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It Takes Two to Tango: Equilibria in a Model of Sales*

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We show that the Varian model of sales with more than two firms has two types of equilibria: a unique symmetric equilibrium, and a continuum of asymmetric equilibria. In contrast, the 2-firm game has a unique equilibrium that is symmetric. For the n-firm case the asymmetric equilibria imply mixed strategies that can be ranked by first-order stochastic dominance. This enables one to rule out asymmetric equilibria on economic grounds by constructing a metagame in which both firms and consumers are players. The unique subgame perfect equilibrium of this metagame is symmetric. *Journal of Economic Literature* Classification Number: 022. © 1992 Academic Press, Inc.

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I. Introduction

It is now well known that there are conditions under which mixed strategy equilibria exist in n-person games where players have payoff functions that are neither quasiconcave nor continuous. For the most part, analyses of these equilibria have focused either on general conditions for existence, as in Dasgupta and Maskin (1986) and Simon (1987), or on the derivation of particular *symmetric* equilibrium mixed strategies, as in Varian (1980). To date, a complete characterization of equilibria in this class of n-person games has not been addressed.[1]

Our initial goal in starting the present line of research was to investigate uniqueness of equilibrium in a wide class of games of this type. It turns out, however, that the question appears to be complex and model specific. Thus, in this paper, we focus exclusively on Varian's (1980) seminal model of sales. While this may seem particularly restrictive, many economic problems, such as the "all-pay auction game,"[2] have a similar structure.

The principal result of this paper is that, when there are more than two firms in the Varian model of sales, there exist a continuum of asymmetric equilibria and a unique symmetric equilibrium. This contrasts sharply with the 2-firm game in which the unique equilibrium is the symmetric one. Despite the multiplicity of equilibria, in all equilibria at least two agents randomize continuously over the union of the supports of the equilibrium price distributions, just as in the 2-firm game. We call this the two-to-tango property.

While the set of equilibria in the Varian model of sales is large, we show that all the asymmetric equilibria imply mixed strategies that can be ranked by first-order stochastic dominance. This ordering enables us to construct a metagame in which both firms and consumers are players and in which asymmetric equilibria are ruled out as subgame perfect equilibria. Intuitively, the asymmetric strategies are not consistent with price dispersion

[1] For particular two-person games, such as the War of Attrition (Hendricks, Weiss and Wilson, 1988), capacity-constrained price setting games (Osborne and Pitchik, 1986a), and price setting with loyal consumers (Narasimhan, 1988), uniqueness has been thoroughly examined. Osborne and Pitchik (1986b) also examine the question of uniqueness in the 3-firm "pure" location model of Hotelling (1929). They show that with a uniform distribution of consumers, in addition to a symmetric mixed strategy equilibrium (see Shaked, 1982), there is a unique (up to symmetry) asymmetric equilibrium within the class of equilibria in which at least one firm uses a pure strategy.

[2] The all-pay auction game is as follows: n players simultaneously bid for the right to buy a prize worth V_i to player i. All players forfeit their bid, and the person submitting the highest bid wins the prize. Moulin (1986) characterizes the symmetric equilibrium of the game when $V_i = 1\ \forall i$. This equilibrium involves mixed strategies that are the mirror image of the mixed strategies set forth in Varian (1980). In Baye, Kovenock, and De Vries (1990), we characterize all equilibria for the all-pay auction game.

because they imply that distributions of prices charged by some firms stochastically dominate those charged by other firms.[3]

II. The Model

Following Varian (1980), consider a market where $n \geq 2$ firms produce a homogeneous product with an identical technology exhibiting weakly declining average cost.[4] The cost curve of each firm is denoted $c(q)$, where q is quantity produced. We assume that there are a large number of consumers, each of whom will purchase one unit of the good if faced with a price less than or equal to a reservation value r, and none of the good if faced with a price greater than r. There are two types of consumers: informed and uninformed. Informed consumers purchase a unit of the good from the store charging the lowest price, as long as this price is below the reservation value. Each uninformed consumer is aware of the price at one firm only, and purchases from that firm if the price is no greater than r. We assume that the same number, U, of uninformed consumers shop at each store, and that $rU - c(U) > 0$.[5] The total number of informed consumers is I.

Firms are assumed to set prices simultaneously. Let $(p_1, \ldots, p_n)$ be the vector of prices charged by the firms, and define $\hat{p}_{-i}$ to be the minimum price charged by any firm other than i, and m_{-i} to be the number of firms charging $\hat{p}_{-i}$. Then firm i's profit is given by

$$\Pi_i(p_1, \ldots, p_n) = \begin{cases} p_i(I + U) - c(I + U) & \text{if} \quad p_i < \hat{p}_{-i} \text{ and } p_i \leq r \\ p_i\left(\dfrac{I}{m_{-i}} + U\right) - c\left(\dfrac{I}{m_{-i}} + U\right) & \text{if} \quad p_i = \hat{p}_{-i} \leq r \\ p_i U - c(U) & \text{if} \quad \hat{p}_{-i} < p_i \leq r \\ 0 & \text{if} \quad p_i > r. \end{cases}$$

Thus, if a firm does not set the lowest price, it services only the U uninformed consumers who shop at the store. If a firm sets the lowest price, it services all of the informed consumers plus the U uninformed

[3] An important by-product of our analysis of this metagame is that we generalize the model of Narasimhan (1988) from 2 firms to n firms.

[4] To focus on essentials, we ignore the entry decision and view n as fixed. Varian assumes strictly declining average cost in order to pin down the equilibrium number of firms when there is free entry.

[5] See Bagnoli (1986) for a discussion of the importance of this assumption.

consumers who shop at the store. In the event several firms tie in charging the lowest price, they share the informed consumers equally.

Varian shows in his Proposition 2 that there is no equilibrium where all firms charge the same price. In a sequence of propositions, he derives the *symmetric* mixed-strategy Nash equilibrium to the game. In the following section we expand the scope of analysis to determine whether there exist asymmetric mixed-strategy equilibria.

III. THE FULL SET OF EQUILIBRIA

In the normal form of the above game, firm i's strategy is $p_i \in [0, \infty)$ and its payoff function is $\Pi_i(p_1, \ldots, p_n)$, $i = 1, \ldots, n$. The complete set of Nash equilibria will be derived in a series of lemmas. In what follows we define

$$\underline{p} \equiv \frac{rU + c(I + U) - c(U)}{I + U}, \tag{D.1}$$

i.e., the price at which a firm selling to both its uninformed consumers and the informed consumers obtains the same profit that it would obtain by charging the reservation price r and selling only to its uninformed consumers. A price below $\underline{p}$ is strictly dominated by setting r.

Let $\underline{s}_i$ and $\overline{s}_i$ denote the lower and upper bounds of firm i's equilibrium price distribution G_i. When $\underline{s}_i = \overline{s}_i$, firm i adopts a pure strategy; otherwise it employs a mixed strategy. Let α_i denote the size of a mass point in i's distribution.

One equilibrium of this game, a symmetric equilibrium, has been analyzed by Varian (1980).[6] Our main result, summarized in the following theorem, is that there is a continuum of asymmetric equilibria as well.

THEOREM 1. *The Varian model of sales possesses two types of equilibria. Either all firms use the same continuous mixed strategy with support* $[\underline{p}, r]$, *or at least two firms randomize over* $[\underline{p}, r]$, *with each other firm i randomizing over* $[\underline{p}, x_i)$, $x_i < r$, *and having a mass point at r equal to* $(1 - G_i(x_i))$.[7] *When two or more firms have a positive density over a common interval they play the same (continuous) mixed strategy over that interval.*

[6] After this paper was completed, it was brought to our attention that Bagnoli (1986) found a finite number of asymmetric equilibria (see Example 2 below).

[7] We could have $x_i < \underline{p}$, in which case the interval $[\underline{p}, x_i)$ is empty and firm i places all mass at r.

To prove the theorem we need a sequence of lemmas.

LEMMA 1. $\forall i\ r \geq \bar{s}_i \geq \underline{s}_i \geq \underline{p} > 0$.

Proof. By setting $p_i = r$ each firm can guarantee itself at least $rU - c(U)$. This rules out prices greater than r, at which firms earn zero. For prices less than $\underline{p}$, $\Pi_i < p(I + U) - c(I + U) = rU - c(U)$. Hence, a firm will never price below $\underline{p}$, as more could be earned by charging r. ■

LEMMA 2. *If* $\exists i, j$ *s.t.* $\bar{s}_i \leq \bar{s}_j$ *and* $\alpha_i(\bar{s}_j) = 0$ *then* $\bar{s}_j = r$. *If* $\bar{s}_i < \bar{s}_j$ $\lim_{p\uparrow r} G_j(p) = G_j(\bar{s}_i)$. *If in addition* $\alpha_i(\bar{s}_i) = 0$ *then* $\lim_{p\uparrow r} G_j(p) = \lim_{p\uparrow \bar{s}_i} G_j(p)$.

Proof. $\Pi_j(\bar{s}_j, G_{-j}) = \bar{s}_j U - c(U) < rU - c(U)$ for $\bar{s}_j < r$. Since the same holds for $\Pi_j(p, G_{-j})$ for $p > \bar{s}_i$ and $p = \bar{s}_i$ if $\alpha_i(\bar{s}_i) = 0$, the claim follows. ■

LEMMA 3. *If* $\bar{s}_1 = \cdots = \bar{s}_m < \bar{s}_{m+1}, \ldots, \bar{s}_n$ *for* $n \geq m \geq 2$ *then* $\exists i \leq m$ *such that* $\alpha_i(\bar{s}_i) = 0$.

Proof. Suppose not. Then any $i \leq m$ has an incentive to undercut $\bar{s}_i$ by small $\varepsilon > 0$. ■

LEMMA 4. *If* $\bar{s}_1 = \cdots = \bar{s}_m < \bar{s}_{m+1}, \ldots, \bar{s}_n$ *for* $n \geq m \geq 2$ *then* $\bar{s}_i = r\ \forall_i$.

Proof. Immediate from Lemmas 2 and 3. ■

LEMMA 5. *There exists no firm i such that* $\bar{s}_i < \bar{s}_j\ \forall j \neq i$.

Proof. Suppose such a firm did exist. If $\alpha_i(\bar{s}_i) = 0$, from Lemma 2 $\lim_{p\uparrow r} G_j(p) = \lim_{p\uparrow \bar{s}_i} G_j(p)$, $\forall j \neq i$, which implies that $\Pi_i(\bar{s}_i, G_{-i}) < \lim_{p\uparrow r} \Pi_i(p, G_{-i})$. If the claim held and $\alpha_i(\bar{s}_i) > 0$ then $\forall j \neq i\ \alpha_j(\bar{s}_i) = 0$, which implies that $\lim_{p\uparrow r} G_j(p) = \lim_{p\uparrow \bar{s}_i} G_j(p)$, leading to a similar contradiction. ■

LEMMA 6. $\bar{s}_i = r\ \forall i$.

Proof. Immediate from Lemmas 4 and 5. ■

Let Π_i^* represent the equilibrium profit of firm $i = 1, \ldots, n$. Then we have:

LEMMA 7. $\Pi_i^* = \Pi_j^*\ \forall i,j$.

Proof. Without loss of generality suppose $\Pi_i^* < \Pi_j^*$. With $\underline{s}_j$ being the lower bound of j's support, $\Pi_i^* < \Pi_j^* = \Pi_j(\underline{s}_j, G_{-j}) \leq \lim_{p_i\uparrow \underline{s}_j} \Pi_i(p_i, G_{-i})$, a contradiction. ■

LEMMA 8. $\Pi_i^* = rU - c(U)\ \forall i$.

Proof. If $\alpha_i(\bar{s}_i) = 0\ \forall i$ we are through. If $\exists j$ such that $\alpha_j(\bar{s}_j) > 0$ then

$\Pi_j^* = rU - c(U)$ from Lemmas 3 and 6, and with firms earning equal profit from Lemma 7, $\Pi_i^* = rU - c(U)\ \forall i$. ■

LEMMA 9. *$\exists i, j$ such that $\underline{s}_i = \underline{s}_j = \underline{p}$.*

Proof. Suppose not. Let $\underline{s}_i$ be the second lowest $\underline{s}$. Then the lowest $\underline{s}$ firm can set a price p slightly below $\underline{s}_i$ and earn $\Pi_i = p(I + U) - c(I + U) > \Pi_i^*$. ■

The previous nine lemmas establish that $\bar{s}_i = r\ \forall i$; there exist two i's, say $i = 1, 2$, such that $\underline{s}_1 = \underline{s}_2 = \underline{p}$; and $\Pi_i^* = rU - c(U)\ \forall i$. We now proceed to pin down the equilibrium distributions. Let $W(p) \equiv p(U + I) - c(U + I)$, $L(p) \equiv pU - c(U)$,

$$A_i \equiv \prod_{\substack{j=1 \\ j\neq i}}^{n} (1 - G_j), \qquad \text{and} \qquad A_{ij} = \prod_{\substack{k=1 \\ k\neq j,i}}^{n} (1 - G_k).$$

LEMMA 10. *There are no point masses on the half open interval $[\underline{p}, r)$.*

Proof. Suppose one of the cumulative distribution functions, say G_i, has a mass point at p_i. Since $\forall p \in [\underline{p}, r)$, $(1 - G_i)A_{ij} > 0$, $(1 - G_i)A_{ij}$ has a downward jump at p_i, $\forall j \neq i$. This follows directly from the monotonicity of the c.d.f.'s. For $p_i > \underline{p}$ this implies that it is worthwhile for j to transfer all mass from an ε-neighborhood above p_i to some δ-neighborhood below p_i. At $p_i = \underline{p}$ it pays for j to transfer mass from an ε-neighborhood above p_i to r. Thus, there would be an ε-neighborhood above p_i in which no other firm j would put mass. But then it cannot be an equilibrium strategy for player i to put mass at p_i. ■

Lemma 11 is a generalization of Varian's Proposition 4.

LEMMA 11. *The integrand*

$$B_i(p_i) \equiv W(p_i)A_i(p_i) + L(p_i)(1 - A_i(p_i)) \tag{D.2}$$

is constant and equal to $rU - c(U)$ at the points of increase of G_i in the half open interval $[\underline{p}, r)$ for all i.

Proof. By Lemma 10 there are no point masses in the interval. Thus, $B_i(p_i)$ is the expected profit of firm i from setting $p_i \in [\underline{p}, r)$. If p_i is a point of increase of G_i then firm i must make its equilibrium profit at p_i. ■

LEMMA 12. *Suppose p is a point of increase of G_i and G_j in $[\underline{p}, r)$. Then $G_i = G_j$ at p.*

Proof. $B_i(p) = B_j(p) = rU - c(U)$. From (D.2) we have

$$W(p)A_{ij}(p)(1 - G_j(p)) + L(p)(1 - A_{ij}(p)(1 - G_j(p))) = rU - c(U).$$

This implies that

$$A_{ij}(p)(1 - G_j(p)) = \frac{rU - c(U) - L(p)}{W(p) - L(p)} = A_{ji}(p)(1 - G_i(p)).$$

Division by $A_{ij}(p) = A_{ji}(p) > 0$ gives $G_j(p) = G_i(p)$. ■

LEMMA 13. *For every i and every point of increase p of G_i in $[\underline{p}, r)$ there is at least one G_j $j \neq i$ such that G_j is increasing at p.*

Proof. Because $B_i(p)$ is constant in a half-open neighborhood about p by Lemma 11, $dB_i(p) = 0$. Suppose contrary to the hypothesis that $dA_i(p) = 0$. Totally differentiating $B_i(p)$ gives

$$A_i dW + (1 - A_i)dL = 0.$$

However, both dW and dL are positive and $A_i(p) \in (0, 1]$. Hence for $dB_i(p)$ to be zero dA_i is necessarily negative. By the monotonicity of the G_j's at least one has to increase. ■

LEMMA 14. *If G_i is strictly increasing on some open subset (x, y), $\underline{p} < x < y < r$, then G_i is strictly increasing on the whole interval $[\underline{p}, y)$.*

Proof. Without loss of generality, suppose, to the contrary, that G_i were constant on (z, x), $\underline{p} \leq z < x$. Then from Lemma 10, $G_i(z) = G_i(x)$. It is evident that there exists an $\varepsilon > 0$ such that on the interval $(x - \varepsilon, x)$ there exist at least two firms, say l and m, with strictly increasing c.d.f.'s over the interval (otherwise mass would be moved up to x by some firm). Thus, for every $p \in (x - \varepsilon, x)$, $B_l(p) = B_m(p) = rU - c(U)$. Furthermore, since there are no mass points in the interval $[\underline{p}, r)$, $B_l(x) = B_m(x) = B_i(x) = rU - c(U)$ which, from arguments similar to those used in proving Lemma 12, implies that $G_l(x) = G_m(x) = G_i(x) < 1$. But with

$$B_i(x) = B_l(x) = B_l(p)\ \forall p \in (x - \varepsilon, x)$$

it must be that $B_i(p) \leq B_l(p)\ \forall p \in (x - \varepsilon, x)$, since such values of p do not lie in i's support. This implies that $A_l(p) \geq A_i(p)$, and hence that $1 - G_i(p) \geq 1 - G_l(p)$. This is a contradiction to the fact that $G_i(x) = G_l(x)$, $G_l(p)$ is increasing on $(x - \varepsilon, x)$ and $G_i(p)$ is constant on $(x - \varepsilon, x)$. ■

LEMMA 15. (It Takes Two to Tango). *At least two firms randomize continuously on* $[\underline{p}, r]$.

Proof. Three cases are possible at r: (i) all firms allocate positive mass at r, (ii) all firms have $G_i(x_i) = 1$ at some $x_i < r$, or (iii) there is at least one firm i that has a positive left-derivative of G_i at r. Cases (i) and (ii) are easily ruled out by previous lemmas. Lemmas 12, 13, and 14 then imply that there are at least two firms that randomize continuously over $[\underline{p}, r]$. ■

LEMMA 16. *Once* G_i *is constant on a subset* (x, y), $\underline{p} \le x < y < r$, *it is constant on* (x, r) *and has a mass point at* r.

Proof. The first part is a direct implication of Lemma 14. The second part follows from Lemma 6. ■

The above lemmas together establish our Theorem 1.

Note that in the case where $n = 2$, Lemmas 12 and 15 and Theorem 1 imply that the equilibrium of Varian's model of sales is unique and symmetric. This illustrates an important property that appears to have implications for other games with discontinuous payoffs. The 2-person game may have a unique equilibrium but the n-person game does not. In 2-person games, in order to make one player indifferent between all pure strategies in its support, the other player's strategy is uniquely determined. In n-player games this is generally not true.[8]

Exact expressions for the equilibrium distributions may be obtained recursively over the interval $[\underline{p}, r]$, conditional on the points at which firms stop randomizing continuously and move remaining mass to r. These expressions are provided in Appendix A. Here we give some instructive examples.

EXAMPLE 1. Symmetric Equilibrium (Varian, 1980). From the proof of Lemma 12, for $n \geq 2$ the symmetric strategies are

$$1 - G = \left[\frac{(r - p)U}{Ip - c(I + U) + c(U)}\right]^{1/(n-1)}.$$

In this case, all firms randomize continuously on the interval $[\underline{p}, r]$, and use the same strategy.

EXAMPLE 2. Pure and Mixed Strategies (Bagnoli, 1986). Completely

[8] Our result thus serves as yet another warning against the common practice of extrapolating from 2 to n.

asymmetric strategies arise when $k \geq 2$ firms randomize over $[\underline{p}, r]$ and $n - k$ firms load all mass at r. The respective strategies are

$$1 - G_i = \left[\frac{(r - p)U}{Ip - c(I + U) + c(U)}\right]^{1/(k-1)} \qquad \text{for } i = 1, \ldots, k$$

$$1 - G_j = \begin{cases} 1 \text{ for } p < r \\ 0 \text{ for } p = r \end{cases} \qquad \text{for } j = k + 1, \ldots, n.$$

EXAMPLE 3. Intermediate Asymmetric Strategies. The final example is a situation where two or more firms randomize over $[\underline{p}, r]$, and other firms randomize over proper subsets $[\underline{p}, x_j]$, $\underline{p} \leq x_j < \bar{r}$. In the case of three firms with strategies H, G, and F, an example of the Nash equilibrium strategies is

$$1 - H = 1 - G = 1 - F = \left[\frac{(r - p)U}{Ip - c(I + U) + c(U)}\right]^{1/2} \qquad \text{for } p \in [\underline{p}, x]$$

$$1 - G = 1 - F = \left[\frac{(r - p)U}{Ip - c(I + U) + c(U)}\right](1 - H(x))^{-1} \qquad \text{for } p \in [x, r]$$

$$1 - H = \begin{cases} 1 - H(x) & \text{for } p \in [x, r) \\ 0 & \text{for } p = r, \end{cases}$$

where

$$1 - H(x) = \left[\frac{(r - x)U}{Ix - c(I + U) + c(U)}\right]^{1/2}.$$

Note that H, G, and F have a kink at x, but not a jump.

To conclude, there are an uncountable infinity of payoff-equivalent equilibrium mixed strategies.

IV. ORDERING THE ASYMMETRIC STRATEGIES

In this section we show that the asymmetric equilibria imply mixed strategies that can be ranked by first-order stochastic dominance. This result will be used in Section V to motivate the symmetric equilibrium as a "reasonable" equilibrium selection by constructing a metagame in which it is the unique subgame perfect Nash equilibrium.

DEFINITION 1. Let F and G be two cumulative distribution functions. F is said to strictly first-order stochastically dominate G if $\int_{-\infty}^{t} dF \leq \int_{-\infty}^{t} dG$ for all t, with strict inequality holding for some t.

THEOREM 2. *If, in a Nash equilibrium, firm i has a larger mass point at r than firm j, then the distribution of prices charged by firm i strictly first-order stochastically dominates the distribution of prices charged by firm j. If two firms load the same mass at r, then their price distributions are stochastically equivalent (i.e., $G_i(p) = G_j(p)\ \forall p$).*

Proof. Let $(G_1, \ldots, G_n)$ be Nash equilibrium mixed strategies, and suppose that firm i loads more mass at r than some firm j. Then by Theorem 1, associated with firms i and j are x_i's and x_j's with $x_i < x_j$ such that firm i randomizes continuously on $[\underline{p}, x_i)$ and loads remaining mass at r, while firm j randomizes continuously on $[\underline{p}, x_j)$ and loads any remaining mass at r. If $x_i = \underline{p}$, firm i loads all mass at r and hence the proposition is trivially proved.

Hence, suppose $\underline{p} < x_i < x_j$. Then G_i and G_j are both strictly increasing for $p \in [\underline{p}, x_i)$, and Lemma 12 reveals that $G_i = G_j$ on $[\underline{p}, x_i)$. Hence $G_i(p) = G_j(p)$ for $p \in [\underline{p}, x_i]$; $G_i(p) < G_j(p)$ for $p \in (x_i, r)$ and $G_i(r) = G_j(r)$. That is, $G_i(p)$ strictly first-order stochastically dominates $G_j(p)$.

If the two firms load the same mass at r, then Theorem 1 implies that the firms randomize on the same interval, say $[\underline{p}, x)$, and load remaining mass at r. Lemma 12 thus implies that the firms have identical distribution functions, so that their strategies are stochastically equivalent. ∎

The basic idea behind Theorem 2 is depicted in Fig. 1 for the case when $n = 3$. Here, firms 2 and 3 randomize continuously on the interval $[\underline{p}, r]$, while firm 1 randomizes continuously on the interval $[\underline{p}, x_1)$ and loads mass at r (see Example 3). On the interval $[\underline{p}, x_1)$ all three firms are equally likely to charge low prices. On the interval $[x_1, r]$, firms 2 and 3 are equally likely to charge low prices, but firms 2 and 3 charge lower prices than firm 1 with probability one, since all of its mass in the interval is at r. Hence, $G_1(p) \leq G_2(p) = G_3(p)$ for all p, with strict inequality holding for $p \in (x_1, r)$.

V. RECONSIDERING THE SYMMETRIC EQUILIBRIUM

An important observation by Varian is that most models of price dispersion imply that some stores consistently charge lower prices than other stores. Varian argues that if price dispersion is to be an equilibrium phenomenon, it must be temporal in nature. According to Varian:

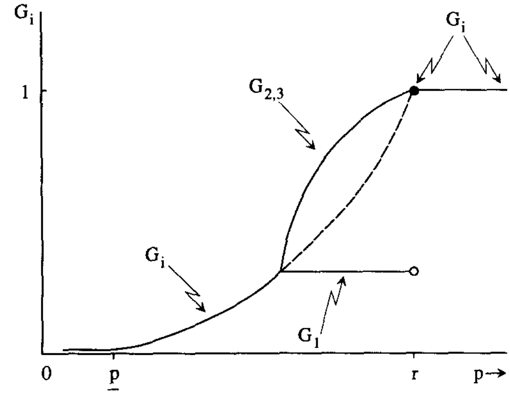

FIG. 1. Stochastic dominance. The figure is drawn on the presumption that $U > I$. This guarantees $G'' > 0$ for $n > 2$, while $G'' < 0$ necessarily for $n = 2$.

> In a market exhibiting temporal price dispersion, we would see each store varying its price over time. At any moment, a cross section of the market would exhibit price dispersion; but because of the intentional fluctuations in price, consumers cannot learn from experience about stores that consistently have low prices, and hence price dispersion may be expected to persist. (p. 651)

While these arguments are intuitively plausible, the Varian game is a one-shot game,[9] and hence the appeal to learning over time as a disciplining device is really outside of the model. The idea behind learning is to allow the consumers to infer the strategies used by the firms. In games with complete information, players know who their opponents are. Hence by considering Nash equilibria in a game of complete information, one circumvents the "learning" story; one simply asks if, given the strategies of the other players, any player has an incentive to deviate. Nash theory says nothing about how agents learn the strategies of opponents, although it does make a nice story. For this reason we offer an extensive form of the Varian game that is consistent with the spirit of Varian's arguments.

Suppose the n firms and $M = nU$ uninformed consumers move simultaneously, the firms choosing their equilibrium price distributions and the uninformed consumers each deciding the identity of the firm to which they will go to make a purchase decision. After the firms and uninformed consumers have moved (the firms having set prices and the uninformed

[9] By Lemma 8 there is a unique Nash equilibrium payoff to the game. Hence, if the game is repeated finitely many times, the firms cannot "cooperate" to improve their payoffs, and it thus makes sense to think of the Varian game being repeated some large (but finite) number of times.

having chosen to which firm to go), the informed consumers decide simultaneously from which, if any, firm to purchase. For simplicity, we assume that $c(q)$ is zero.[10]

We now establish

THEOREM 3. *The unique subgame perfect equilibrium in the extensive form sales game is the symmetric equilibrium.*[11]

The proof proceeds by several lemmata. Before these are stated, some remarks are in order. Note that the only proper subgames of the game start at the first node along each of the paths at which an informed consumer must make a decision where to shop. Subgame perfection requires that all informed consumers buy from one of the firms setting the lowest price. If the uninformed consumers allocate themselves equally across firms and the firms do not play the symmetric equilibrium then they must play one of the asymmetric equilibria. By Theorem 2 there will exist some firm, say firm 1, whose distribution stochastically dominates some other firm's distribution. This implies that it is not a best response for uninformed consumers to shop at the first firm,[12] since, on average, they will pay lower prices by shopping elsewhere. Suppose, then, that the uninformed consumers do not allocate themselves equally across firms. Let U_i be the number of uninformed consumers allocating themselves to firm i. We deal first with the case where $U_1 < U_2 < U_3 \leq \ldots \leq U_n$. Degenerate cases where one of the strict inequalities adjacent to U_2 is weak require a separate analysis. This is carried out in Appendix B.

Let $\underline{p}_i \equiv U_i r/(I + U_i)$, $i = 1, \ldots, n$. By assumption $\underline{p}_1 < \underline{p}_2 < \underline{p}_3 \leq \cdots \leq \underline{p}_n$. It is easily shown that Lemmas 1 through 6 from Section III hold for this case, where in Lemma 1 we insert $\underline{p}_i$ in place of $\underline{p}$ in both the statement and proof, and the proof of Lemma 3 is altered in an obvious fashion. We replace the remaining lemmas of Section III with the lemmas that follow. Henceforth, let $\underline{s}$ denote the lower bound of the union of the supports of the firms' equilibrium price distributions.

LEMMA 7′. $\underline{s} \geq \underline{p}_2$.

Proof. Firm i would never put mass below $\underline{p}_i$ since setting price equal

[10] We rule out for now mixed strategies on the part of uninformed consumers, although this does not affect the nature of the outcome because firms care only about the expected number of uninformed consumers that they serve.

[11] If $c(q)$ were to exhibit sufficiently increasing returns to scale, i.e., if $c(I + M)/(I + M) < r$ while $c(I)/I > r$, there would also be asymmetric equilibria whereby one firm monopolizes the market. Our assumption that $c(q)$ exhibits constant returns to scale rules out this type of equilibrium.

[12] Note that an uninformed consumer's payoff is linear in price, so that such a consumer will go to a store with the lowest expected price.

to r strictly dominates such a strategy. Firm 1 clearly has no incentive to put mass in the interval $[\underline{p}_1, \underline{p}_2)$. ■

LEMMA 8′. *All firms other than firm* 1 *must place a mass point at* r.

Proof. By Lemma 6, $\bar{s}_i = r \; \forall i$. Since $\underline{s} \geq \underline{p}_2 > \underline{p}_1$, firm 1 must have equilibrium profit Π_1^* of at least $(I + U_1)\underline{p}_2 > rU_1$. Thus, firm 1 cannot have a mass point at r since, by Lemma 3, some firm must put no mass at r, in which case firm 1 would be undercut at r with certainty and earn rU_1 there. Since $\Pi_1^* > rU_1$, in every neighborhood below r firm 1 must undercut every other firm with positive probability. Thus, every firm but firm 1 must put a mass point at r. ■

LEMMA 9′. $\forall i \neq 1 \; \Pi_i^* = rU_i$.

Proof. Immediate from Lemmas 3 and 8′. ■

LEMMA 10′. $\underline{s} = \underline{p}_2$ *and* $\underline{s}_1 = \underline{s}_2 = \underline{p}_2$.

Proof. From Lemma 7′ $\underline{s} \geq \underline{p}_2$. Suppose $\underline{s} > \underline{p}_2$. By undercutting $\underline{s}$ by an arbitrarily small amount firm 2 could earn arbitrarily close to $(I + U_2)\underline{s} > rU_2 = \Pi_2^*$, a contradiction. Thus, $\underline{s} = \underline{p}_2$. The second part of the claim is straightforward. ■

LEMMA 11′. *There are no point masses on the half open interval* $[\underline{p}_2, r)$.

Proof. Similar to the proof of Lemma 10, inserting $\underline{p}_2$ for $\underline{p}$, and noting that if firm 1 has a mass point at $\underline{p}_2$, firm 2 will move mass up to r, while if firm 2 has a mass point at $\underline{p}_2$ firm 1 will move mass slightly below $\underline{p}_2$. ■

LEMMA 12′. $B_i(P_i) \equiv (I + U_i)p_iA_i(p_i) + U_ip_i(1 - A_i(p_i))$ *is constant and equal to* Π_i^* *at the points of increase of* G_i *in* $[\underline{p}_2, r)$ *for al i.* $B_i(p_i) \leq \Pi_i^*$ *if* p_i *is not a point of increase in* $[\underline{p}_2, r)$.

Proof. Similar to Lemma 11. ■

LEMMA 13′. $\forall p \in [\underline{p}_2, r) \; \exists i_1, i_2$ *such that* $\forall \varepsilon > 0 \; G_i(p + \varepsilon) - G_i(p - \varepsilon) > 0$, $i = i_1, i_2$.

Proof. Immediate. ■

LEMMA 14′. $\underline{s}_i = r \; \forall i > 2$.

Proof. Without loss of generality assume $\underline{s}_3 = \min_{i \geq 3} \underline{s}_i$. Suppose $\underline{s}_3 \neq r$. Then there exists an initial interval of increase $[\underline{s}_3, \underline{s}_3 + \varepsilon)$ in which $B_3(\mathrm{p}) = \Pi_3^* = U_3r = (I + U_3)pA_3(p) + U_3p(1 - A_3(p))$. Thus

$$U_3 = A_3(p)\frac{p}{r - p}I \qquad \forall p \in [\underline{s}_3, \underline{s}_3 + \varepsilon).$$

But since G_1 and G_2 are increasing on $[\underline{p}_2, \underline{s}_3)$,

$$U_2 = A_2(\underline{s}_3)\frac{\underline{s}_3}{r - \underline{s}_3}I.$$

Since for $\underline{s}_3 < r$: $A_2(\underline{s}_3) = \prod_{j\neq 2}(1 - G_j(\underline{s}_3)) > \prod_{j\neq 3}(1 - G_j(\underline{s}_3)) = A_3(\underline{s}_3)$ we have a contradiction to the fact that $U_2 < U_3$. Thus, $\underline{s}_3 = r$. ■

We have thus shown that if $U_1 < U_2 < U_3 \leq \cdots \leq U_n$ then firms 1 and 2 will continuously randomize over $[\underline{p}_2, r)$, with firm 2 having a mass point at r and all other firms setting price equal to r with probability one. This cannot comprise a subgame perfect equilibrium because the uninformed consumers shopping at firms 2 through n would prefer to defect to firm 1, given the strategies played. While the degenerate cases where $U_2 = U_3 = \cdots = U_m$, $m \leq n$, will generally lead to multiple equilibria, the stochastic dominance rankings apply, and as long as all the firms are not symmetric some uninformed consumers would want to defect. This case is covered in Appendix B.

VI. Conclusion

This paper has derived the complete set of equilibria in Varian's (1980) model of sales. In addition to the well known symmetric equilibrium, Theorem 1 reveals the existence of a continuum of asymmetric Nash equilibrium mixed strategies. While the set of equilibria is thus very large, Theorem 2 revealed that the set of equilibrium mixed strategies can be ranked by first-order stochastic dominance. We then constructed an extensive form of the model which, along with the stochastic dominance ranking, yields Varian's symmetric equilibrium as the unique subgame perfect Nash equilibrium.

The basic technology used to characterize the complete set of equilibria to the sales game may also be used to characterize the full set of equilibria in other games with discontinuous payoffs. For example, results similar to those in section III and IV reveal a continuum of equilibrium in the all pay auction (see Moulin, 1986, or Weber, 1985). A detailed examination of the implications of the asymmetric equilibria in the all pay auction for lobbying and patent races is contained in Baye, Kovenock, and De Vries (1990).

Appendix A

The exact expressions for the equilibrium distribution functions in Theorem 1 may be obtained recursively as follows: If we assume, without loss of generality, that firms $1, \ldots, m$, $m \geq 2$, randomize over $[\underline{p}, r]$, with firms $m+1, \ldots, n$ randomizing over $[\underline{p}, x_i)$, $x_n \leq x_{n-1} \leq \ldots \leq x_{m+1} < r$, then

(a) $\forall p \in [\underline{p}, x_n)$

$$1 - G_i(p) = \left[\frac{(r-p)U}{Ip - c(I+U) + c(U)}\right]^{1/(n-1)} \qquad \forall i.$$

(b) For $j = n, n-1, \ldots, m+2$ and $\forall p \in [x_j, x_{j-1})$

$$1 - G_i(p) = \left[\frac{(r-p)U}{Ip - c(I+U) + c(U)}\right]^{1/(j-2)} \left[\prod_{k=j}^{n} (1 - G_k(x_k))\right]^{-1/(j-2)} \qquad i = 1, \ldots, j-1$$

$$1 - G_i(p) = 1 - G_i(x_i) \qquad i = j, \ldots, n.$$

(c) $\forall p \in [x_{m+1}, r)$

$$1 - G_i(p) = \left[\frac{(r-p)U}{Ip - c(I+U) + c(U)}\right]^{1/(m-1)} \left[\prod_{k=m+1}^{n} (1 - G_k(x_k))\right]^{-1/(m-1)} \qquad i = 1, \ldots, m$$

$$1 - G_i(p) = 1 - G_i(x_i) \qquad i = m+1, \ldots, n$$

(d) $1 - G_i(r) = 0 \qquad \forall i.$

Appendix B

This appendix deals with degenerate rankings of $U_1, \ldots, U_n$ in the proof of Theorem 3. We first deal with the case where $U_1 < U_2 = U_3 = \cdots = U_m < U_{m+1} \leq, \ldots, \leq U_n$ for some $3 \leq m \leq n$. It is easily seen that for this case the previously altered versions of Lemmas 1 through 6 hold, as do Lemmas 7′ through 9′. Lemmas 11′ through 13′ also continue to hold (with an obvious alteration in the labelling of players in the proof of Lemma 11′). Lemmas 10′ and 14′ must be altered (slightly) as follows; the proofs require only a minor change in the labelling of players.

LEMMA 10″. $\underline{s} = \underline{p}_2$. *There exists at least one firm i, $2 \leq i \leq m$, such that* $\underline{s}_i = \underline{p}_2$.

LEMMA 14″. $\underline{s}_i = r \ \forall i > m$.

If $n > m$ we are through in our proof that such an allocation of consumers cannot be part of a subgame perfect Nash equilibrium; firms $m + 1, \ldots, n$ place all mass at r while other firms place mass below r, which contradicts the fact that $U_n > U_j \; \forall j \leq m$.

Suppose then that $n = m$, so that $U_1 < U_2 = U_3 = \ldots = U_n$. The following versions of Lemmas 12 and 14 hold for firms $2, \ldots, n$.

LEMMA 15″. *Suppose p is a point of increase of G_i and G_j in $[\underline{p}_2, r]$, $i, j \in \{2, \ldots, n\}$. Then $G_i = G_j$ at p.*

Proof. Same as proof of Lemma 12. ∎

LEMMA 16″. *If G_i, $i \in \{2, \ldots, n\}$, is strictly increasing on some open subset (x, y), $\underline{p}_2 < x < y < r$, then G_i is strictly increasing on the whole interval $[\underline{p}_2, y)$.*

Proof. Similar to proof of Lemma 14 where one of the firms l, m must be an element of $\{2, \ldots, n\}$ and this firm is used throughout the continuation of the proof. ∎

Lemma 16″, together with Lemmas 10″ and 13′, imply the following:

LEMMA 17″. *At least one of the firms $2, \ldots, n$ must randomize on the interval $[\underline{p}_2, r]$.*

We are now in a position to show that the indicated allocation of consumers cannot be part of a subgame perfect equilibrium. To do this we show that G_1 is strictly first-order stochastically dominated by G_i, $i \in \{2, \ldots, n\}$.

LEMMA 18″. *$\underline{s}_1 = \underline{p}_2$, and for every price $\underline{p}_2 < p < r$ in the support of G_1, $G_1(p) > G_i(p)$, $i \in \{2, \ldots, n\}$.*

Proof. From Lemma 17″ at least one of the firms $2, \ldots, n$ has support $[\underline{p}_2, r]$. Without loss of generality, suppose this is firm 2. From Lemmas 3 and 8′, firm 1 does not have a mass point at r, and from Lemma 11′ no firm has a mass point in $[\underline{p}_2, r)$. Thus, there exists some point $p \in (\underline{p}_2, r)$ at which $G_1(p)$ is increasing. At any such point

$$B_1(p) \geq \underline{p}_2(I + U_1)$$

since the right-hand side is what firm 1 can obtain by charging $\underline{p}_2$. Rearranging this expression, we obtain

$$A_1(p) \geq [\underline{p}_2(I + U_1) - \quad pU_1]/pI.$$

From Lemmas 9′ and 12′

$$A_2(p) = (r - p)U_2/pI.$$

Recalling that $\underline{p}_2 = rU_2/(I + U_2)$ we may subtract A_2 from A_1 to obtain

$$A_1(p) - A_2(p) \geq (U_2 - U_1)(p - \underline{p}_2)/pI > 0,$$

where the strict right-hand inequality follows from the assumption that $U_2 > U_1$ and $p > \underline{p}_2$. Thus, at any point of increase of G_1 in the interval $(\underline{p}_2, r)$, $A_1 > A_2$. This directly implies that $G_1 > G_2$ for any such point. But since G_2 has support $[\underline{p}_2, r]$ and G_1 has no mass points, this implies that $\underline{s}_1 = \underline{p}_2$. Furthermore, since for any other firm $i \in \{2, \ldots, n\}$ and for any $p \in [\underline{p}_2, r]$, $G_2(p) \geq G_i(p)$, we have the claim. ■

An immediate consequence of Lemma 18″ and the fact that G_1 has no mass points is that $G_1(p) \geq G_2(p)$ for every p in $[\underline{p}_2, r]$, with strict inequality on the open interval. This contradicts the fact that $U_1 < U_2$, so the given allocation of consumers cannot be part of a subgame perfect equilibrium.[13]

The remaining cases to be covered, where $U_1 = U_2 = \ldots = U_m < U_{m+1} \leq \ldots \leq U_n$ for $2 \leq m \leq n - 1$, require a mixture of the analysis of the symmetric case and asymmetric case. It can be shown that firms 1 through m may play any m-firm equilibrium of the type outlined in Theorem 1, while firms $m + 1$ through n put all mass at r. Since at least two of the firms among $\{1 \ldots, m\}$ put all probability mass below r, this cannot be a subgame perfect equilibrium allocation of consumers.

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[13] Note that in our proof we did not need to demonstrate that the support of G_1 is $[\underline{p}_2, r]$. Nevertheless, this is indeed the case.

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