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Saving under rank-dependent utility*

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Summary. In this note we use the rank-dependent utility (RDU) model to analyze saving decisions. The RDU model enables us to separate the effects of pessimism and optimism on saving from that of concavity of the utility function. While pessimism induces more saving, the importance of this effect is shown to depend upon properties of the utility function such as prudence and temperance.

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Since the pioneering work by Drèze and Modigliani (1972), Leland (1968), and Sandmo (1970), many articles – both theoretical and empirical – have been devoted to the analysis of optimal saving when uncertainty about future income prevails. It is fair to say that almost all models related to this problem have been developed in the expected utility (EU) framework.

Because the expected utility (EU) model seems to be unsatisfactory from a descriptive point of view (see Starmer, 2000, for a review), we investigate here how a departure from this model affects the traditional results on optimal saving. In this note we use the rank-dependent utility model (RDU) (Quiggin, 1981, 1982; Yaari, 1987) to develop our argument. Our choice of RDU rests upon two reasons. First RDU is now recognized as the prominent alternative to expected utility (along with prospect theory). Second, RDU enables the analyst to incorporate and disentangle the effects of concavity of the utility function on the one hand (which corresponds

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to the notion of risk aversion in the EU model) and of probability weighting on the other hand (where probability weighting can be used to incorporate the effect of pessimism or optimism).

While theory has well documented the effect of concavity of the utility function, the effect of probability weighting is much less discussed. It is common knowledge that psychological attitudes other than those reflected by concavity of utility affect intertemporal choices. RDU is particularly suited to analyze the joint effects on saving of concavity of utility and of probability weighting and to express the specific role of psychological attitudes other than the standard one captured by concave utility.

The note also displays another, perhaps surprising, result. While we use a RDU model, it turns out that concepts more recently developed in the EU framework, such as prudence and temperance (Kimball, 1990, 1992), are also useful to indirectly analyze the effect of pessimism on optimal saving.

Our note is organized as follows. In Section 1, we briefly present the results on saving in a EU framework, using the standard two-period model. In Section 2 we expose the basic RDU model, while its implications for saving are detailed in Section 3, which is the central section of this note. A short conclusion is presented in Section 4.

1 The EU model

Consider a two-period model involving current (c_1) and future (c_2) consumption. For the sake of simplicity, the interest rate and the individual's discount factor of future utility are both taken to be equal to zero.

While current income (y_1) is certain, future income (y_2) is random and its expected value is denoted by $\bar{y}_2$. For ease of illustration, we will mainly consider the case where y_2 takes on two different values. We discuss the generalization to more than two outcomes at the end of Section 3. If the random variable $\tilde{y}_2$ takes only two values y_{2a} and y_{2b} $(y_{2a} < y_{2b})$ with probabilities p and $1-p$ respectively, the problem of an EU maximizer is to choose c_1 so as to maximize:

$$U(c_1) + [pU((y_1 - c_1) + y_{2a}) + (1-p)U((y_1 - c_1) + y_{2b})] \tag{1.1}$$

where U is assumed to be increasing and strictly concave.

The first-order condition (F.O.C.) for a maximum is :

$$U'(c_1) - [pU'(y_1 - c_1 + y_{2a}) + (1-p)U'(y_1 - c_1 + y_{2b})] = 0 \tag{1.2}$$

Concavity of U guarantees that the second-order condition (S.O.C.) is satisfied.

The prudence premium ψ (Kimball, 1990) is defined as the solution of

$$E[U'(\tilde{y}_2)] = U'(\bar{y}_2 - \psi), \tag{1.3}$$

where E is the expectations operator. That is, the prudence premium does to marginal utility what the risk premium π, defined by $E[U(\tilde{y}_2)] = U(\bar{y}_2 - \pi)$, does to total utility. Using the definition of the prudence premium (1.2) can be rewritten as :

$$U'(c_1) - U'(y_1 - c_1 + \bar{y}_2 - \psi(\bar{c}_2) = 0 \tag{1.4}$$

implying that the optimal c_1 (denoted c_1^*) is given by:

$$c_1^* = \frac{y_1 + \bar{y}_2 - \psi\left(y_1 + \bar{y}_2 - c_1{}^*\right)}{2} \tag{1.5}$$

If there were no risk on future income ψ would be equal to zero and there would be perfect smoothing of consumption (Gollier, 2001).

When U'' is negative, which corresponds to risk aversion under EU, and U''' is positive, which corresponds to prudence under EU, the future income risk makes ψ positive and so the presence of future income risk reduces current consumption and stimulates saving. In a sense – in the EU model – ψ characterizes the impact of future income risk on saving for a risk averse and prudent individual.

2 The RDU model

In EU, final outcomes are transformed by the utility function but the probabilities are not transformed. The RDU model generalizes EU because it also allows for a distortion of probabilities. When there are just two different outcomes, Gul's (1991) theory of disappointment aversion (DA) is a special case of RDU. Hence, the results that we will derive for the case where y_2 can take on just two outcomes also hold under DA.

In RDU, a pessimistic decision maker assigns to the worst outcome (y_{2a}) a transformed probability $(w\,(p))$ that is larger (smaller) than p while an optimistic decision maker does the reverse. Consequently under RDU the problem is to maximize:

$$U\left(c_1\right) + \left[w\left(p\right)U\left(y_1 - c_1 + y_{2a}\right) + \left(1 - w\left(p\right)\right)U\left(y_1 - c_1 + y_{2b}\right)\right] \tag{2.1}$$

In the derivation of Eq. (2.1), we have ordered the outcomes in ascending order of preference, because subsequent notation becomes easier. Other papers have chosen to order the outcomes in descending order and then apply the dual of w, $w^*(p) = 1 - w(1-p)$ for all p, to incorporate probability weighting. Notice that the transformed probability associated with $1-p$ – the probability of the best outcome – is $1 - w(p)$ so that the sum of the transformed probabilities is equal to one. It follows that a pessimist (optimist) inflates (deflates) the likelihood of the worst outcome and deflates (inflates) that of the best outcome.

The F.O.C. belonging to (2.1) is:

$$U'\left(c_1\right) - \left[w\left(p\right)U'\left(y_1 - c_1 + y_{2a}\right) + \left(1 - w\left(p\right)\right)U'\left(y_1 - c_1 + y_{2b}\right)\right] = 0 \tag{2.2}$$

and the optimal c_1 is denoted $\hat{c}_1$. The question of interest is then to compare $\hat{c}_1$ with the previous value c_1^* discussed in Section 1 (see Eq. (1.5)). The difference between $\hat{c}_1$ and c_1^* indicates the impact of pessimism or optimism beyond that of concavity of U on consumption and saving. In the next section we characterize c_1^* in such a way that it can be easily compared with $\hat{c}_1$.

3 The impact of probability transformation

In order to evaluate and interpret the impact of pessimism or optimism we define a sure amount of money, D, by the solution of:

$$\begin{aligned} &w(p)\, U'(y_1 - c_1 + y_{2a}) + (1 - w(p))\, U'(y_1 - c_1 + y_{2b}) \\ &= pU'(y_1 - c_1 + y_{2a} - D) + (1 - p)\, U'(y_1 - c_1 + y_{2b} - D) \end{aligned} \tag{3.1}$$

D can be interpreted as a willingness to pay (WTP) for shifting the decision weight of the worst outcome from $w(p)$ to p while maintaining expected marginal utility constant. Traditionally, WTP is defined so as to maintain expected total utility constant. Here, however, because we are looking at optimal decisions, marginal utility instead of total utility is the relevant concept. As we show below, the value of D plays a crucial role in analysing the impact of pessimism. Besides, as indicated in the introduction, we show that the value of D depends upon concepts developed in the EU framework such as prudence and temperance.

It can easily be shown that $w(p) > p$ implies that D is positive. Indeed, as $w(p)$ increases above p the left hand side of (3.1) increases because more weight is attached to the highest level of marginal utility. To restore the equality, the right hand side of (3.1) must be increased, which is obtained by reducing wealth because of decreasing marginal utility. Of course, if $w(p) = p$, $D = 0$, and under optimism $(w(p) < p)$ D is strictly negative.

Given the definition of D, (2.2) can be rewritten as:

$$U'(c_1) - [pU'(y_1 - c_1 + y_{2a} - D) + (1 - p)\, U'(y_1 - c_1 + y_{2b} - D)] = 0 \tag{3.2}$$

and we can now apply the notion of prudence as in Section 1 to find:

$$U'(c_1) - U'(y_1 - c_1 + \bar{y}_2 - D - \psi(y_1 - c_1 + \bar{y}_2 - D)) = 0 \tag{3.3}$$

so that :

$$\hat{c}_1 = \frac{y_1 + \bar{y}_2 - D - \psi(y_1 + \hat{c}_1 + \bar{y}_2 - D)}{2} \tag{3.4}$$

Equation (3.4) compared with Eq. (1.5) shows the importance of D in the determination of $\hat{c}_1$. Indeed, D has two effects on $\hat{c}_1$: a direct one through its presence as the third term in the numerator of $\hat{c}_1$ and an indirect one through its influence on ψ.

Because D is so central to the understanding of the effect of probability weighting, we now give a simple interpretation of its value. In the appendix we show that through successive Taylor series we can approximate D by:

$$D \approx [(w(p) - p)(y_{2b} - y_{2a})] \cdot \left(\frac{U''(y_1 + \bar{y}_2 - c_1)}{U''(y_1 + \bar{y}_2 - c_1 - T)} \right) \tag{3.5}$$

where T is the "temperance premium" defined by the condition:

$$E[U''(\tilde{y}_2)] = U''(\bar{y}_2 - T) \tag{3.6}$$

It is well known that $U''' > 0$ and $U'''' < 0$ imply that T is positive. Kimball (1992) gives arguments why it is plausible that the temperance premium is positive.

Before interpreting Eq. (3.5), it is worth pointing out that our approximation of D is coherent with its general properties discussed earlier. Indeed, it follows from the approximation in (3.5) that:

$$w(p) \gtreqless p \Leftrightarrow D \gtreqless 0.$$

Given these results D can be easily interpreted. It is made up of two terms. The term in brackets in Eq. (3.5) corresponds to the change in expected wealth induced by the transition from p to $w(p)$, i.e., the change in expected wealth caused by pessimism or optimism. Indeed, with the weights $(w(p), 1 - w(p))$, expected wealth is equal to $y_1 - c_1 + w(p)y_{2a} + (1 - w(p))y_{2b}$. With the weights $(p, 1 - p)$, expected wealth is equal to $y_1 - c_1 + py_{2a} + (1 - p)y_{2b}$. The difference between these two expected wealth levels is precisely the term in brackets in (3.5). Under pessimism (optimism), this term measures by how much expected wealth falls (increases) when p is replaced by $w(p) > p$ $(w(p) < p)$. The effect of pessimism (optimism) on expected wealth is then weighted in (3.5) by the term in parentheses, which is a strictly positive number since it is the ratio of two values of U''.

If $U''' = 0$ (i.e., when U is quadratic) the term that multiplies the brackets is equal to one. For most other commonly used utility functions for which U''' is positive and U'''' negative $(T > 0)$ the term in brackets is multiplied by a positive number smaller than one. Hence, properties of U up to its fourth derivative influence the monetary value D that is equivalent to the transformation of probabilities in the RDU model.

Now that the value of D is understood, we can return to the analysis of the difference between $\hat{c}_1$ and c_1^*, which measures the specific impact of pessimism (or optimism) on current consumption and saving beyond that of concavity of U. If U is quadratic, ψ is equal to zero and $\hat{c}_1 - c_1^*$ is equal to $-\frac{D}{2}$ where D is simply the change in expected wealth induced by the probability transformation. Hence, pessimistic decision makers with quadratic utility increase their saving by one half of the fall in expected wealth due to their inflation of p into $w(p)$.

For other decision makers with $U''' > 0$ and $U'''' < 0$, the impact of pessimism has two components. The direct one $\left(-\frac{D}{2}\right)$ is still at work, however D is now smaller than the change in expected wealth because it is weighted by a number smaller than 1 (see above). Besides, we must also take into consideration the impact of D on the prudence premium (ψ) which can go either way depending upon the behavior of ψ with respect to wealth. When ψ is constant at all wealth levels then D has no indirect effect on saving. However, when ψ is declining in wealth (the more standard assumption) then the indirect effect reinforces the direct one to stimulate saving.

Throughout the paper we assumed that y_2 could just take on two values. However, the decomposition of D in a term that reflects the change in expected wealth due to pessimism or optimism and the ratio $\frac{U''(y_1+\bar{y}_2-c_1)}{U''(y_1+\bar{y}_2-c_1-T)}$ also holds when y_2 takes on more than two values. Recall that when y_2 takes on just two distinct values, DA is a special case of RDU and (3.5), therefore, also holds under DA. Under DA, the term in brackets in (3.5) measures the change in expected wealth due to disappointment aversion. When y_2 takes on more than two distinct outcomes, DA is no longer a special case of RDU. It can be shown, however, that in this case D can also be decomposed in a term that reflects the change in expected wealth due to disappointment aversion and the ratio $\frac{U''(y_1+\bar{y}_2-c_1)}{U''(y_1+\bar{y}_2-c_1-T)}$. Proofs of the above claims can be found on www.bmg.eur.nl/personal/bleichrodt/publications.

4 Conclusion

As is well known, saving is influenced by other factors than concavity of the utility function such as pessimism or optimism. The EU model has exclusively considered the impact of concavity of U, which – in that model – is interpreted as risk aversion.

Rank dependent utility can incorporate through probability weighting the impact of pessimism. We have exploited this feature in this paper. Besides, we have shown that more recent concepts in EU theory – such as prudence and temperance – can be used to obtain a monetary equivalent of the effect of probability transformation. In this way the specific impact of pessimism on saving can be given a natural and simple monetary interpretation.

Appendix

From Eq. (3.1) we write :

$$\begin{aligned} &w(p)\,U'(w_a - c_1) + (1 - w(p))\,U'(w_b - c_1) \\ &= pU'(w_a - c_1 - D) + (1 - p)\,U'(w_b - c_1 - D) \end{aligned} \tag{A.1}$$

where $w_a \equiv y_1 + y_{2a}$ and $w_b \equiv y_1 + y_{2b}$.

The right hand side of (A.1) is approximately equal to:

$$\begin{aligned} &pU'(w_a - c_1) + (1 - p)\,U'(w_b - c_1) \\ &- D\left[pU''(w_a - c_1) + (1 - p)\,U''(w_b - c_1)\right] \end{aligned} \tag{A.2}$$

which can be rewritten as:

$$pU'(w_a - c_1) + (1 - p)\,U'(w_b - c_1) - DU''(\bar{w} - c_1 - T) \tag{A.2$'$}$$

where the concept of temperance is used and where $\bar{w}$ stands for $pw_a + (1 - p)\,w_b$.

Introducing the result of (A.2′) into (A.1) we obtain:

$$(w(p) - p)\,(U'(w_a - c_1) - U'(w_b - c_1)) \cong -DU''(\bar{w} - c_1 - T) \tag{A.3}$$

Then expanding $U'(w_a - c_1)$ and $U'(w_b - c_1)$ around $U'(\bar{w} - c_1)$, we get:

$$U'(w_a - c_1) \cong U'(\bar{w} - c_1) + (w_a - \bar{w})\,U''(\bar{w} - c_1) \tag{A.4a}$$

$$U'(w_b - c_1) \cong U'(\bar{w} - c_1) + (w_b - \bar{w})\,U''(\bar{w} - c_1) \tag{A.4b}$$

implying that:

$$U'(w_a - c_1) - U'(w_b - c_1) \cong (w_a - w_b)\, U''(\bar{w} - c_1) \tag{A.4c}$$

Then finally:

$$D \cong [(w(p) - p)(w_b - w_a)] \cdot \frac{U''(\bar{w} - c_1)}{U''(\bar{w} - c_1 - T)} \tag{A.5}$$

which is Eq. (3.5) since $w_b - w_a = y_{2b} - y_{2a}$ and since $\bar{w} = y_1 + \bar{y}_2$.

References

Drèze, J., Modigliani, F.: Consumption decisions under uncertainty. Journal of Economic Theory **5**, 308–335 (1972)

Gollier, C.: The economics of risk and time. Boston: MIT Press 2001

Gul, F.: A theory of disappointment aversion. Econometrica **59**, 667–686 (1991)

Kimball, M.: Precautionary motives for holding assets. In: Newmann, P., Falwell, J. (eds.) The new palgrave dictionary of money and finance, pp. 158–161. London: MacMillan 1992

Kimball, M. S.: Precautionary savings in the small and in the large. Econometrica **58**, 53–73 (1990)

Leland, H.: Saving and uncertainty: The precautionary demand for saving. Quarterly Journal of Economics **82**, 465–473 (1968)

Quiggin, J.: Risk perception and risk aversion among Australian farmers. Australian Journal of Agricultural Economics **25**, 160–169 (1981)

Quiggin, J.: A theory of anticipated utility. Journal of Economic Behavior and Organization **3**, 323–343 (1982)

Sandmo, A.: The effect of uncertainty on saving decisions. Review of Economic Studies **37**, 353–360 (1970)

Starmer, C.: Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk. Journal of Economic Literature **28**, 332–382 (2000)

Yaari, M. E.: The dual theory of choice under risk. Econometrica **55**, 95–115 (1987)