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A CHARACTERIZATION OF QUALITY-ADJUSTED LIFE-YEARS UNDER CUMULATIVE PROSPECT THEORY

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Quality-adjusted life-years (QALYs) are the most common utility measure in medical decision analysis and economic evaluations of health care. This paper presents an axiomatization of QALYs under cumulative prospect theory (CPT), currently the most influential model for decision under uncertainty. Because the set of health states need not be endowed with a natural topology that is connected, we first show how existing CPT characterizations can be extended to a class of outcome sets for which no connected natural topology is given. We then characterize QALY models with linear, power, and exponential utility for duration. Finally, we define loss aversion for multiattribute utility theory and characterize the QALY models under general and constant loss aversion. The measurement of QALYs belongs to the general field of multiattribute utility theory. Hence, our results can be generalized to other multiattribute decision contexts and they thereby contribute to the development of multiattribute utility theory under cumulative prospect theory.

This paper presents characterizations of quality-adjusted life-years (QALYs) under cumulative prospect theory. QALYs are the most common outcome measure in medical decision analysis and economic evaluations of health care (Gold et al. 1996, Drummond et al. 1997). They provide a simple way to trade off the two main dimensions of health, duration, and health status. QALYs are tractable and easy to communicate to decision makers. A disadvantage of QALYs is that they represent individual preferences over health profiles only under strong assumptions.

Axiomatic foundations for QALY utility models have been studied under the assumptions of expected utility (Pliskin et al. 1980, Maas and Wakker 1994, Bleichrodt et al. 1997, Miyamoto 1999) and rank-dependent utility (Bleichrodt and Quiggin 1997, Miyamoto, 1999). This paper extends this work to cumulative prospect theory (Tversky and Kahneman 1992). Cumulative prospect theory (CPT) is currently the most influential model for decision under uncertainty. CPT characterizes two major deviations from expected utility: probability transformation, the nonlinear weighting of probabilities, and loss aversion, the tendency to overweight outcomes that are perceived as losses relative to outcomes that are perceived as gains. Both probability transformation and loss aversion are well-documented in the empirical literature (see Tversky and Kahneman 1992, Starmer 2000, and the references therein).

We consider chronic health states. If health status is constant, the QALY domain is a Cartesian product, $\mathcal{T} \times \mathcal{H}$, where $\mathcal{T}$ is an interval of survival durations and $\mathcal{H}$ is a set of health states. In many applications of QALYs, $\mathcal{H}$ is a finite set of health states with no connected natural topology given. Hence, we would like our representation theorems to include the possibility that $\mathcal{T} \times \mathcal{H}$ is not endowed with a natural topology that is connected. Previous characterizations of CPT assumed a connected outcome set (Luce and Fishburn 1991, Wakker and Tversky 1993, Luce and Fishburn 1995, Luce 2000). We show that under an assumption that is entirely evident in the medical context, the zero-condition, $\mathcal{T} \times \mathcal{H}$ is connected in the order topology. This result makes it possible to extend previous representation theorems to a class of outcome sets for which no connected topology is

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naturally given. Our results are related to Fishburn (1981) and Gonzales (1996, 2000), which study additive representability for Cartesian products in which not every attribute set is connected. However, Fishburn and Gonzales do not consider CPT, they use an algebraic instead of a topological approach, and they do not use the zero-condition.

We derive three QALY models under cumulative prospect theory: the linear QALY model, the power QALY model, and the exponential QALY model. The linear QALY model, in which utility for duration is linear, is the most widely used QALY model. The assumption of linear utility for duration is sometimes weakened to accommodate empirical evidence of nonlinear utility for duration and to permit discounting and risk aversion. We consider the two most widely used nonlinear utility functions: the power function and the exponential function.

After characterizing the three QALY models we define loss aversion for multiattribute utilities. The definition of loss aversion is not straightforward because the magnitude of gains cannot be directly compared to the magnitude of losses due to differences in the decision weights for gains and losses. Our definition does not preclude that loss aversion varies over outcomes. In empirical work it is more convenient to assume that loss aversion is constant. We, therefore, also characterize a special class of the three QALY models, to which we refer as the *decomposable QALY models,* in which loss aversion is constant. The conditions we impose to characterize the decomposable QALY models allow us to weaken the axioms used to derive CPT by axioms that imply a more general rank- and state-dependent additive utility model when the number of states of nature is at least three. CPT is derived from this general model and the conditions used to characterize the decomposable QALY models.

The measurement of QALYs belongs to the general domain of multiattribute utility theory (Keeney and Raiffa 1976). Previous characterizations of cumulative prospect theory assumed single-attribute utility functions (Tversky and Kahneman 1992, Wakker and Tversky 1993). Dyckerhoff (1994) and Miyamoto and Wakker (1996) studied multiattribute utility theory without expected utility foundations, but only for outcomes of the same sign. Zank (2001), like us, derived results on multiattribute utility theory under cumulative prospect theory. The present study differs in two respects from Zank (2001). First, in Zank (2001) all outcomes have quantitative attributes that constitute connected sets. In our decision framework, one of the attributes, health status, is qualitative and need not be connected. Gonzales (2000) mentions several decision contexts besides medical decision making in which attribute sets on which no connected topology is given are important. A second difference with Zank (2001) is that we define and characterize loss aversion for multiattribute utility theory.

In what follows, §1 gives the notation and the assumptions that are required for CPT representations in a one-dimensional domain. Section 2 gives the notation and the assumptions that are required for CPT representations in a multiattribute domain like health outcomes. Section 3 presents a representation theorem that extends Wakker and Tversky's (1993) axiomatization of CPT to a class of outcome sets for which no connected natural topology is given. In §4, the linear, power, and exponential QALY models are defined under CPT. Characterizations of these models are given in §5. In §6, general loss aversion is defined, and the three QALY models are characterized under general loss aversion. Section 7 characterizes the three decomposable QALY models. Proofs are given in the appendix.

1. Notation and structural assumptions for uniattribute domains. Let $\mathcal{S} = \{1, \ldots, n\}$ be a finite *state space*. Subsets of $\mathcal{S}$ are *events*. $\mathcal{X}$ denotes the set of *outcomes*. A *prospect* is a function from $\mathcal{S}$ to $\mathcal{X}$. Let $\mathcal{F} = \mathcal{X}^n$ denote the set of prospects. For $f \in \mathcal{F}$, f_i is the outcome if the ith state obtains. Given a prospect $f \in \mathcal{F}$, a state $j \in \mathcal{S}$, and an outcome $x \in \mathcal{X}$, we denote by $x_j f$ the prospect f with f_j replaced by x. Given prospects f, $g \in \mathcal{F}$, and an event $A \subseteq \mathcal{S}$, $g_A f$ denotes the prospect f with f_j replaced by g_j for all $j \in A$; (x, A, y) denotes the binary prospect, which gives x if event A obtains and y otherwise.

Let $\succcurlyeq$ denote a preference relation over $\mathcal{F}$. $\succcurlyeq$ is a *weak order* if it is transitive and complete. The relations $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ are defined as usual. Outcomes are identified with constant prospects. The preference relation $\succcurlyeq$ satisfies *outcome monotonicity* if for all $f, g \in \mathcal{F}$, $f_j \succcurlyeq g_j$ for all $j \in \mathcal{S}$ implies $f \succcurlyeq g$, with strict preference holding if there is a $j \in \mathcal{S}$ for which $f_j \succ g_j$.

Outcomes are defined with respect to a designated outcome $x_0 \in \mathcal{X}$. Any outcome $y \sim x_0$ is a *reference outcome*. An outcome $x \in \mathcal{X}$ is a *gain* if $x \succ x_0$, a *loss* if $x \prec x_0$, a *nonloss* if $x \succcurlyeq x_0$, and a *nongain* if $x \preccurlyeq x_0$. For any $f \in \mathcal{F}$, let f^+ denote the prospect such that $f_j^+ = f_j$ if $f_j \succcurlyeq x_0$ and $f_j^+ = x_0$ if $f_j \prec x_0$. Similarly, let f^- denote the prospect such that $f_j^- = f_j$ if $f_j \preccurlyeq x_0$ and $f_j^- = x_0$ if $f_j \succ x_0$. The prospect f^+ denotes the nonloss part of f, and the prospect f^- denotes the nongain part of f. Let $\mathcal{X}^+$ be the set $\{x \in \mathcal{X}\colon x \succcurlyeq x_0\}$ and $\mathcal{X}^-$ the set $\{x \in \mathcal{X}\colon x \preccurlyeq x_0\}$. $\mathcal{F}^+$ is the set of *nonloss prospects* $\{f \in \mathcal{F}\colon f_j \succcurlyeq x_0$ for all $j \in \mathcal{S}\}$. $\mathcal{F}^-$ is the set of *nongain prospects* $\{f \in \mathcal{F}\colon f_j \preccurlyeq x_0$ for all $j \in \mathcal{S}\}$.

A prospect is *rank-ordered* if $f_1 \succcurlyeq \cdots \succcurlyeq f_n$. For each prospect there exists a permutation ρ such that $f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}$. For each permutation ρ, $\mathcal{F}_\rho = \{f \in \mathcal{F}\colon f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}\}$. For $A \subseteq \mathcal{S}$, the set $\mathcal{F}^A$ contains those prospects that yield nonlosses on A and nongains on A^c. $\mathcal{F}_\rho^A = \mathcal{F}_\rho \cap \mathcal{F}^A$. Subsets of sets $\mathcal{F}_\rho^A$ are *sign-comonotonic*.

A real function $V\colon \mathcal{F} \to \mathbb{R}$ *represents* $\succcurlyeq$ on $\mathcal{F}$ if $V(f) \geq V(g)$ iff $f \succcurlyeq g$. A *capacity* W on $\mathcal{S}$ is a function on $2^{\mathcal{S}}$ such that $W(\varnothing) = 0$, $W(S) = 1$, and if $A \supset B$, then $W(A) \geq W(B)$. The *CPT functional* assigns a CPT value to each prospect, defined next. The *CPT value* of a prospect $f \in \mathcal{F}_\rho^A$ with $A = \{\rho(1), .., \rho(k)\}$ for some $k \leq n$ is

$$CPT(f) = \sum_{i=1}^{k} \pi^+_{\rho(i)} U(f_{\rho(i)}) + \sum_{i=k+1}^{n} \pi^-_{\rho(i)} U(f_{\rho(i)}), \tag{1}$$

with

$$\pi^+_{\rho(i)} = W^+(\rho(1), \ldots, \rho(i)) - W^+(\rho(1), \ldots, \rho(i-1)), \tag{2a}$$

and

$$\pi^-_{\rho(i)} = W^-(\rho(i), \ldots, \rho(n)) - W^-(\rho(i+1), \ldots, \rho(n)), \tag{2b}$$

where W^+ and W^- are capacities for gains and losses, respectively, and U is a real-valued *utility function* over $\mathcal{X}$.

Define $[x; y] \succ^* [v; w]$ if there exist prospects f, g and a state j such that $x_j f \succcurlyeq y_j g$ and $v_j f \prec w_j g$ and the four prospects $\{x_j f,\ y_j g,\ v_j f,\ w_j g\}$ are sign-comonotonic. A state j is *nonnull* on a sign-comonotonic set $\mathcal{F}_\rho^A$ if there exist $f_j x, f_j y \in \mathcal{F}_\rho^A$ such that $f_j x \succ f_j y$. Define $[x; y] \succcurlyeq^* [v; w]$ if there exist prospects f, g and a state j such that $x_j f \succcurlyeq y_j g$ and $v_j f \preccurlyeq w_j g$ and the four prospects $\{x_j f, y_j g, v_j f, w_j g\}$ are sign-comonotonic and state j is nonnull on the sign-comonotonic set containing these prospects. As usual, $\preccurlyeq^*$ and $\prec^*$ denote the reversed relations and $\sim^*$ denotes the intersection of $\succcurlyeq^*$ and $\preccurlyeq^*$. Wakker and Tversky (1993) showed that under CPT the star relations order utility differences, i.e., $[x; y] \succcurlyeq^* [v; w]$ iff $U(x) - U(y) \geq U(v) - U(w)$.

The preference relation $\succcurlyeq$ satisfies *sign-comonotonic trade-off consistency*, or *trade-off consistency* for short, if there exist no outcomes $x, y, u, v \in \mathcal{X}$ such that both $[x; y] \succ^* [u; v]$ and $[u; v] \succcurlyeq^* [x; y]$. The preference relation $\succcurlyeq$ satisfies *gain-loss consistency* if for all $f, g \in \mathcal{F}$, $f^+ \sim g^+$, $f^- \sim g^-$, and $f \sim x_0$ implies $g \sim x_0$. The preference relation $\succcurlyeq$ is *truly mixed* if there exists a prospect $f \in \mathcal{F}$ with $f^+ \succ x_0$ and $f^- \prec x_0$. This definition obviously implies that $n \geq 2$. If $\mathcal{X}$ is a connected topological space and $\succcurlyeq$ is truly mixed, Wakker and Tversky (1993) showed that CPT represents preferences over prospects with U continuous, U a ratio scale, and W^+ and W^- uniquely determined iff $\succcurlyeq$ is a continuous weak order that satisfies trade-off consistency and gain-loss consistency.

2. Notation and structural assumptions for multiattribute domains. $\mathscr{X}$ is a Cartesian product of the set of durations $\mathscr{T} = [0, M]$, $M > 0$, and the set of health states $\mathscr{H}$. $\mathscr{H}$ is a general set. We assume that the reference outcome $x_0 = (t_0, h_0) \succ (0, h)$ for all $h \in \mathscr{H}$. Health status is assumed to be *essential*, i.e., there exist $h_1, h_2 \in \mathscr{H}$ such that for some $t \in \mathscr{T}$, $(t, h_1) \succ (t, h_2)$. Essentialness of duration is defined similarly and is implied by the assumption that $(t_0, h_0) \succ (0, h)$ for all $h \in \mathscr{H}$. The preference relation $\succcurlyeq$ is *continuous in duration* if for all f, $g \in \mathscr{F}$, for all $h \in \mathscr{H}$, and for all $j \in \mathscr{S}$ the sets $\{t \in \mathscr{T}\colon (t, h)_j f \succcurlyeq g\}$ and $\{t \in \mathscr{T}\colon (t, h)_j f \preccurlyeq g\}$ are closed. The preference relation $\succcurlyeq$ is *monotonic in duration* if for all $h \in \mathscr{H}$, and $t_1, t_2 \in \mathscr{T}$, $t_1 > t_2$ implies $(t_1, h) \succ (t_2, h)$.

Our representation theorems and proofs will use notation for substructures of a multiattribute domain. For any $h \in \mathscr{H}$, let $\mathscr{X}_h = \{(t, h)\colon t \in \mathscr{T}\}$. Similarly, for any $t \in \mathscr{T}$, let $\mathscr{X}_t = \{(t, h)\colon h \in \mathscr{H}\}$. Let $\mathscr{F}_j = \mathscr{X}_j^n$, $j = h, t$. For $i = +, -$ and $j = h, t$, $\mathscr{X}_j^i = \mathscr{X}_j \cap \mathscr{X}^i$, $\mathscr{F}_j^i = \mathscr{F}_j \cap \mathscr{F}^i$, and for each permutation ρ, $\mathscr{F}_{\rho, j} = \mathscr{F}_\rho \cap \mathscr{F}_j$, $\mathscr{F}_{\rho, j}^i = \mathscr{F}_{\rho, j} \cap \mathscr{F}^i$. There may exist $h \in \mathscr{H}$ for which there is no $t \in \mathscr{T}$ such that $(t, h) \sim x_0$. However, if $\succcurlyeq$ is continuous in duration and there exists a $t \in \mathscr{T}$ such that $(t, h) \succ x_0$, then there exists an $s \in \mathscr{T}$ such that $(s, h) \sim x_0$ because $\mathscr{X}_h$ is connected. Define $r(h) = s$ with s the duration such that $(s, h) \sim x_0$. If $\succcurlyeq$ is monotonic in duration, then s is unique. The duration $r(h)$ is the *reference level of duration with respect to h*. Let $\mathscr{H}^0 = \{h \in \mathscr{H}\colon r(h) \text{ exists}\}$. $\mathscr{H} - \mathscr{H}^0$ denotes the complement of $\mathscr{H}^0$ in $\mathscr{H}$.

In later sections we will study conditions under which a utility function U exhibits loss aversion. This construct is meaningful only if the outcome domain has a quantitative measure, e.g., money or life years. We, therefore, consider loss aversion only for duration, not for health status. In the context of QALY models, loss aversion is conditional on a health state. The utility function *U is loss averse with respect to health state $h \in \mathscr{H}^0$* if

$$U(r(h)+x, h) - U(r(h)+y, h) \leq U(r(h)-y, h) - U(r(h)-x, h) \tag{3}$$

whenever $x > y > 0$ and $r(h)+x, r(h)-x \in \mathscr{T}$. The existence of $r(h)$ is ensured because h is in $\mathscr{H}^0$. In other words, given $h \in \mathscr{H}^0$, the utility function is steeper for losses than for corresponding gains. *Loss aversion* holds if U is loss averse with respect to every $h \in \mathscr{H}^0$. If $h \in \mathscr{H} - \mathscr{H}^0$, then $x_0 \succcurlyeq (t, h)$ for all $t \in \mathscr{T}$ so Equation (3) is vacuously satisfied.

Let us summarize the structural assumptions made throughout the paper.

Structural assumption 1. $\mathscr{S}$ is a finite state space. $\mathscr{X} = \mathscr{T} \times \mathscr{H}$, $\mathscr{T} = [0, M]$ for some $M \in \mathbb{R}^+$, $\mathscr{H}$ is general. Health status is essential. There exists a preference relation $\succcurlyeq$ over $\mathscr{F} = \mathscr{X}^n$. The reference outcome x_0 satisfies $x_0 \succ (0, h)$ for all $h \in \mathscr{H}$.

3. CPT for an outcome set that is not a connected topological space. Because $\mathscr{H}$ is general, we cannot assume that $\mathscr{X}$ is endowed with a natural topology that is connected. Hence, we cannot use Wakker and Tversky's (1993) Theorem 6.3 to infer the existence of a CPT representation for $\succcurlyeq$ over $\mathscr{F}$. We can, however, extend this theorem to a domain with no connected natural topology given, provided that $\succcurlyeq$ satisfies the *zero-condition*, i.e., for all $h, h' \in \mathscr{H}$: $(0, h) \sim (0, h')$. The zero-condition is self-evident in the medical context because $(0, h)$ and $(0, h')$ are indistinguishable under the interpretation of time as survival duration (Miyamoto and Eraker 1988, Bleichrodt et al. 1997, Miyamoto et al. 1998). The zero-condition implies that all sets $\mathscr{X}_h$ overlap in the preference order with respect to points of the form $(0, h)$.

The next two lemmas are the main mathematical steps in our analysis. Lemma 3.1 is central in extending earlier representations that concerned only connected topological spaces to the outcome set of this paper for which no connected natural topology is given. Let $T_\succcurlyeq$ denote the order topology on $\mathscr{X}$.

LEMMA 3.1. *Suppose that structural assumption* 1 *holds. If $\succcurlyeq$ is a weak order that satisfies the zero-condition, then $T_\succcurlyeq$ is connected.*

Lemma 3.2 is the most intricate part in the mathematical proof. Let $\mathcal{T}^n_{\succcurlyeq}$ denote the product topology on $\mathcal{F}$.

LEMMA 3.2. *Suppose that structural assumption* 1 *holds. If* $\succcurlyeq$ *is a weak order that is continuous in duration and monotonic in duration and that satisfies outcome monotonicity and the zero-condition, then* $\succcurlyeq$ *is continuous w.r.t.* $\mathcal{T}^n_{\succcurlyeq}$ *on* $\mathcal{F}$.

Note that we did not presuppose continuity with respect to a connected product topology. Lemma 3.2 shows, however, that the preference relation is continuous with respect to the product topology of the order topologies within every component. That is, the product topology of the order topologies within every component is finer than the order topology of the overall preference relation.

THEOREM 3.3. *Suppose that structural assumption* 1 *holds, that* $\succcurlyeq$ *is truly mixed, and that* $\succcurlyeq$ *satisfies the zero-condition. Then the following two statements are equivalent*:

(1) *CPT holds with a utility function that is continuous in duration and increasing in duration and with positive decision weights.*

(2) $\succcurlyeq$ *is a weak order that is continuous in duration and monotonic in duration and that satisfies outcome monotonicity, trade-off consistency and, if* $n = 2$, *gain-loss consistency.*

4. The QALY models.

4.1. The linear QALY model. Suppose that CPT holds. The *linear QALY model* holds if $t \mapsto U(t, h)$ is linear both for gains and for losses. Formally, if $t < r(h)$ or if $r(h)$ does not exist, then

$$U(t, h) = \lambda(h) \cdot \nu(h) \cdot t - k, \tag{4}$$

and if $t \geq r(h)$, then

$$U(t, h) = \nu(h) \cdot t + \alpha(h), \tag{5}$$

with $\nu(h)$ and $\lambda(h)$ positive functions of h. The function value $\lambda(h)$ reflects different sensitivity for losses than for gains. The function value $\alpha(h)$ ensures continuity of U at the reference level of duration. The scaling constant k ensures that $U(x_0) = 0$. Note that $U(0, h) = -k$ for all h, so that this family automatically satisfies the zero-condition. Conversely, the zero-condition ensures that k is independent of h. The scaling $U(x_0) = 0$ is common in prospect theory. In medical decision making it is more common to have $U(0, h) = 0$ for all $h \in \mathcal{H}$.

4.2. The power QALY model. Suppose that CPT holds. The *power QALY model* holds if $t \mapsto U(t, h)$ is a member of the log/power family for gains and a possibly different member of the log/power family for losses. The latter family will only include the positive powers because zero is contained in its domain and negative powers and the logarithm are not defined at zero. Formally, for each $h \in \mathcal{H}$ there exist $\theta(h)$, $\tau(h)$ such that if $t < r(h)$ or if $r(h)$ does not exist, then

$$U(t, h) = \lambda(h) \cdot \nu(h) \cdot t^{\tau(h) \cdot \theta(h)} - k \quad \text{with } \tau(h) \cdot \theta(h) > 0, \tag{6}$$

and if $t \geq r(h)$, then

$$U(t, h) = \nu(h) \cdot t^{\theta(h)} + \alpha(h) \quad \text{if } \theta(h) > 0, \tag{7a}$$

$$U(t, h) = \nu(h) \cdot \log(t) + \alpha(h) \quad \text{if } \theta(h) = 0, \tag{7b}$$

$$U(t, h) = -\nu(h) \cdot t^{\theta(h)} + \alpha(h) \quad \text{if } \theta(h) < 0. \tag{7c}$$

$\nu(h)$, $\lambda(h)$, $\alpha(h)$, and k are positive, and their interpretation is as in the linear QALY model. The function value $\tau(h)$ reflects different curvature for losses than for gains. The cases $\tau(h)\cdot\theta(h)=0$ and $\tau(h)\cdot\theta(h)<0$ are excluded because zero is contained in the domain of $\mathscr{T}$. It is permissible that Equation (7a) holds for some $h\in\mathscr{H}^0$ and Equation (7b) or (7c) for other $h\in\mathscr{H}^0$. Because $U(0,h)=-k$ for all h, the family satisfies the zero-condition.

4.3. The exponential QALY model. Suppose that CPT holds. The *exponential QALY model* holds if $t\mapsto U(t,h)$ is a member of the linear/exponential family for gains and a possibly different member of the linear/exponential family for losses. Formally, for each $h\in\mathscr{H}$ there exist $\varphi(h)$ and $\tau(h)$ such that if $t<r(h)$ or if $r(h)$ does not exist, then

(8a) $$U(t,h)=\lambda(h)\cdot\nu(h)\cdot(e^{\tau(h)\cdot\phi(h)\cdot t}-1)-k \quad \text{if } \tau(h)\cdot\varphi(h)>0,$$

(8b) $$U(t,h)=\lambda(h)\cdot\nu(h)\cdot t-k \quad \text{if } \tau(h)\cdot\varphi(h)=0,$$

(8c) $$U(t,h)=-\lambda(h)\cdot\nu(h)\cdot(e^{\tau(h)\cdot\phi(h)\cdot t}-1)-k \quad \text{if } \tau(h)\cdot\varphi(h)<0,$$

and if $t\geq r(h)$, then

(9a) $$U(t,h)=\nu(h)\cdot e^{\phi(h)\cdot t}+\alpha(h) \quad \text{if } \varphi(h)>0,$$

(9b) $$U(t,h)=\nu(h)\cdot t+\alpha(h) \quad \text{if } \varphi(h)=0,$$

(9c) $$U(t,h)=-\nu(h)\cdot e^{\phi(h)\cdot t}+\alpha(h) \quad \text{if } \varphi(h)<0.$$

The interpretation of the parameters is similar as in the power QALY model. Again, it is permissible that Equation (8a) holds for some $h\in\mathscr{H}$ and Equation (8b) or (8c) for other $h\in\mathscr{H}$ and that Equation (9a) holds for some $h\in\mathscr{H}^0$ and Equation (9b) or (9c) for other $h\in\mathscr{H}^0$. Because $U(0,h)=-k$ for all h, the family satisfies the zero-condition.

5. Characterization of the QALY models under CPT. The preference relation $\succcurlyeq$ satisfies *constant sensitivity* on $\mathscr{X}^+$ if for all $h\in\mathscr{H}$ and for all (s,h), (t,h), $(s+\delta,h)$, $(t+\delta,h)\in\mathscr{X}^+$, and for all $\delta\in\mathbb{R}$, $[(s+\delta,h);(t+\delta,h)]\succ^*[(s,h);(t,h)]$ is excluded. Constant sensitivity on $\mathscr{X}^-$ is defined similarly.

For a given $f\in\mathscr{F}_h$, let $\delta\cdot f$ denote the operation defined by $\delta\cdot f=((\delta\cdot t_1,h),\ldots,(\delta\cdot t_n,h))$, $\delta>0$, whenever $\delta\cdot f\in\mathscr{F}_h$. That is, duration in each state of the world is multiplied by a common positive constant δ. The preference relation $\succcurlyeq$ satisfies *constant proportional risk aversion* on $\mathscr{F}^+$ if for all $h\in\mathscr{H}$ and for all $\delta\in\mathbb{R}^+$, if $f,\ g,\ \delta\cdot f,\ \delta\cdot g\in\mathscr{F}_h^+$, then $f\succcurlyeq g$ iff $\delta\cdot f\succcurlyeq\delta\cdot g$. Constant proportional risk aversion on $\mathscr{F}^-$ is defined similarly.

For a given $f\in\mathscr{F}_h$, let $\delta+f$ denote the operation defined by $\delta+f=((\delta+t_1,h),\ldots,(\delta+t_n,h))$, $\delta\in\mathbb{R}$ whenever $\delta+f\in\mathscr{F}_h$. That is, a common constant δ is added to duration in each state of the world. The preference relation $\succcurlyeq$ satisfies *constant absolute risk aversion on* $\mathscr{F}^+$ if for all $h\in\mathscr{H}$ and for all $\delta\in\mathbb{R}$, if $f,g,\delta+f,\delta+g\in\mathscr{F}_h^+$, then $f\succcurlyeq g$ iff $\delta+f\succcurlyeq\delta+g$. Constant absolute risk aversion on $\mathscr{F}^-$ is defined similarly.

THEOREM 5.1. *Suppose that statement* (1) *of Theorem* 3.3 *holds. Then*

(a) *The linear QALY model holds iff* $\succcurlyeq$ *satisfies the zero-condition and constant sensitivity holds on* $\mathscr{X}^-$ *and on* $\mathscr{X}^+$.

(b) *The power QALY model holds iff* $\succcurlyeq$ *satisfies the zero-condition and constant proportional risk aversion holds on* $\mathscr{F}^-$ *and on* $\mathscr{F}^+$.

(c) *The exponential QALY model holds iff* $\succcurlyeq$ *satisfies the zero-condition and constant absolute risk aversion holds on* $\mathscr{F}^-$ *and on* $\mathscr{F}^+$.

6. The QALY models with general loss aversion. Let us now turn to loss aversion. We extend Wakker and Tversky's (1993) definition of loss aversion to multiattribute utility functions. Suppose that CPT holds with U continuous in duration and increasing in duration. Let $\omega_h(t) = U(t, h)$ be a utility function over duration defined with health status held constant at $h \in \mathscr{H}$. Loss aversion in the sense of Equation (3) means that for all $h \in \mathscr{H}^0$, and for all $r(h)+t_1,\ r(h)-t_1 \in \mathscr{T}$ with $t_1 > t_2 \geq 0$: $\omega_h(r(h)+t_1) - \omega_h(r(h)+t_2) < \omega_h(r(h)-t_2) - \omega_h(r(h)-t_1)$. Let $r(h)+t_1,\ r(h)-t_1 \in \mathscr{T}$ with $t_1 > t_2 \geq 0$ and let $h \in \mathscr{H}^0$. Let f and g be sign-comonotonic prospects such that $f^+ \sim (r(h)+t_1, h)$, $f^- \sim (r(h)-t_1, h)$, $g^+ \sim (r(h)+t_2, h)$, and $g^- \sim (r(h)-t_2, h)$. If t_1 and t_2 are not too extreme, it is possible to find such prospects. Substitution gives $\text{CPT}(f^+) = \omega_h(r(h)+t_1)$, $\text{CPT}(f^-) = \omega_h(r(h)-t_1)$, $\text{CPT}(g^+) = \omega_h(r(h)+t_2)$, $\text{CPT}(g^-) = \omega_h(r(h)-t_2)$, and, because $\text{CPT}(f) = CPT(f^+) + \text{CPT}(f^-)$ and $\text{CPT}(g) = \text{CPT}(g^+) + \text{CPT}(g^-)$, loss aversion holds iff $f \preccurlyeq g$.

If t_1 and t_2 are too extreme to find such prospects f and g, then it may be possible to find $t_1' > t_2' \geq 0$ and $t_1'' > t_2'' \geq 0$ close to zero such that $[(r(h)+t_1', h);\ r(h)+t_2', h)] \sim^* [(r(h)+t_1, h);\ (r(h)+t_2, h)]$ and $[(r(h)-t_2'', h);\ (r(h)-t_1'', h)] \sim^* [(r(h)-t_2, h);\ (r(h)-t_1, h)]$. That is, we try to copy $[(r(h)+t_1, h);\ (r(h)+t_2, h)]$ and $[(r(h)-t_2, h); (r(h)-t_1, h)]$ to a neighborhood of $r(h)$ in which the required f and g exist. Copies need not exist for arbitrary t_1 and t_2, but if $t_1 - t_2$ is sufficiently small the required copies can be found.

From the $\sim^*$ relations we obtain $\omega_h(r(h)+t_1') - \omega_h(r(h)+t_2') = \omega_h(r(h)+t_1) - \omega_h(r(h)+t_2)$ and $\omega_h(r(h)-t_2'') - \omega_h(r(h)-t_1'') = \omega_h(r(h)-t_2) - \omega_h(r(h)-t_1)$. Because t_1', t_1'', t_2', and t_2'' are all close to zero we can, by continuity in duration and the assumption of the truly mixed case, find sign-comonotonic prospects f and g that satisfy $f^+ \sim (r(h)+t_1', h)$, $g^+ \sim (r(h)+t_2', h)$, $f^- \sim (r(h)-t_1'', h)$, and $g^- \sim (r(h)-t_2'', h)$. It follows that loss aversion holds iff $f \preccurlyeq g$.

THEOREM 6.1. *Suppose that the assumptions of Theorem* 5.1 *hold. Then the linear, power, and exponential QALY models satisfy loss aversion iff for all $h \in \mathscr{H}^0$ there are no $(r(h)+t_1, h)$, $(r(h)+t_2, h)$, $(r(h)+t_1', h)$, $(r(h)+t_2', h) \in \mathscr{X}_h^+$, $(r(h)-t_1, h)$, $(r(h)-t_2, h)$, $(r(h)-t_1'', h)$, $(r(h)-t_2'', h) \in \mathscr{X}_h^-$, with $t_1 > t_2 \geq 0$, $t_1' > t_2' \geq 0$, $t_1'' > t_2'' \geq 0$ and $f, g \in \mathscr{F}_\rho^A$ such that $[(r(h)+t_1, h);\ (r(h)+t_2, h)] \sim^* [(r(h)+t_1', h);\ (r(h)+t_2', h)]$, $[(r(h)-t_2, h);\ (r(h)-t_1, h)] \sim^* [(r(h)-t_2'', h);\ (r(h)-t_1'', h)]$, $f^+ \sim (r(h)+t_1', h)$, $g^+ \sim (r(h)+t_2', h)$, $f^- \sim (r(h)-t_1'', h)$, $g^- \sim (r(h)-t_2'', h)$, and $f \succ g$.*

7. The decomposable QALY models. The *decomposable linear, power, and exponential QALY models* are defined by setting $\lambda(h) = \lambda$ in Equation (4); $\lambda(h) = \lambda$, $\theta(h) = \theta$, and $\tau(h) = \tau$ in Equation (6); $\theta(h) = \theta$ in Equations (7a)–(7c); $\lambda(h) = \lambda$, $\phi(h) = \phi$, and $\tau(h) = \tau$ in Equations (8a)–(8c); and $\phi(h) = \phi$ in Equations (9a)–(9c). In the power and exponential QALY models, defined in §4, the utility of duration still depends on health status. In the decomposable QALY models, the utility of duration is independent of health status and $U(t, h)$ has been "truly" decomposed into a utility function over health status and a utility function over duration. This explains our naming of these models. Because λ is also independent of health status, the decomposable QALY models imply constant loss aversion.

In case $n \geq 3$, the conditions that we impose to characterize the decomposable QALY models allow us to weaken the assumption that CPT holds. In particular, trade-off consistency need no longer be imposed. Instead we assume a more general utility model, defined in Lemma A.1 in the appendix. CPT can be derived from this general model and the conditions used to characterize the decomposable QALY models.

7.1. Characterization. Health status is *preferentially independent* of duration if for all $t_1, t_2 \in \mathcal{T}$, and for all $h_1, h_2 \in \mathcal{H}$: $(t_1, h_1) \succcurlyeq (t_1, h_2)$ iff $(t_2, h_1) \succcurlyeq (t_2, h_2)$. The preference relation $\succcurlyeq$ satisfies *attribute monotonicity* if $\succcurlyeq$ satisfies monotonicity in duration and health status is preferentially independent of duration. *Tail independence* holds if $[x_A f \succcurlyeq x_A g$ iff $y_A f \succcurlyeq y_A g]$ whenever $A = \{\rho(1), \ldots, \rho(k)\}$ or $A = \{\rho(m), \ldots, \rho(n)\}$ for some $k, m \in \mathcal{S}$ and all prospects in question are from the same set $\mathcal{F}_{\rho, h}$.

The preference relation $\succcurlyeq$ satisfies *weak utility independence* on $\mathcal{F}^-$ if for all events $A \subseteq \mathcal{S}$, for all $h, h' \in \mathcal{H}$ and for all $s, t, w \in \mathcal{T}$ with $s, t, w \le r(h)$ if $h \in \mathcal{H}^0$ and $s, t, w \le r(h')$ if $h' \in \mathcal{H}^0$, $(s, h) \sim ((t, h), A, (w, h))$ iff $(s, h') \sim ((t, h'), A, (w, h'))$.

The preference relation $\succcurlyeq$ satisfies *mixed utility independence* if for all events $A \subseteq \mathcal{S}$, for all $h, h' \in \mathcal{H}^0$, for all $s, w \in \mathcal{T}$ with $s, w \ge \max\{r(h), r(h')\}$, and for all $t, z \in \mathcal{T}$ with $t, z \le \min\{r(h), r(h')\}$, $((s, h), A, (t, h)) \sim ((w, h), A, (z, h))$ iff $((s, h'), A, (t, h')) \sim ((w, h'), A, (z, h'))$.

For $h \in \mathcal{H}^0$, mixed utility independence implies weak utility independence. However, mixed utility independence has no implications for $h \in \mathcal{H} - \mathcal{H}^0$. Therefore, we have to impose both mixed utility independence and weak utility independence on $\mathcal{F}^-$ in the next theorem.

ASSUMPTION 7.1. *The preference relation $\succcurlyeq$ satisfies weak utility independence on $\mathcal{F}^-$ and mixed utility independence. If $n = 2$ then statement* (2) *of Theorem* 3.3 *holds. If $n \ge 3$, then $\succcurlyeq$ is a weak order that is continuous in duration and satisfies outcome monotonicity, attribute monotonicity, and tail independence.*

THEOREM 7.2. *Suppose that structural assumption* 1 *holds and that $\succcurlyeq$ is truly mixed. Then*

(a) *CPT holds with the utility function equal to the decomposable linear QALY model iff Assumption* 7.1 *holds, $\succcurlyeq$ satisfies the zero-condition, and constant sensitivity holds on $\mathcal{X}^-$ and on $\mathcal{X}^+$.*

(b) *CPT holds with the utility function equal to the decomposable power QALY model iff Assumption* 7.1 *holds, $\succcurlyeq$ satisfies the zero-condition, and constant proportional risk aversion holds on $\mathcal{F}^-$ and on $\mathcal{F}^+$.*

(c) *CPT holds with the utility function equal to the decomposable exponential QALY model iff Assumption* 7.1 *holds, $\succcurlyeq$ satisfies the zero-condition, and constant absolute risk aversion holds on $\mathcal{F}^-$ and on $\mathcal{F}^+$.*

Appendix: Proofs.

PROOF OF LEMMA 3.1. The following proof does not impose any restriction on $\mathcal{H}$, and this set can be completely general. Consider the order topology $T_{\succcurlyeq}$ on $\mathcal{X}$, i.e. the smallest topology containing all sets $\{x \in \mathcal{X}: x \succ y\}$ and $\{x \in \mathcal{X}: x \prec y\}$. The preference relation $\succcurlyeq$ on $\mathcal{X}$ is continuous with respect to this topology. We show that $T_{\succcurlyeq}$ is connected.

A union of an arbitrary collection of connected sets with nonempty intersection is connected again. For any topology, each element of the space is contained in a maximal connected set, its *topological component*. Each topological component is closed. The topological components partition the space. Consider, for any arbitrary $h \in \mathcal{H}$, the topological component containing $(0, h)$. This set contains:

(a) $\mathcal{X}_{h'}$ for each $h' \in \mathcal{H}$ for which $(0, h')$ is contained in the component, because $\mathcal{X}_{h'}$ is connected.

(b) $(0, h')$ for each $h' \in \mathcal{H}$ because $(0, h') \sim (0, h)$ and every closed set that contains $(0, h)$ also contains $(0, h')$.

Because of (a) and (b), the topological component of $(0, h)$ is the whole set $\mathcal{X}$. $\mathcal{X}$ is connected with respect to the order topology indeed. $\square$

PROOF OF LEMMA 3.2. Consider the product topology $T^n_{\succcurlyeq}$ on the set of prospects $\mathcal{F} = \mathcal{X}^n$, which is also connected. We show that $\succcurlyeq$ on the set of prospects is continuous with respect to $T^n_{\succcurlyeq}$.

Consider $\{g' \in \mathcal{F}: g' \succ g\}$ for some $g \in \mathcal{F}$. We prove that this set is open with respect to $\mathcal{T}^n_{\succcurlyeq}$. Let f be an element of this set, i.e., $f \succ g$.

CLAIM 1. *If f_1 is nonminimal, then we can find $f_1' \prec f_1$ such that still $(f_1', f_2, \ldots, f_n) \succ g$.*

PROOF. The outcome f_1 is nonminimal and, hence, is of the form (t, h) for some $t > 0$. Assume that $((0, h), f_2, \ldots, f_n) \preccurlyeq g$ (otherwise we are done). By continuity in duration, connectedness of $\mathcal{X}_h$, outcome monotonicity, and monotonicity in duration, there exists $0 < t' < t$ such that $((t', h), f_2, \ldots, f_n) \succ g$. Take $f_1' = (t', h)$. Q.E.D.

By induction, we can find $f' \succ g$ with $f_j' \prec f_j$ for all nonminimal f_j and $f_j' = f_j$ for all minimal f_j. Define $\mathcal{A}_j = \mathcal{X}$ whenever f_j is minimal, and $\mathcal{A}_j = \{\alpha \in \mathcal{X}: \alpha \succ f_j'\}$ for all j with f_j nonminimal. Because of outcome monotonicity, $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is a subset of $\{g' \in \mathcal{F}: g' \succ g\}$ that contains f and that is open with respect to the product topology $T^n_{\succcurlyeq}$. It follows that $\{g' \in \mathcal{F}: g' \succ g\}$ is open with respect to $T^n_{\succcurlyeq}$. Similarly, each set $\{g' \in \mathcal{F}: g' \prec g\}$ is open with respect to the product topology for each g. $\square$

PROOF OF THEOREM 3.3. It is easily verified that CPT with positive decision weights and a utility function that is continuous in duration and increasing in duration implies that $\succcurlyeq$ is continuous in duration and monotonic in duration and that $\succcurlyeq$ is a weak order that satisfies outcome monotonicity, trade-off consistency, and gain-loss consistency. By Lemma 3.1, the order topology on $\mathcal{X}$, $T_{\succcurlyeq}$, is connected. Hence, $T^n_{\succcurlyeq}$, is connected. By Lemma 3.2, $\succcurlyeq$ on $\mathcal{F}$ is continuous with respect to $T^n_{\succcurlyeq}$. $\succcurlyeq$ is truly mixed and $\succcurlyeq$ is a weak order that satisfies outcome monotonicity, trade-off consistency, and, if $n = 2$, gain-loss consistency. By Theorem 6.3 and Observation 8.1 in Wakker and Tversky (1993), CPT holds. U is continuous in duration because $\succcurlyeq$ is continuous in duration. U is increasing in duration because $\succcurlyeq$ is monotonic in duration. The decision weights are positive by outcome monotonicity. $\square$

PROOF OF THEOREM 5.1. The "only if" parts of statements (a)–(c) are easily verified. We prove the "if" parts. If $h \in \mathcal{H} - \mathcal{H}^0$, then $(t, h) \preccurlyeq x_0$ for all $t \in \mathcal{T}$, and thus sign-dependence does not affect preferences. CPT with all outcomes nongains is both a special case of Bleichrodt and Quiggin's (1997) general rank-dependent utility model and of Miyamoto's (1988) generic utility model. Hence, Part (a) follows from Theorem 2 in Bleichrodt and Quiggin (1997) and Parts (b) and (c) from Theorems 3 and 4, respectively, in Miyamoto (1988).

Suppose that $h \in \mathcal{H}^0$. Let U be scaled such that $U(r(h), h) = 0$ and define $\omega_h^-(t) = U(t, h)$ for $t < r(h)$ and $\omega_h^+(t) = U(t, h)$ for $t \geq r(h)$.

PROOF OF THE "IF" PART OF PART (a) FOR h IN $\mathcal{H}^0$. By constant sensitivity and Corollary 9.3 in Wakker and Tversky (1993), for each $h \in \mathcal{H}^0$, $i = +, -$, $\omega_n^i(t)$ is both convex and concave and thus linear on $\mathcal{X}_h^i$. Thus, $\omega_h^i(t) = \alpha^i(h) + \nu^i(h) \cdot t$, with $\nu^i(h)$ positive and $\alpha^i(h)$ real. $\alpha^+(h)$ is chosen so as to establish continuity at $r(h)$. By the zero-condition, $\alpha^-(h)$ is independent of h. Let $\alpha^-(h) = -k$ for all $h \in \mathcal{H}^0$ and for some $k > 0$. Setting $\nu(h) = \nu^+(h)$, $\alpha(h) = \alpha^+(h)$, and $\lambda(h) = \nu^-(h)/\nu^+(h)$ gives the desired representation. Q.E.D.

PROOF OF THE "IF" PART OF PART (b) FOR h IN $\mathcal{H}^0$. By Theorem 3 in Miyamoto (1988), constant proportional risk aversion implies that for each $h \in \mathcal{H}^0$, $i = +, -$, $\omega_h^i(t)$ is either power, $\omega_h^i(t) = \mathrm{sgn}(\theta^i(h)) \cdot \nu^i(h) \cdot t^{\theta^i(h)} + \alpha^i(h)$, or logarithmic, $\omega_h^i(t) = \nu^i(h) \cdot \log(t) + \alpha^i(h)$ on $\mathcal{X}_h^i$, with $\nu^i(h)$ positive, $\alpha^i(h)$ real, and $\theta^i(h)$ a nonzero real. $\alpha^+(h)$ is chosen so as to establish continuity at $r(h)$. The logarithmic and the negative power function are excluded for $\omega_h^-(t)$, because 0 is in the domain of $\omega_h^-(t)$ and these functions are undefined at 0. Thus for each $h \in \mathcal{H}^0$, $\theta^-(h)$ is positive. By the zero-condition, $\alpha^-(h)$ is independent of h. Let $\alpha^-(h) = -k$ for all $h \in \mathcal{H}^0$ and for some $k > 0$. Define for all $h \in \mathcal{H}^0$, $\nu(h) = \nu^+(h)$,

$\alpha(h) = \alpha^+(h)$, $\lambda(h) = \nu^-(h)/\nu^+(h)$. If $\omega_h^+(t)$ is logarithmic, the representation follows from setting $\theta(h) = \theta^-(h)$, $\tau(h) = 1$. If $\omega_h^+(t)$ is a negative or positive power function, setting $\theta(h) = \theta^+(h)$, $\alpha(h) = \alpha^+(h)$, and $\tau(h) = \theta^-(h)/\theta^+(h)$ gives the desired representation. Q.E.D.

PROOF OF THE "IF" PART OF PART (c) FOR h IN H^0. By Theorem 4 in Miyamoto (1988), constant absolute risk aversion implies that for each $h \in \mathscr{H}^0$, $i = +, -$, $\omega_h^i(t)$ is either exponential, $\omega_h^i(t) = \text{sgn}(\varphi^i(h)) \cdot \nu^i(h) \cdot e^{\varphi^i(h) \cdot t} + \alpha^i(h)$, or linear, $\omega_h^i(t) = \nu^i(h) \cdot t + \alpha^i(h)$, on $\mathscr{X}_h^i$ with $\nu^i(h)$ positive and $\alpha^i(h)$ real. $\alpha^+(h)$ is chosen so as to establish continuity at $r(h)$. Define $g^*(h) = \alpha^-(h)$ if $\omega_h^-(t)$ is linear and $g^*(h) = \text{sgn}(\varphi^-(h)) \cdot \nu^-(h) + \alpha^-(h)$ if $\omega_h^-(t)$ is exponential. Then $\omega_h^-(t) = \nu^-(h) \cdot t + g^*(h)$ or $\omega_h^-(t) = \text{sgn}(\varphi^i(h)) \cdot \nu^i(h) \cdot (e^{\varphi^i(h) \cdot t} - 1) + g^*(h)$. By the zero-condition, $\alpha^-(h)$ is independent of h. Let $\alpha^-(h) = -k$ for all $h \in \mathscr{H}^0$ and for some $k > 0$. Define for all $h \in \mathscr{H}^0$, $\nu(h) = \nu^+(h)$, $\alpha(h) = \alpha^+(h)$, and $\lambda(h) = \nu^-(h)/\nu^+(h)$. If $\omega_h^-(t)$ and $\omega_h^+(t)$ are both exponential, then the representation follows from setting $\varphi(h) = \varphi^+(h)$, $\alpha(h) = \alpha^+(h)$, and $\tau(h) = \varphi^-(h)/\varphi^+(h)$. If $\omega_h^-(t)$ is exponential and $\omega_h^+(t)$ is linear, the representation follows from setting $\varphi(h) = \varphi^-(h)$, $\alpha(h) = \alpha^+(h)$, and $\tau(h) = 1$. If $\omega_h^-(t)$ is linear and $\omega_h^+(t)$ is exponential, the representation follows from setting $\varphi(h) = \varphi^+(h)$ and $\alpha(h) = \alpha^+(h)$. If $\omega_h^-(t)$ and $\omega_h^+(t)$ are both linear, the representation follows from part (a). $\square$

PROOF OF THEOREM 6.1. Follows from applying Proposition 9.4 in Wakker and Tversky (1993) to each $\mathscr{F}_h$. $\square$

PROOF OF THEOREM 7.2. The "only if" parts of statements (a)–(c) are easily verified. We prove the "if" parts. We first establish that a CPT representation exists for $\succcurlyeq$ on $\mathscr{F}$. For $n = 2$ this follows from Theorem 3.3. Hence, let $n > 2$. Let ρ be a permutation and let $h \in \mathscr{H}$.

An outcome $x \in \mathscr{X}$ is *maximal* if for no other outcome $y \in \mathscr{X}$ we have $y \succ x$. An outcome $x \in \mathscr{X}$ is *minimal* if for no other outcome $y \in \mathscr{X}$ we have $x \succ y$. An *extreme prospect* is a prospect that either assigns to each state a maximal outcome or to each state a minimal outcome.

LEMMA A.1. *There exist additive functions $V_{j,\rho,h}$: $\mathscr{X} \to \mathbb{R}$, $j \in \mathscr{S}$, that are monotonic in duration and continuous in duration such that $f \mapsto \sum_{j \in \mathscr{S}} V_{j,\rho,h}(f_j)$ represents $\succcurlyeq$ on $\mathscr{F}_{\rho,h} \backslash \{$extreme prospects$\}$. The $V_{j,\rho,h}$ are unique up to positive affine transformations with a common unit.*

PROOF. By Lemma 3.1, the order topology on $\mathscr{X}$, $T_\succcurlyeq$, is connected. Hence, $T_\succcurlyeq^n$ is connected. By Lemma 3.2, $\succcurlyeq$ on $\mathscr{F}$ is continuous with respect to $T_\succcurlyeq^n$. Lemma A.1 now follows from Lemma A.2 in Wakker and Zank (2002) and Corollary C.5 in Chateauneuf and Wakker (1993). By Proposition 3.5 in Wakker (1993), the representation can be extended to the entire set $\mathscr{F}_{\rho,h}$ if the $V_{j,\rho,h}$ are linearly related. Q.E.D.

LEMMA A.2. *On $\mathscr{F}_{\rho,h}$ the representation in Lemma* A.1 *is the restriction of a CPT form.*

PROOF. If $h \in \mathscr{H}^0$, define, for all $j \in \mathscr{S}$, $V_{j,\rho,h}(r(h)) = 0$. Let $i = +, -$. Partition $\mathscr{S}$ into $A = \{1, \ldots, m\}$ and $A^c = \{m+1, \ldots, n\}$ and consider the sets of prospects $\mathscr{M}_{\rho,h}^i = \{f \in \mathscr{F}_{\rho,h}^i$: for all $i, j \in A$, $f_i = f_j$ and for all $r, s \in A^c$, $f_r = f_s\}$. Elements of $\mathscr{M}_{\rho,h}^i$ will be denoted by $(x, y)_m$ where $f_i = x$ for $j \in \{1, \ldots, m\}$ and $f_j = y$ for $j \in \{m+1, \ldots, n\}$. On $\mathscr{M}_{\rho,h}^i$, $\succcurlyeq$ is represented by $(x, y)_m \mapsto V_{1,m,\rho,h}^i(x) + V_{m+1,n,\rho,h}^i(y)$ with $V_{1,m,\rho,h}^i = \sum_{j=1}^m V_{j,\rho,h}$ and $V_{m+1,n,\rho,h}^i = \sum_{j=m+1}^n V_{j,\rho,h}$.

By Theorem 2 in Bleichrodt and Quiggin (1997) and constant sensitivity on $\mathscr{X}^i$, or by Theorem 2 in Miyamoto and Wakker (1996) and constant proportional risk aversion on $\mathscr{F}^i$, or by Theorem 1 in Miyamoto and Wakker (1996) and constant absolute risk aversion on $\mathscr{F}^i$, $V_{1,m,\rho,h}^i$ and $V_{m+1,n,\rho,h}^i$ are linear with respect to each other on $\mathscr{F}_{\rho,h}^i$. That is, $V_{1,m,\rho,h}^i = a_{1,m,\rho,h}^i \cdot V_{m+1,n,\rho,h}^i + b_{1,m,\rho,h}^i$, $a_{1,m,\rho,h}^i > 0$, $b_{1,m,\rho,h}^i$ real.

Because $V^i_{1,m,\rho,h}$ and $V^i_{m+1,n,\rho,h}$ are linearly related, we can extend the additive representation to the entire set $\mathcal{F}_{\rho,h}$. Let $U^+_{\rho,h}(x) = \sum_{j=1}^n V_{j,\rho,h}(x)$ where all x are gains and let $U^-_{\rho,h}(x) = \sum_{j=1}^n V_{j,\rho,h}(x)$ where all x are losses. Define $V^i_{1,m,\rho,h} = \pi^i_{1,m,\rho,h} \cdot U^i_{\rho,h}$ and $V^i_{m+1,n,\rho,h} = \pi^i_{m+1,n,\rho,h} \cdot U^i_{\rho,h}$, where $\pi^i_{1,m,\rho,h} = 1/(1+a^i_{1,m,\rho,h})$ and $\pi^i_{m+1,n,\rho,h} = a^i_{1,m,\rho,h}/(1+a^i_{1,m,\rho,h})$ are uniquely defined, positive decision weights that sum to one. The uniqueness of the decision weights follows from the common unit of the additive representation. Positivity follows from outcome monotonicity.

We can now define a $\text{CPT}^i_{\rho,h}$ representation on $\mathcal{F}^i_{\rho,h}$ as in the proofs of Lemmas 7 and 8 in Zank (2001). For all $f \in \mathcal{F}_{\rho,h}$, the additive representation, obtained earlier, is the sum of a CPT^+ and a CPT^- functional. It is, therefore, a CPT functional on $\mathcal{F}_{\rho,h}$, and we denote this representation by $\text{CPT}_{\rho,h}$ henceforth.

If $h \in \mathcal{H} - \mathcal{H}^0$, all outcomes are losses. If all outcomes are of the same sign, CPT coincides with rank-dependent utility. By Theorem 2 in Bleichrodt and Quiggin (1997) and constant sensitivity on $\mathcal{X}^-$, or by Theorem 2 in Miyamoto and Wakker (1996) and constant proportional risk aversion on $\mathcal{F}^-$, or by Theorem 1 in Miyamoto and Wakker (1996) and constant absolute risk aversion on $\mathcal{F}^-$, rank-dependent utility represents $\succcurlyeq$ on $\mathcal{F}_{\rho,h}$. Q.E.D.

Lemma A.3. *On $\mathcal{F}^-_\rho$ the representation in Lemma* A.1 *is the restriction of a CPT form.*

Proof. Rescale $U^-_{\rho,h}$ such that $U^-_{\rho,h}(0,h) = 0$ for all $h \in \mathcal{H}$. Let $h, h' \in \mathcal{H}$. By weak utility independence, for all $s, t, w \in \mathcal{T}$ with $s, t, w \le r(h)$ if $h \in \mathcal{H}^0$ and $s, t, w \le r(h')$ if $h' \in \mathcal{H}^0$, $(s,h) \sim ((t,h), A, (w,h))$ iff $(s,h') \sim ((t,h'), A, (w,h'))$. Because the equivalences are independent of h, $\text{CPT}^-_{\rho,h}$ and $\text{CPT}^-_{\rho,h'}$ are related by a positive affine transformation. Hence, for a given $h'' \in \mathcal{H}$, $\text{CPT}^-_{\rho,h} = \nu^-_\rho(h) \cdot \text{CPT}^-_{\rho,h''} + \alpha^-(h)$. Because the decision weights have to sum to one, they are unique and for all $h, h' \in \mathcal{H}$ and for all $j \in \mathcal{S}$, $\pi^-_{j,\rho,h} = \pi^-_{j,\rho,h'}$. Hence, for a given $h'' \in \mathcal{H}$, $U^-_{\rho,h} = \nu^-_\rho(h) \cdot U^-_{\rho,h''} + \alpha^-(h)$. By the zero-condition, $0 = U^-_{\rho,h}(0) = \nu^-_\rho(h) \cdot U^-_{\rho,h''}(0) + \alpha^-(h) = \alpha^-(h)$. Select a $h \in \mathcal{H}$ and define $\omega^-_\rho(t) = U^-_{\rho,h}(t,h)$. Rescale $U^-_{\rho,h}$ such that $U^-_{\rho,h}(0,h) = -k$ for some $k \in \mathbb{R}^+$ and $U^-_{\rho,h}(r(h),h) = 0$ if $r(h)$ exists. Then for all $h \in \mathcal{H}$, $U^-_{\rho,h}(t,h) = \nu^-_\rho(h) \cdot \omega^-_\rho(t) - k$. By constant sensitivity (constant proportional risk aversion, constant absolute risk aversion) $\omega^-_\rho(t)$ is linear (positive power, linear/exponential). The function $\omega^-_\rho(t)$ is independent of h and, therefore, we obtain a CPT representation on $\mathcal{F}^-_\rho$. Q.E.D.

Lemma A.4. *On $\mathcal{F}^+_\rho$ the representation in Lemma* A.1 *is the restriction of a CPT form.*

Proof. By the assumptions of the truly mixed case and outcome monotonicity, $\mathcal{H}^0$ contains at least one h. If $\mathcal{H}^0$ contains just one health state, then we obtain a CPT representation on $\mathcal{F}^+_\rho$ by defining $U^+_\rho = U^+_{\rho,h}$ and for all $j \in \mathcal{S}$, $\pi^+_{j,\rho} = \pi^+_{j,\rho,h}$.

Suppose that $\mathcal{H}^0$ contains at least two health states. Let $h, h' \in \mathcal{H}^0$ and let $A \in \mathcal{S}$. By mixed utility independence, for all $s, w \in \mathcal{T}$ with $s, w \ge \max\{r(h), r(h')\}$, and for all $t, z \in \mathcal{T}$ with $t, z \le \min\{r(h), r(h')\}$, $((s,h), A, (t,h)) \sim ((w,h), A, (z,h))$ iff $((s,h'), A, (t,h')) \sim ((w,h'), A, (z,h'))$. Because the equivalences are independent of h, $\text{CPT}_{\rho,h}$ and $\text{CPT}_{\rho,h'}$ are related by a positive affine transformation. We showed in Lemma A.3 that $\text{CPT}^-_{\rho,h}$ and $\text{CPT}^-_{\rho,h'}$ are related by a positive affine transformation. Hence, by mixed utility independence, $\text{CPT}^+_{\rho,h}$ and $\text{CPT}^+_{\rho,h'}$ are also related by a positive affine transformation. Consequently, for a given $h'' \in \mathcal{H}^0$, $\text{CPT}^+_{\rho,h'} = \nu^+_\rho(h) \cdot \text{CPT}^+_{\rho,h''} + \alpha^+(h)$. Because the decision weights sum to one, they are unique. Therefore, for all $h, h' \in \mathcal{H}$ and for all $j \in \mathcal{S}$, $\pi^+_{j,\rho,h} = \pi^+_{j,\rho,h'}$. Hence, for a given $h'' \in \mathcal{H}^0$, $U^+_{\rho,h} = \nu^+_\rho(h) \cdot U^+_{\rho,h''} + \alpha^+(h)$. Select a $h \in \mathcal{H}^0$ and define $\omega^+_\rho(t) = U^+_{\rho,h}(t,h)$. Then for all $h \in \mathcal{H}^0$, $U^+_{\rho,h}(t) = \nu^+_\rho(h) \cdot \omega^+_\rho(t) + \alpha^+(h)$. By constant sensitivity (constant proportional risk aversion, constant absolute risk aversion) $\omega^+_\rho(t)$ is linear (logarithmic/power, linear/exponential). The function $\omega^+_\rho(t)$ is independent of h and, therefore, we obtain a CPT representation on $\mathcal{F}^+_\rho$. Q.E.D.

Hence, for all $f \in \mathcal{F}_\rho$, the additive representation, obtained earlier, is the sum of a CPT^+ and a CPT^- functional. It is, therefore, a CPT functional on $\mathcal{F}_\rho$, and we denote this representation by CPT_ρ henceforth.

LEMMA A.5. *For any two different permutations ρ and ρ', the CPT representations on $\mathcal{F}_\rho$ and $\mathcal{F}_{\rho'}$ coincide on common domain in the sense that the utility function U is the same for both sets and the capacities coincide on common domain.*

PROOF. If $\mathcal{F}_\rho \cap \mathcal{F}_{\rho'}$ contains nonconstant prospects, then this follows from fixing the scales of U_ρ and $U_{\rho'}$. If $\mathcal{F}_\rho \cap \mathcal{F}_{\rho'}$ contains only constant prospects then, because $n > 2$, we can construct a sequence of permutations $\rho_1, \ldots, \rho_n$ such that $\mathcal{F}_\rho \cap \mathcal{F}_{\rho_1}$, $\mathcal{F}_{\rho_1} \cap \mathcal{F}_{\rho_2}, \ldots,$ $\mathcal{F}_{\rho_n} \cap \mathcal{F}_{\rho'}$ all contain nonconstant prospects. By the first case, the CPT representations are identical. Q.E.D.

The proof that the CPT representation thus defined is representing on $\mathcal{F}$, the definition of the unique capacity, and the derivation of the uniqueness results are similar to Zank (2001, pp. 76–77).

We finally derive the decomposable QALY models. Because we have a CPT representation on $\mathcal{F}$, we can apply the proof of Theorem 5.1 to derive the linear QALY model (if constant sensitivity holds on $\mathcal{X}^-$ and on $\mathcal{X}^+$), the power QALY model (if constant proportional risk aversion holds on $\mathcal{F}^+$ and on $\mathcal{F}^-$) or the exponential QALY model (if constant absolute risk aversion holds on $\mathcal{F}^+$ and on $\mathcal{F}^-$). The parameters τ, θ, and φ in Equations (6)–(9) are independent of h because $\omega^-(t)$ and $\omega^+(t)$ are independent of h. It remains to be shown that $\lambda(h) = \nu^-(h)/\nu^+(h)$ is independent of h. If $h \in \mathcal{H} - \mathcal{H}^0$, there is nothing to prove, so let $h \in \mathcal{H}^0$. If $\mathcal{H}^0$ contains only one health state, then the representation follows from setting $\lambda = \lambda(h)$. So let there be at least two health states in $\mathcal{H}^0$. Let $h, h' \in \mathcal{H}^0$ and let $A \subseteq \mathcal{S}$. By continuity in duration and connectedness of $\mathcal{T}$ we can find $s, y \in \mathcal{T}$ with $s, y \geq \max\{r(h), r(h')\}$ and $t, z \in \mathcal{T}$ with $t, z \leq \min\{r(h), r(h')\}$ such that $((s, h), A, (t, h)) \sim ((y, h), A, (z, h))$. Evaluation by CPT gives

$$\begin{aligned} &w^+(A) \cdot (\nu(h) \cdot \omega^+(s) + \alpha(h)) + w^-(A^c) \cdot (\lambda(h) \cdot \nu(h) \cdot \omega^-(t) - k) \\ &\quad = w^+(A) \cdot (\nu(h) \cdot \omega^+(y) + \alpha(h)) + w^-(A^c) \cdot (\lambda(h) \cdot \nu(h) \cdot \omega^-(z) - k), \end{aligned}$$

where ω^- and ω^+ are either linear, power or logarithmic, or exponential. Solving for λ gives

$$\lambda(h) = \frac{w^+(A)(\omega^+(s) - \omega^+(y))}{w^-(A^c)(\omega^-(z) - \omega^-(t))}.$$

By mixed utility independence also, $((s, h'), A, (t, h')) \sim ((y, h'), A, (z, h'))$ and thus

$$\begin{aligned} &w^+(A) \cdot (\nu(h') \cdot \omega^+(s) + \alpha(h')) + w^-(A^c) \cdot (\lambda(h') \cdot \nu(h') \cdot \omega^-(t) - k) \\ &\quad = w^+(A) \cdot (\nu(h') \cdot \omega^+(y) + \alpha(h')) + w^-(A^c) \cdot (\lambda(h') \cdot \nu(h') \cdot \omega^-(z) - k). \end{aligned}$$

Hence,

$$\lambda(h') = \frac{w^+(A)(\omega^+(s) - \omega^+(y))}{w^-(A^c)(\omega^-(z) - \omega^-(t))},$$

which shows that for all $h, h' \in \mathcal{H}^0$, $\lambda(h) = \lambda(h')$. □

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