(\sum_{i=1}^{k}\hat{A}_i\right)x^{-\hat{\alpha}} &= k^{-\hat{\alpha}}\left(\sum_{i=1}^{k}\frac{m_i}{n}\left(Z_{(m+1)}\right)^{\hat{\alpha}}\right)x^{-\hat{\alpha}} \\ &= k^{-\hat{\alpha}}\frac{\left(\sum_{i=1}^{k} m_i\right)}{n}\left(Z_{(m+1)}\right)^{\hat{\alpha}}x^{-\hat{\alpha}} \\ &= k^{1-\hat{\alpha}}\hat{A}x^{-\hat{\alpha}}. \end{aligned}$$

We can adapt this pooling method to the market model with little modification. Pooling the series $\{R\}$,$\{Q_1\}, \ldots$,$\{Q_k\}$, one can use the same procedure as in the case of cross-independence.[4] For the estimation of the tail index one again uses Equation (15), where in this case $\{Z\} = \{R_{r1}, \ldots, R_{rn}, Q_{11}, \ldots, Q_{1n}, \ldots, Q_{k1}, \ldots, Q_{kn}\}$. Estimators for the scales are

$$\hat{A}_i = \frac{m_i}{n} \left(Z_{(m+1)}\right)^{\hat{\alpha}}, i = 1, \ldots, \ k \text{ and } r,$$

where $m_i$ is such that

$$X_{i(1)} \geq \ldots \geq X_{i(m_i)} \geq Z_{(m+1)} \geq X_{i(m_i+1)} \geq \ldots \geq X_{i(n)},$$

where $X_i$ can be $R$ or $Q_i$.

In case the tail indexes differ across securities and risk factors, the above can be easily adapted to estimation on individual series. There is, however, considerable evidence that the tail indexes are comparable for equities from the S&P 500 index [see, e.g., Jansen and De Vries (1991) and Hyung and De Vries (2002)]. Therefore we decided to proceed on the basis of the assumption that the tail indexes are equal.

## 4 EMPIRICAL ANALYSIS OF THE DIVERSIFICATION SPEED

We now apply our theoretical results to the daily returns of a set of stocks. In order to estimate the parameters of the market model we choose the Standard & Poor's 500 index as a representation of the market factor. This is certainly not the market portfolio as in the capital asset pricing model (CAPM); nevertheless, the S&P 500 index represents about 80% of the total market capitalization. To see the effects of portfolio diversification, we choose 15 stocks arbitrarily from the S&P 100 index in March 2001. We use the daily returns (close-to-close data), including cash dividends. The data were obtained from Datastream. The data span runs from January 2, 1980, through March 6, 2001, giving a sample size of $n = 5{,}526$. Thus more than 20 years of daily data are considered, including the short-lived 1987 crash. All results are in terms of the excess returns above the risk-free interest rate (three-month U.S. Treasury bills).

The summary statistics for each stock return series and the market factor are given in Table 1. On an annual basis, the excess returns hover around 7.5% and have comparable second moments. The excess returns all exhibit considerably higher than normal kurtosis. This latter feature is also captured by the estimates of the tail index $\alpha$ in Table 2. In this table we report tail index and scale estimates using the individual series, counter to the pooling method outlined above. This is done in order to show that the tail indexes are indeed rather similar, while there is considerable variation in the scales. This motivates the single-tail index, heterogeneous scale model implemented in the other tables. Table 2 also gives the beta estimates for the market model.

[4] The determination of the parameters $\beta_i$ and the residuals $Q_i$ entering in the definition of the market model is done by regressing the stock returns on the market return. The coefficient $\beta_i$ is thus given by the ordinary least squares estimator, which is consistent as long as the residuals are white noise and have zero mean and finite variance. The idiosyncratic noise $Q_i$ is obtained by subtracting $\beta_i$ times the market return to the stock return.

**Table 1** Selected stocks and summary statistics of excess returns.

| Series | Name | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ |
|---|---|---|---|---|---|
| m | S&P 500 Index | .0747 | 2.52 | −2.31 | 55.49 |
| 1 | ALCOA | .0707 | 4.84 | −0.26 | 13.39 |
| 2 | AT&T | .0392 | 4.33 | −0.35 | 16.41 |
| 3 | Black & Decker | −.0168 | 5.61 | −0.32 | 10.57 |
| 4 | Campbell Soup | .0897 | 4.37 | 0.28 | 9.06 |
| 5 | Disney (WALT) | .0981 | 4.86 | −1.30 | 29.82 |
| 6 | Entergy | .0454 | 4.06 | −0.97 | 23.66 |
| 7 | General Dynamics | .0764 | 4.53 | 0.26 | 10.24 |
| 8 | Heinz HJ | .0968 | 3.99 | 0.11 | 6.35 |
| 9 | Johnson & Johnson | .1053 | 4.08 | −0.32 | 9.45 |
| 10 | Merck | .1212 | 3.96 | −0.03 | 6.31 |
| 11 | Pepsico | .1170 | 4.43 | −0.04 | 7.82 |
| 12 | Ralston Purina | .1077 | 4.08 | 0.70 | 15.41 |
| 13 | Sears Roebuck | .0542 | 4.91 | −0.24 | 16.83 |
| 14 | United Technologies | .0851 | 4.19 | −0.10 | 6.83 |
| 15 | Xerox | −.0423 | 5.48 | −1.78 | 33.74 |

Observations cover January 1, 1980–March 6, 2001, giving 5,526 daily observations. The $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_4$ denote the sample mean, standard error, skewness, and kurtosis of annualized excess returns, respectively. The estimates are reported in terms of the excess returns above the risk-free interest rate (U.S. three-month Treasury bill).

**Table 2** Left-tail parameter estimates.

| Series | $\alpha$ | $A$ | $m$ |
|---|---|---|---|
| $R_m$ | 2.963 | 2.522 | 298 |
| 1 | 3.789 | 110.117 | 113 |
| 2 | 2.785 | 7.953 | 289 |
| 3 | 3.220 | 58.601 | 136 |
| 4 | 3.505 | 48.766 | 68 |
| 5 | 2.549 | 6.211 | 496 |
| 6 | 1.981 | 1.339 | 682 |
| 7 | 3.218 | 27.687 | 140 |
| 8 | 3.404 | 25.811 | 197 |
| 9 | 3.377 | 23.663 | 292 |
| 10 | 4.035 | 104.724 | 62 |
| 11 | 3.789 | 103.171 | 71 |
| 12 | 3.136 | 14.106 | 190 |
| 13 | 3.166 | 28.244 | 256 |
| 14 | 4.335 | 288.036 | 66 |
| 15 | 2.098 | 2.999 | 537 |

The values in columns $\alpha$, A, and m are, respectively, the tail index, the scale parameter, and the estimated optimal number of order statistics.

**Table 3** Left-tail parameter estimates.

| | Excess returns | | | Residuals | |
|---|---|---|---|---|---|
| Series | $A$ | $m$ | $\beta$ | $A$ | $m$ |
| T | 23.0 | 1609 | – | 19.6 | 1021 |
| $R_m$ | – | – | 1 | 4.3 | 15 |
| 1 | 26.2 | 122 | 0.877 | 24.7 | 86 |
| 2 | 19.5 | 91 | 0.929 | 15.2 | 53 |
| 3 | 46.4 | 216 | 0.938 | 42.2 | 147 |
| 4 | 22.7 | 106 | 0.719 | 19.5 | 68 |
| 5 | 24.0 | 112 | 1.012 | 22.1 | 77 |
| 6 | 14.4 | 67 | 0.475 | 14.9 | 52 |
| 7 | 25.3 | 118 | 0.710 | 25.0 | 87 |
| 8 | 16.3 | 76 | 0.640 | 14.9 | 52 |
| 9 | 13.9 | 65 | 0.927 | 10.6 | 37 |
| 10 | 15.7 | 73 | 0.854 | 11.5 | 40 |
| 11 | 24.2 | 113 | 0.867 | 18.7 | 65 |
| 12 | 15.0 | 70 | 0.669 | 16.4 | 57 |
| 13 | 29.0 | 135 | 1.074 | 17.5 | 61 |
| 14 | 20.2 | 94 | 0.895 | 13.2 | 46 |
| 15 | 32.4 | 151 | 0.949 | 26.7 | 93 |

The values in row T give estimates from the pooled series imposing scale homogeneity. The values in rows $R_{m,}$ 1, 2, … , 15 give estimates for the market returns and the individual stock series for the total excess returns and the residual parts. The values in columns A and m are the scale parameter and the estimated optimal number of order statistics imposing identical tail indexes. The values in column $\beta$ are the market model beta.

In Table 3, computations proceed by using the pooling method, assuming identical tail indexes for all risk components. We report the estimates of the scale parameter $A$ and the optimal number of order statistics $m$. Both are calculated for the series of excess returns and for the (constructed) orthogonal residuals from the market model (using the betas). The tail index estimate using all excess returns is 3.163, while when we use all the residuals the tail index is 3.246. The scale parameter estimates, however, differ considerably as these range between 14.4 and 46.4 for the excess returns, and are between 4.3 and 42.2 for market returns and residuals. We note that the scale estimates for the excess returns using the pooling method are more homogeneous than when using the individual series approach from Table 2.

The effects of portfolio diversification are reported in Table 4. The downside risk measure is the probability of a loss in excess of the VaR level $s$; we report at four different loss levels ($s$ = 7.10, 11.69, 13.33, and 15.97).[5] Four different levels of portfolio aggregation are considered: 1 stock, 5 stocks, 10 stocks, and 15 stocks. The numbers in row *EMP* are the probabilities from the empirical distribution function

[5] We chose this particular set of VaR values from the 5.0%, 1.0%, 0.5%, and 0.25% quantiles of the market returns.

**Table 4** Lower tail probabilities in percentages.

| s | −7.10 | | | | −11.69 | | | |
|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 5 | 10 | 15 | 1 | 5 | 10 | 15 |
| *EMP* | 4.995 | 1.195 | 0.633 | 0.579 | 0.995 | 0.253 | 0.145 | 0.145 |
| *NOR* | 7.325 | 0.934 | 0.225 | 0.198 | 0.817 | 0.005 | 0.000 | 0.000 |
| *FAT* | 6.551 | 1.181 | 0.741 | 0.706 | 0.988 | 0.265 | 0.185 | 0.171 |
| $CD_p$ | – | 0.633 | 0.392 | 0.423 | – | 0.125 | 0.078 | 0.084 |
| *s* | −13.33 | | | | −15.97 | | | |
| $k$ | 1 | 5 | 10 | 15 | 1 | 5 | 10 | 15 |
| *EMP* | 0.489 | 0.163 | 0.109 | 0.127 | 0.235 | 0.109 | 0.090 | 0.090 |
| *NOR* | 0.309 | 0.000 | 0.000 | 0.000 | 0.051 | 0.000 | 0.000 | 0.000 |
| *FAT* | 0.603 | 0.179 | 0.129 | 0.118 | 0.304 | 0.104 | 0.078 | 0.071 |
| $CD_p$ | – | 0.082 | 0.051 | 0.055 | – | 0.046 | 0.028 | 0.030 |

The entries in row *EMP* are the probabilities from the empirical distribution. The rows *NOR* row and *FAT* report the probabilities calculated directly from the parameters of the averaged series itself, where in the former case one uses the presumption of normality and in the latter case regular variation is imposed. The numbers in rows *CDp* are the probabilities estimated using the pooled series. The $k$ denotes the number of individual stocks included in the averaged series, and $s$ is the loss quantile. Probabilities are written in Percentage format.

of the total return series. The normal law is often used as the workhorse distribution model in finance, even though it does not capture the characteristic tail feature of the data. Therefore, in the rows labeled *NOR*, we give the probabilities from the normal model-based formula, using the mean and variance estimates from the averaged series. The estimated values in rows *FAT* were obtained by the heavy-tailed model using the averaged total excess returns $\sum_{i=1}^{k} R_i/k$. The rows *CDp* give the probability estimates from the pooled series on the basis of Equation (12), assuming the heterogeneous scale model. One notes that the normal model does well in the center, but performs poorly as one moves into the tail part. In contrast, the averaged series in rows *FAT* is always quite close to the empirical distribution function in the tail area. This shows that the heavy-tailed model is much better at capturing the tail properties. If we turn to the last rows, one notes that the model in Equation (12) does capture a considerable part of the tail risk of the portfolio, but that there is a gap between the tail risk that is explained by the model and which is left unexplained. This is further interpreted below.

To judge these results and to study the speed of diversification, a graphical exposition is insightful. In Figures 1 and 2 we show the diversification speed of the risk level by plotting the probability of loss for two different VaR levels against the number of securities that are included in the portfolio.[6] Figure 1 is for the 7.10 VaR level and Figure 2 is for the 15.97 VaR level. The top line gives the total amount of tail risk by means of the empirical distribution function. The gray

[6] The order by which the securities are included corresponds to the numbering in Table 1.

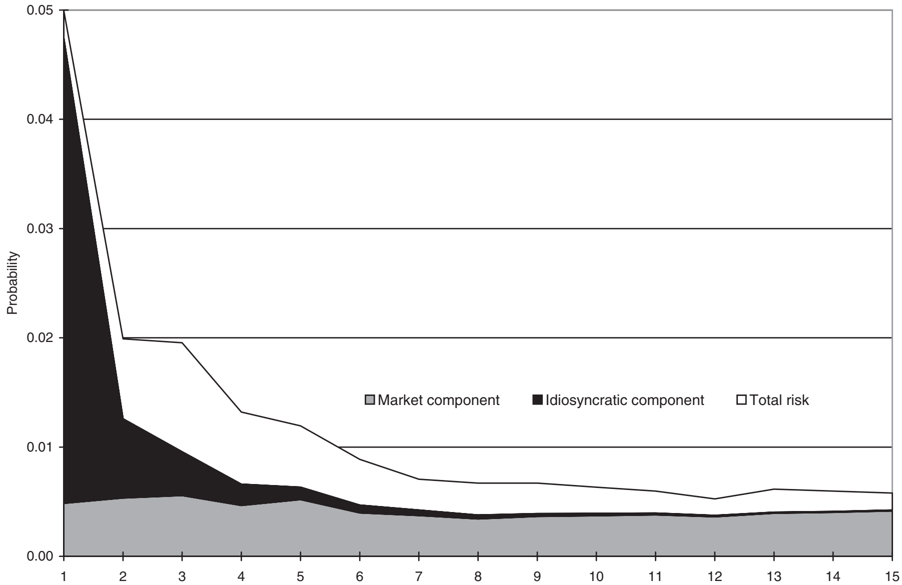


**Figure 1** Downside risk decomposition at s = −7.10 (fat-tailed case).

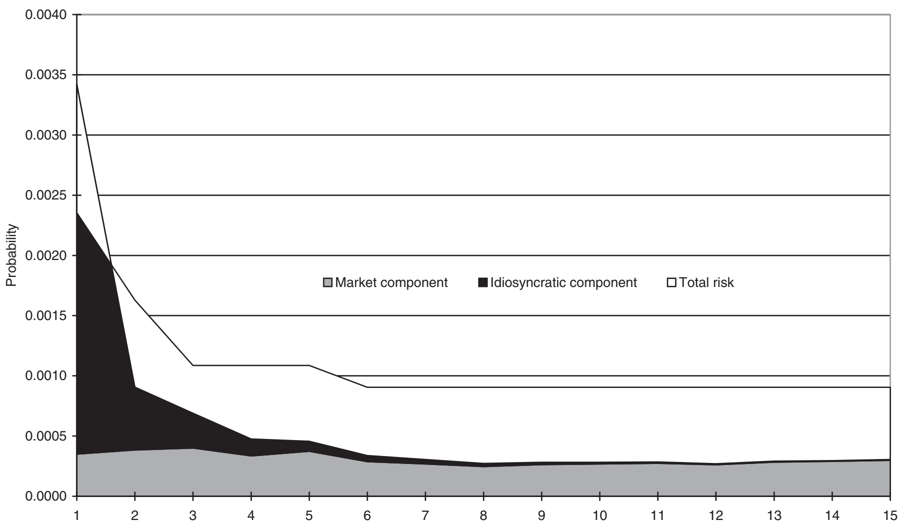


**Figure 2** Downside risk decomposition at s = −15.97 (fat-tailed case).

area constitutes the market risk component, while the black area contains the idiosyncratic risk from Equation (12). Note that the idiosyncratic risk is basically eliminated once the portfolio includes about seven stocks. To put this result into perspective, we also provide a graph for the speed of diversification concerning the variance (see Figure 3). This is a global risk measure, and under independence

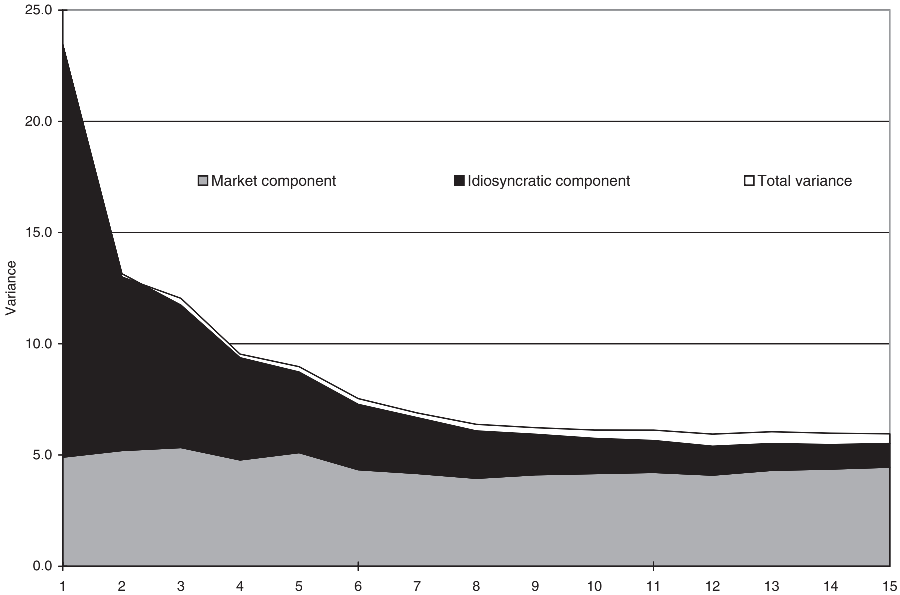


**Figure 3** Variance decomposition.

the variance of the idiosyncratic part should decline linearly in $k$. As can be seen from this latter figure, it takes approximately twice the number of securities to eliminate the variance part contributed by the idiosyncratic risk [cf. Elton and Gruber (1995)]. Note that this corroborates the rate given in Equation (6) and the value of $\alpha \simeq 3$ as in Table 2 (while the variance declines at speed 1). Interestingly, as noted at the end of the previous paragraph, another remarkable difference between the last figure and the first two figures is the size of the residual risk driven by the factors other than the market factor. While this component is relatively minor for the variance risk measure, it is even larger than the market risk component for the downside risk measure. This points to the presence of another factor, $F$, uncorrelated with $R$ as in Equation (13). This other factor induces a small correlation between the residuals (see Figure 3). This small correlation notwithstanding, the other factor appears important with respect to the downside risk. In future research we hope to relate this factor to economic variables.

Next we compare the VaR diversification speed under the normal model with the fat-tailed model. To plot the VaR diversification speed, we now look in the VaR $k$ space. From Equation (14) it is clear one cannot separate the market part from the idiosyncratic part due to the power $1/\alpha$. Nevertheless, one can first plot the VaR level doing as if only the market factor were relevant (e.g., this would be the case if the idiosyncratic risks have a higher tail index compared to the market index). The market factor is from Equation (14):

$$x = (\frac{1}{k}\sum_{i=1}^{k}\beta_i)[A_r]^{1/\alpha}\overline{p}^{-1/\alpha}. \tag{16}$$

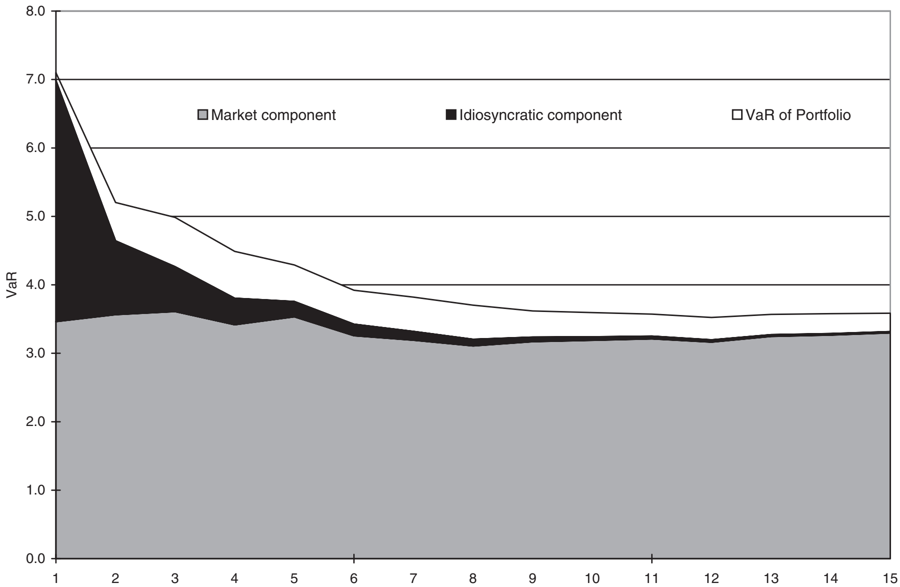


**Figure 4** VaR decomposition at $p = 0.05$ (fat-tailed case).

The next line plots the combined effect, market factor, and idiosyncratic components, which simply is Equation (14). Third, one plots the empirical quantile function as more assets are added. Similarly one can proceed in this fashion under the assumption that the returns follow the normal distribution.

Figures 4–7 show the decreasing level of VaR for the given probability. Figure 4 is for the 0.05 probability level, and Figure 5 is for the 0.0025 probability level for a fat-tailed distribution. The top line gives the total amount of VaR by means of the empirical distribution function. The gray area constitutes the VaR level from the market risk component, as in Equation (16), while the black area plus the gray area displays Equation (14). Figure 6 is for the 0.05 probability level, and Figure 7 is for the 0.0025 probability level for a normal distribution. These figures clearly display the theoretical prediction of Equation (9), that the VaR diversification speed for the idiosyncratic risk is lower for the normal model than for the fat-tailed model.

## 5 OUT OF SAMPLE, OUT OF PORTFOLIO

The semiparametric approach we followed to construct the downside risk measure can also be used to go beyond the sample. We consider two possible applications of this technique which might be of use to risk managers. The first application asks the question how much extra diversification benefits could be derived from adding more securities, without having observations on these securities. By making an assumption regarding the value of the average beta and the average scale of the residual risk factors in the enlarged portfolio, one

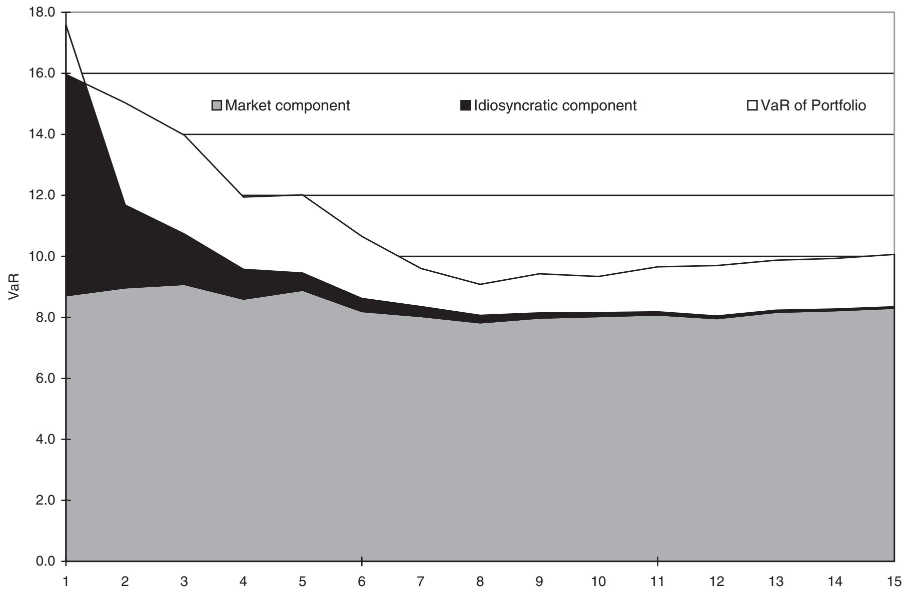


**Figure 5** VaR decomposition at $p$ = 0.0025 (fat-tailed case).

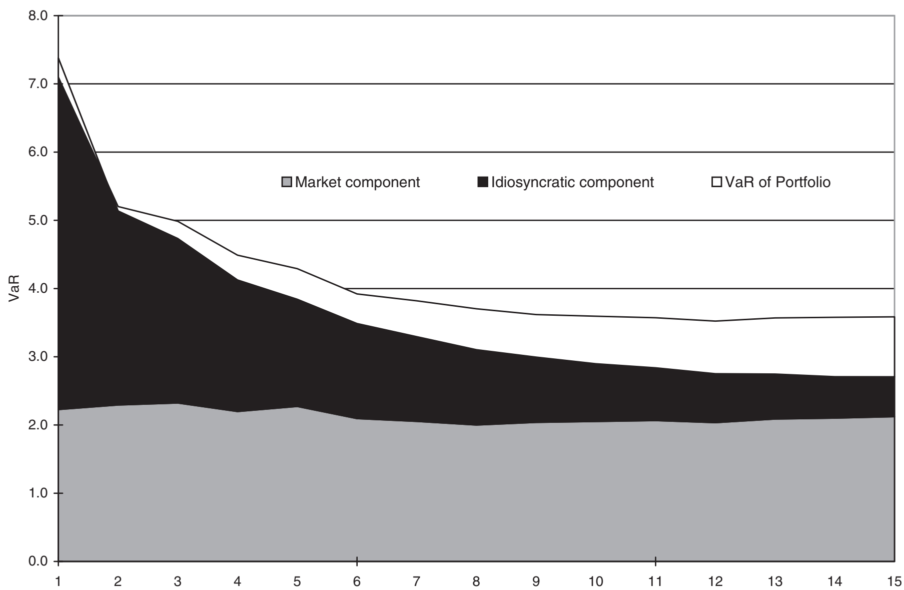


**Figure 6** VaR decomposition at $p$ = 0.05 (normal case).

can use Equation (12) to extrapolate to larger than sample size portfolios. A second application is to increase the loss levels at which one wants to evaluate the downside risk level beyond the worst case in the sample. Moreover, even at

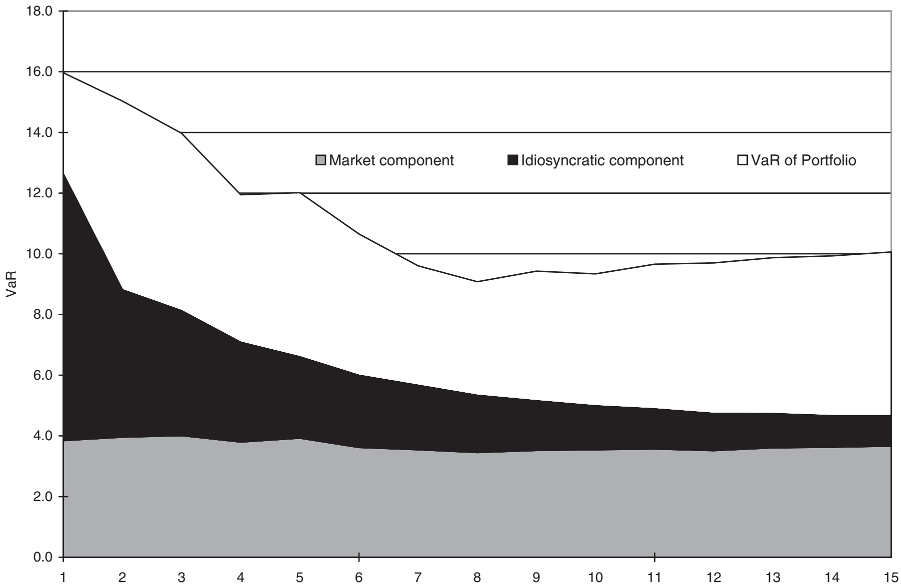


**Figure 7** VaR decomposition at $p = 0.0025$ (normal case).

the border of the sample, our approach has real benefits. By its very nature, the empirical distribution is bounded by the worst case and hence has its limitations, since the worst case is a bad estimator of the quantile at the $1/n$ probability level (and vice versa). Thus increasing the loss level $x$ in Equation (12) beyond the worst case gives an idea about the risk of observing even higher losses.

In Table 5, the block denoted as Case I summarizes some information from Table 4. The Case III block addresses the first application by increasing the number of securities $k$ beyond the sample value of 15. We assumed the following average beta values: $\overline{\beta} = 0.7$, 0.83, and 0.9. The Case II block increases the loss return level. In Table 4 we used 15.97 as the highest loss level. Above this level, many securities have no observations. There is one equity with much higher loss returns and we used this one to provide the "out of sample" loss levels of 22.03, 25.21, 33.69, and 40.45, respectively. To interpret Case III, note that the inclusion of more stocks that have a close correlation with the market component increases the loss probability for a given VaR level. For example, consider a portfolio of $k = 30$ stocks, at the $-15.97$ quantile, when $\overline{\beta} = 0.7$ the probability is 0.0169, but when $\overline{\beta} = 0.9$ the probability increases to 0.0381.

## 6 CONCLUSION

Risk managers use portfolios to diversify away the unpriced risk of individual securities. In this article we study the benefits of portfolio diversification with respect to extreme downside risk, or the VaR risk measure. The risk of a security is decomposed into a part that is attributable to the market risk and an independent

**Table 5** Lower tail probabilities: Beyond the sample and market.

| s | | −7.10 | −11.69 | −13.33 | −15.97 | −22.03 | −25.21 | −33.69 | −40.45 |
|---|---|---|---|---|---|---|---|---|---|
| | % | 5.0 | 1.0 | 0.5 | 0.25 | 0.090 | 0.054 | 0.018 | 0.009 |
| *k* | | CASE I | | | | CASE II | | | |
| | *EMP* | 1.1946 | .2534 | .1629 | .1086 | .0362 | .0362 | .0181 | .0181 |
| 5 | *FAT* | 1.1900 | .2660 | .1798 | .1045 | .0397 | .0265 | .0111 | .0064 |
| | *CDp* | .6093 | .1205 | .0789 | .0439 | .0154 | .0100 | .0039 | .0021 |
| | *EMP* | .6335 | .1448 | .1086 | .0905 | .0181 | .0181 | .0181 | .0181 |
| 10 | *FAT* | .6800 | .1490 | .1001 | .0578 | .0217 | .0144 | .0060 | .0034 |
| | *CDp* | .3914 | .0774 | .0507 | .0282 | .0099 | .0064 | .0025 | .0014 |
| | *EMP* | .5792 | .1448 | .1267 | .0905 | .0181 | .0181 | .0181 | .0181 |
| 15 | *FAT* | .7087 | .1722 | .1189 | .0712 | .0286 | .0195 | .0086 | .0051 |
| | *CDp* | .4227 | .0836 | .0547 | .0304 | .0107 | .0069 | .0027 | .0015 |
| | | CASE III | | | | | | | |
| | *CDp1* | .2375 | .0470 | .0307 | .0171 | | | | |
| 20 | *CDp2* | .4190 | .0829 | .0543 | .0302 | | | | |
| | *CDp3* | .5318 | .1052 | .0689 | .0383 | | | | |
| | *CDp1* | .2359 | .0467 | .0305 | .0170 | | | | |
| 25 | *CDp2* | .4175 | .0826 | .0541 | .0301 | | | | |
| | *CDp3* | .5302 | .1049 | .0687 | .0382 | | | | |
| | *CDp1* | .2350 | .0465 | .0304 | .0169 | | | | |
| 30 | *CDp2* | .4166 | .0824 | .0540 | .0300 | | | | |
| | *CDp3* | .5294 | .1047 | .0686 | .0381 | | | | |

The entries in rows *EMP* are the probabilities from the empirical distribution. The numbers in rows *FAT* are the probabilities calculated directly from the parameters of the average series itself. The numbers in row *CDp* are the probabilities from the fat-tail market model of Equation (12). The numbers in rows *CDp1, 2,* and *3* are calculated by imposing $\beta = 0.7$, 0.8358, and 0.9, respectively. The $k$ denotes the number of individual stocks included in the averaged series, and $s$ gives the loss quantile. Probabilities are written in percentage format.

risk factor. The independent part consists of an idiosyncratic part and a second common factor. Two different measures for diversification effects are studied. The VaR diversification speed measure holds the probability level constant and gives the rate of change by which the VaR declines as more securities are added to the portfolio, while the diversification speed of the risk level holds the VaR level constant and measures the decline in the probability level. For the VaR diversification speed measure, we argued fat-tailed distributed idiosyncratic risk factors should go down at a higher speed than normal distributed idiosyncratic risk factors. This theoretical prediction was also found empirically to be the case. Furthermore, we provide predictions for the downside risk diversification benefits beyond the range of the empirical distribution function.

This research can be extended in several directions. Given the large gaps in Figures 1 and 3 between the total downside risk and the market factor downside risk contribution, it is of interest to see whether one can identify the remaining risk factors *F*, as in Equation (13). Moreover, one would like to explain why these remaining risk factors are relatively unimportant for the global risk measure such as the variance. Moreover, the above analysis may explain why many investors seem to hold not-so-well-diversified portfolios if a global risk measure like the variance is used as the yardstick.

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