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Reference-dependent expected utility with incomplete preferences

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ABSTRACT

An important reason why people deviate from expected utility is reference-dependence of preferences, implying loss aversion. Bleichrodt [Bleichrodt H. (2007). Reference-dependent utility with shifting reference points and incomplete preferences. *Journal of Mathematical Psychology*, *51*, 266–276] argued that in the empirically realistic case where the reference point is always an element of the decision maker's opportunity set, reference-dependent preferences have to be taken as incomplete. This incompleteness is a consequence of reference-dependence and is different in nature from the type of incompleteness usually considered in the literature. It cannot be handled by existing characterizations of reference-dependence, which all assume complete preferences. This paper presents new preference foundations that extend reference-dependent expected utility to cover this case of incompleteness caused by reference-dependence. The paper uses intuitive axioms that are easy to test. Two special cases of reference-dependent expected utility are also characterized: one model in which utility is decomposed into a normative and a psychological component and one model in which loss aversion is constant. The latter model has been frequently used in empirical research on reference-dependence.

1. Introduction

An important reason why decisions under uncertainty deviate from expected utility is reference-dependence. People do not evaluate outcomes as absolute amounts, as expected utility assumes, but as gains and losses relative to a reference point and are more sensitive to losses than to gains. Reference-dependence is empirically well-established and it is an important factor in explaining people's attitudes towards risk (Rabin, 2000). There is also growing evidence that reference-dependence can explain a variety of field data (Camerer, 2000). Markowitz (1952) already suggested reference-dependence as a way to solve the puzzle posed by Friedman and Savage (1948) that people simultaneously buy insurance and lotteries. The best-known theory of reference-dependence is prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992), currently the main descriptive theory of decision under uncertainty.

In Bleichrodt (2007), I explained that reference-dependence often requires incompleteness of preferences. This follows because in many decision situations the reference point is always one of the options that are available to the decision maker (for empirical evidence see for example Hershey and Schoemaker (1985), Robinson, Loomes, and Jones-Lee (2001), and van Osch, van den Hout, and Stiggelbout (2006)). Then, if two alternatives are both less preferred to the reference point, a preference between these alternatives cannot be observed. This incompleteness, stemming from reference-dependence, is different in nature from the incompleteness commonly studied in the literature, which reflects indecisiveness, confusion, and lack of information of the decision maker.

Because previous characterizations of reference-dependence in decision under uncertainty assume complete preferences,[1] they cannot handle the above situation and extensions are required that allow for incomplete preferences. The purpose of this paper is to present such extensions. The paper builds on a general additive reference-dependent utility model with incomplete preferences that was derived in Bleichrodt (2007). It starts by strengthening one of the conditions in Bleichrodt (2007) so that a separation between utility and beliefs is achieved. Two general models of reference-dependent preferences under uncertainty then follow: one in which the reference point is fixed and one in which the reference point can vary across decisions. The first model is an extension of Sugden's (2003) reference-dependent subjective expected utility model to the case of incomplete preferences. As in Sugden (2003), the reference point can be any act, certain or uncertain. The second model, which I will refer to as *reference-dependent expected utility*, allows in addition for the possibility of shifts in the reference point. As was formally shown by Wakker (2005, Observation 4.4 and Theorem 4.5) such reference point shifts are necessary to explain the commonly observed deviations from expected utility. Many studies show empirical evidence of

[1] For example Köbberling and Wakker (2003), Luce and Fishburn (1991), Schmidt (2003), Sugden (2003), Tversky and Kahneman (1992), Wakker and Tversky (1993),Zank (2001).

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shifts in reference points in decision under uncertainty. See for example Bateman, Munro, Rhodes, Starmer, and Sugden (1997), Post, Van den Assem, Baltussen, and Thaler (2008), and for a meta-analysis Kühberger (1998). The central preference condition used to derive reference-dependent expected utility is intuitive and easy to test through Wakker and Deneffe's (1996) tradeoff method.

Then two special cases of reference-dependent expected utility will be considered. In the first model, utility is decomposed into two terms: a basic utility, which can be interpreted as the normative component of utility or the standard economic concept of utility and a function reflecting the impact of additional psychological factors on utility, in particular loss aversion. While loss aversion is general in the first special case, in the second model it is captured by a single parameter. Tversky and Kahneman (1991) referred to this case as *constant loss aversion*. Constant loss aversion is commonly assumed in empirical research on reference-dependence and in applications thereof.

The paper proceeds as follows. Section 2 gives notation and states the main assumptions. Section 3 gives preference foundations for reference-dependent expected utility with a fixed reference point and with shifting reference points. Section 4 characterizes the special case in which utility is decomposed into a normative and a psychological component. Section 5 characterizes constant loss aversion. Section 6 concludes. All proofs are in the Appendix.

2. Notation and assumptions

2.1. Notation

Consider a decision maker who faces uncertainty: there is a finite number, n, of *states of nature*, exactly one of which will occur. Probabilities for the states of nature may but need not be known. The restriction to finite states is made for simplicity; the extension to infinite states can be achieved by using tools developed in Wakker (1993) and is similar to Section 5 in Köbberling and Wakker (2003). The set of states is denoted by $\mathcal{S}$, elements of which are denoted by i, j, k. $\mathcal{C}$ denotes a set of consequences or *outcomes*, elements of which are denoted by α, β, γ. The decision problem is to choose between acts. Each act is an n-tuple of outcomes, one for each state of the world. Formally, an *act* is a function from $\mathcal{S}$ to $\mathcal{C}$. The set of acts is denoted as $\mathcal{F} = \mathcal{C}^n$. $(f_1, \ldots, f_n)$ denotes the act which results in outcome f_j if state of nature j obtains. By $\alpha_j f$ denote the act f with f_j replaced by $\alpha \in \mathcal{C}$, i.e., $\alpha_j f = (f_1, \ldots, f_{j-1}, \alpha, f_{j+1}, \ldots, f_n)$.

Let $r \in \mathcal{F}$ denote a *reference act*. Each act can serve as a reference act. Hence, the reference act can yield different reference outcomes for different states of nature, which is contrary to most previous characterizations of prospect theory in which the reference act yields the same outcome for each state of nature.

Let $\succeq_r$ denote the preference relation over $\mathcal{F}$ when r is the reference point. Bleichrodt (2007) explains how such a preference relation can be derived from a choice function. $\succeq_r$ is assumed to be transitive. As usual, $\succ_r$ denotes strict preference, $\sim_r$ indifference, and $\preceq_r$ and $\prec_r$ the reversed preference relations.

A *constant act* yields the same outcome for each state of nature: act f is constant if there is some $\alpha \in \mathcal{C}$ such that $f(s) = \alpha$ for all $s \in \mathcal{S}$. This will be written as $f = \alpha$. Given that outcomes are identified with constant acts, the preference relation also applies to outcomes and we can write $\alpha \succeq_r \beta$ in case $f \succeq_r g$ with $f = \alpha$ and $g = \beta$.

2.2. Preference conditions

For $r \in \mathcal{F}$, let B_r be the set of pairs of acts for which, judged from r, a preference can be observed. That is, $B_r = \{(f, g) \in \mathcal{F} \times \mathcal{F} : f \succeq_r r \text{ or } g \succeq_r r\}$. In Bleichrodt (2007),[2] I characterized the following representation, first for one fixed reference point $\succeq_r$ and then for the empirically more realistic case where the reference point could shift across decisions (Theorem 3.2).

Definition 1. There exist functions $V_j : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, where the index j indicates the state of nature, such that

a. $(f, g) \in B_r$ iff $\sum_{j=1}^n V_j(f_j, r_j) \geq 0$ or $\sum_{j=1}^n V_j(g_j, r_j) \geq 0$
b. for all $(f, g) \in B_r, f \succeq_r g$ iff $\sum_{j=1}^n V_j(f_j, r_j) \geq \sum_{j=1}^n V_j(g_j, r_j)$
c. $V_j(r_j, r_j) = 0$ for all j.
d. for all $j \in \{1, \ldots, n\}$, $V_j(\cdot, r_j)$ is increasing in its first argument (i.e. represents $\succeq_r$): for all $\alpha, \beta \in \mathcal{C}$, $V_j(\alpha, r_j) \geq V_j(\beta, r_j)$ iff $\alpha \succeq_r \beta$.
e. The V_j are continuous in their first argument and their range is $\mathbb{R}$.

Furthermore the V_j are joint ratio scales: they can be replaced by functions $W_j, j = 1, \ldots, n$ if and only if there exists a positive σ such that $W_j = \sigma V_j$ for all j. In the case of shifting reference points, the functions $V_j(\cdot, r_j)$ are independent of r: for all reference points r and r' and for all j, if $r_j = r'_j$ then $V_j(\cdot, r_j) = V_j(\cdot, r'_j)$. In the case of shifting reference points, the V_j are decreasing in their second argument.

The above representation can be interpreted as a *state-dependent reference expected utility* representation. In this paper I will impose additional conditions that ensure that the representation is state-independent and utility and beliefs can be separated. I will distinguish the first case, where there is one fixed reference point, from the second, where reference points can shift, in what follows by adding the adjective *for a fixed reference point r*.

Definition 2. *Reference-dependent expected utility* holds if in Definition 1 the V_j can be written as $p_j U$ where the p_j are unique positive subjective probabilities that sum to one and U: $\mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a *utility function*.

The following definitions are similar to those in Bleichrodt (2007) where they are discussed in more detail.

Definition 3. For a given reference act r, *r-upper completeness* holds if (i) $r \sim_r r$ and (ii) for all $f, g \in \mathcal{F}$, if $f \succeq_r r$ or $g \succeq_r r$, then either $f \succeq_r g$ or $g \succeq_r f$ holds; if $r \succ_r f$ and $r \succ_r g$ then neither $f \succeq_r g$ nor $g \succeq_r f$ holds.

Hence, a preference between two acts can only be observed if at least one of them is weakly preferred to the reference act. State j is *nonnull with respect to r* if there exist $(\alpha_j f, f) \in B_r$ such that $\alpha_j f \succ_r f$. Intuitively, nonnullness means that a state matters to the decision maker. I assume throughout that for all r, there are at least three states and all states are nonnull with respect to r.

Definition 4. *Reference-independence for outcomes* holds if for all $\alpha, \beta \in \mathcal{C}$ and for all $r, r' \in \mathcal{F}$, if $(\alpha, \beta) \in B_r \cap B_{r'}$ then $\alpha \succeq_r \beta$ iff $\alpha \succeq_{r'} \beta$.

Reference-independence for outcomes implies that preferences over outcomes are reference-independent. This seems plausible for one-dimensional outcomes such as the amount of money or life durations. For multi-dimensional outcomes it is less obvious. The condition implies that preferences over outcomes do not depend on the reference act and, hence, preferences over outcomes will henceforth be denoted simply as $\succeq$.

Definition 5. *Weak monotonicity* holds if for all $(f, g) \in B_r, f_j \succeq g_j$ for all j implies that $f \succeq_r g$.

[2] The notation in Bleichrodt (2007) is slightly different because there I considered general multi-attribute decision making.

Definition 6. *Solvability* holds if for all acts f, g, r, with $g \succeq_r r$ and for all states j there exists an outcome α such that $\alpha_j f \sim_r g$.

Consider the order topology on $\mathcal{C}$, which is generated by the sets $\{\alpha \in \mathcal{C} : \alpha \succ \beta\}$ and $\{\alpha \in \mathcal{C} : \alpha \prec \beta\}$, where $\beta \in \mathcal{C}$. Let $\mathcal{F}$ be endowed with the product topology.

Definition 7. *Preference continuity* holds if for all acts f and r, the sets $\{g \in \mathcal{F} : g \succeq_r f\}$ and $\{g \in \mathcal{F} : g \preceq_r f\}$ are closed in $\mathcal{F}$.

2.3. Tradeoff consistency

I will now introduce the main condition used in this paper. For outcomes α, β, γ, δ write

$$\alpha \ominus \beta \sim^*_\mu \gamma \ominus \delta$$

if there exist acts f and g, a state of nature j, and a reference act r with $r_j = \mu$ such that

$$\alpha_j f \sim_r \beta_j g \quad \text{and} \quad \gamma_j f \sim_r \delta_j g.$$

Intuitively, the $\sim^*$ relations can be seen as capturing strength of preference. Judged from μ obtaining α instead of β is an equally good tradeoff as obtaining γ instead of δ because both exactly offset the receipt of g_i instead of f_i in all other states i. Bleichrodt (2007) also used a $\sim^*$ relation. The difference with the condition used in Bleichrodt (2007) is that in the present formulation comparisons across states of nature are allowed. In Bleichrodt (2007) the relations were defined by looking at one state only.

We impose the following consistency condition on the $\sim^*_\mu$ relations.

Definition 8. *Tradeoff consistency* holds if improving an outcome in any $\sim^*_\mu$ relationship breaks that relationship. That is, if $\alpha \ominus \beta \sim^*_\mu \gamma \ominus \delta$ and $\alpha \ominus \beta \sim^*_\mu \gamma \ominus \delta'$ both hold then we must have $\delta \sim \delta'$.

In words, when the strength of preference of α over β is equal to the strength of preference of γ over δ and also to the strength of preference of γ over δ', then δ and δ' should be equally attractive. Tradeoff consistency ensures two things, first that the preference relations $\succeq_r$ have additive representations and, second, that the utility functions for different states can be taken to be proportional. Additivity results, in fact, from the weaker version of tradeoff consistency that was introduced in Bleichrodt (2007). Proportionality, i.e., the separation of utility and beliefs, follows from the strengthening of tradeoff consistency considered in this paper. In the presence of weak monotonicity, transitivity, and solvability, tradeoff consistency implies Savage's (1954) sure-thing principle for indifferences.[3] The reverse does not hold; tradeoff consistency is a stronger condition than the sure-thing principle.

An important advantage of using tradeoff consistency as a condition in axiomatizations, besides its intuitive appeal, is that it can easily be tested empirically through measurements of utility by the tradeoff method (Wakker & Deneffe, 1996). For empirical applications of the tradeoff method see, for example, Abdellaoui (2000), Abdellaoui, Vossmann, and Weber (2005), Bleichrodt and Pinto (2000, 2005), Etchart-Vincent (2004), and Fennema and van Assen (1999).

3. Reference-dependent expected utility

3.1. Fixed reference point

We are now in a position to characterize reference-dependent expected utility for a fixed reference point r.

Theorem 1. *Consider a given reference alternative $r \in \mathcal{F}$. Let there be at least three states of nature, which are all nonnull with respect to r. The following two statements are equivalent for $\succeq_r$:*

1. *The order topology on $\mathcal{C}$ is connected, $\succeq_r$ is transitive and satisfies r-upper completeness, reference-independence for outcomes, weak monotonicity, solvability, preference continuity, and tradeoff consistency.*
2. *reference-dependent expected utility holds for a fixed reference point. Furthermore, the p_j are uniquely determined and U is a ratio scale.* □

Sugden (2003)axiomatized a model that is similar to reference-dependent expected utility with a fixed reference point r. This was an important contribution as Sugden's model allowed for any act, constant or nonconstant to serve as the reference act. Theorem 1 differs in two ways from Sugden's axiomatization. First, Sugden (2003) does not allow for incompleteness, which restricts the applicability of his model. If the reference act always belongs to the individual's opportunity set then Sugden's result cannot be applied because it then uses unobservable inputs. Second, Sugden's axioms are somewhat complex and it is not immediately obvious how they can be tested. By contrast, tradeoff consistency is intuitive and easy to test empirically.

3.2. Shifting reference points

The next step is to extend Theorem 1 to $\{\succeq_r : r \in \mathcal{F}\}$, i.e. to the empirically more interesting case where the reference act can vary across decision contexts. The following additional three conditions, which are discussed in Bleichrodt (2007), achieve this.

Definition 9. *Upper completeness* holds if r-upper completeness holds for all r.

Definition 10. *Reference monotonicity* holds when for all acts f and r and for all outcomes α, β, δ, $\beta \preceq \alpha$, $\delta_j f \sim_{\alpha_j r} \alpha_j r$ implies $\delta_j f \succeq_{\beta_j r} \beta_j r$.

In Theorem 1, the subjective probabilities p_j and the utility function $U(\cdot, r_j)$ in Theorem 1 depend on the given reference act r. This makes the model too general to yield predictions. The final condition achieves independence of subjective probabilities and utility from the reference act.

Definition 11. *Neutral independence* holds if for all $f, g \in \mathcal{F}$ and for all $\alpha, \beta \in \mathcal{C}$, $\alpha_j f \sim_{\alpha_j r} \alpha_j g$ implies $\beta_j f \sim_{\beta_j r} \beta_j g$.

Theorem 2. *Let there be at least three states of nature, which are all nonnull with respect to every $r \in \mathcal{F}$. The following two statements are equivalent for $\{\succeq_r : r \in \mathcal{F}\}$:*

1. *The order topology on $\mathcal{C}$ is connected, $\succeq_r$ is transitive and satisfies upper completeness, reference-independence for outcomes, weak monotonicity, reference monotonicity, solvability, preference continuity, tradeoff consistency, and neutral independence.*
2. *Reference-dependent expected utility holds. Furthermore, the p_j are uniquely determined and U is a ratio scale.* □

4. Separating "Normative Utility" and "Psychological Utility"

Next a special case of reference-dependent expected utility will be derived in which the utility function U can be decomposed as

$$U(f_j, r_j) = F(u(f_j) - u(r_j)), \tag{1}$$

with $u : \mathcal{C} \to \mathbb{R}$ continuous and F continuous and strictly increasing and $F(0) = 0$. Moreover, u is an interval scale, unique up to unit and origin, and F is a ratio scale.

[3] The proof is similar to the proof of Lemma 2 in Bleichrodt (2007).

In this decomposition, u is a *basic utility function* (Köbberling & Wakker, 2005), which expresses the individual's attitude towards outcomes. The basic utility function is reference-independent and may be interpreted as the normative component of utility. The function F models the impact of additional psychological factors, such as numerical sensitivity and loss aversion. Hence, (1) separates normative and psychological utility. Köszegi and Rabin (2006) called F a "universal gain–loss function". Sugden (2003) referred to (1) as a satisfaction-change decomposition and showed that it can explain important empirical deviations from expected utility. Fishburn (1992, Theorem 2) proposed a slightly more general model for additive skew-symmetric nontransitive preferences, in which the functions F and u are state-dependent.

To derive (1), two new conditions must be imposed. Suppose that $\alpha \succ \beta \succ \gamma$ and that $U(\alpha, \beta) = U(\kappa, \lambda)$ and $U(\beta, \gamma) = U(\lambda, \mu)$. That is, the gain of getting α instead of β is as attractive as the gain of getting κ instead of λ and the gain of getting β instead of γ is as attractive as the gain of getting λ instead of μ. Then it seems plausible that the gain of getting α instead of γ should also be equally attractive as the gain of getting κ instead of μ. That is, $\alpha\gamma$, the "concatenation" of the preference intervals $\alpha\beta$ and $\beta\gamma$ should match $\kappa\mu$, the concatenation of the preference intervals $\kappa\lambda$ and $\lambda\mu$. The next condition ensures this.

Definition 12. The *concatenation condition* holds if for all j, if $\alpha_j f \sim_{\beta_j r} \beta_j r$, $\kappa_j f \sim_{\lambda_j r} \lambda_j r$, $\beta_j g \sim_{\gamma_j r} \gamma_j r$, $\kappa_j g \sim_{\mu_j r} \mu_j r$, and $\alpha_j h \sim_{\gamma_j r} \gamma_j r$ then also $\kappa_j h \sim_{\mu_j r} \mu_j r$.

The second new condition is of a technical nature and extends preference continuity.

Definition 13. *Reference act continuity* holds if for all acts $f \in \mathcal{F}$ the sets $\{r \in \mathcal{F} : f \succeq_r r\}$ and $\{r \in \mathcal{F} : f \preceq_r r\}$ are both closed in $\mathcal{F}$.

Theorem 3. *Let the conditions of Theorem 2 [1] hold. Then the following two statements are equivalent.*

1. *The concatenation condition, and reference act continuity hold.*
2. *Reference-dependent expected utility [with a fixed reference point r] holds with $U(f_j, r_j) = F(u(f_j) - u(r_j))$.* □

Sugden (2003, Theorem 2) gave a preference foundation for the decomposition (1) when reference-dependent preferences are complete. Theorem 3 generalizes Sugden's result even when preferences are complete and the reference point is fixed. First, Sugden (2003) assumed richness of the state space, which is not assumed here. Second, the outcome set is $\mathbb{R}^+$ in Sugden (2003), whereas the set of outcomes in this paper is more general, being any set whose order topology is connected. Finally, for $\mathcal{C} = \mathbb{R}^+$, Sugden's conditions imply our conditions, but the reverse is not true, as we prove in the Appendix. The concatenation condition is also easier to test empirically than Sugden's conditions S3 (gain/loss symmetry) and S4 (gain/loss additivity) because it is entirely defined in terms of indifferences.

5. Constant loss aversion

Loss aversion is general in (1). For empirical purposes and for applications, it is often desirable to restrict loss aversion. Indeed, in most empirical analyses loss aversion is characterized by a single parameter (e.g. Bleichrodt, Pinto, and Wakker (2001), Schmidt and Traub (2002) and Pennings and Smidts (2003)). Preference foundations for this case will be given in this section.

Constant loss aversion (Tversky & Kahneman, 1991) holds if in Theorem 2 $U(f_j, r_j) = u(f_j) - u(r_j)$ when $f_j \succeq r_j$ and $U(f_j, r_j) = \lambda(u(f_j) - u(r_j))$ when $f_j \preceq r_j$. The parameter λ captures the attitude towards gains and losses and can be interpreted as a loss aversion coefficient. Constant loss aversion is the special case of Theorem 3 where F is both linear for gains and linear for losses, but may be different for gains than for losses. Constant loss aversion models have been proposed by Tversky and Kahneman (1991, 1992), Shalev (2000), and Köbberling and Wakker (2005), but none of these papers provided preference foundations for constant loss aversion.

Consider two acts f and g and suppose that for some state of nature j the outcomes of both acts are gains when judged from reference act r. An implication of constant loss aversion is that if r_j is changed into ρ_j but the change is such that f_j and g_j remain gains while for all other states of nature the reference act does not change, i.e. $r_i = \rho_i$ for all $i \neq j$, then the preference between f and g is not affected by the change in the reference point. Hence, we may say that under constant loss aversion, reference-independence holds for gains. Similarly, reference-independence then holds for losses.

Let us now formalize the above implication.

Definition 14. *Reference-independence for gains* holds if for all r, for all ρ_j, for all $f, g \in B_r \cap B_{\rho_j r}$, and for all j, if $f_j \succeq r_j, f_j \succeq \rho_j, g_j \succeq r_j$, and $g_j \succeq \rho_j$, then $f \succeq_r g$ iff $f \succeq_{\rho_j r} g$. *Reference-independence for losses* holds if for all r, for all ρ_j, for all $f, g \in B_r \cap B_{\rho_j r}$ and for all j, if, $r_j \succeq f_j$, $\rho_j \succeq f_j$, $r_j \succeq g_j$, and $\rho_j \succeq g_j$, then $f \succeq_r g$ iff $f \succeq_{\rho_j r} g$.

Theorem 4. *Let the conditions of Theorem 3 hold. Then the following two statements are equivalent.*

1. *reference-independence for gains and reference-independence for losses both hold.*
2. *constant loss aversion holds.* □

6. Conclusion

This paper has extended models of reference-dependence in decision under uncertainty to the case where preferences are incomplete and the reference point can shift across decisions. Incompleteness follows when the reference point is one of the available acts, a decision context that is empirically realistic. Shifting reference points are required to explain the common deviations from expected utility. The preference conditions used are intuitive and easy to test. The paper has also given a preference foundation of a model with constant loss aversion, a common assumption in empirical research and practical applications.

Let me conclude by pointing out two avenues for future research. First, this paper focused on reference-dependence and did not consider nonadditive decision weighting, the other deviation from expected utility modeled by prospect theory. Second, I have not considered the question how reference points are formed. Like previous studies, I took the reference point as exogenously given. Providing preference foundations for models with endogenous reference points will likely involve studying incomplete preferences. For example, Hershey and Schoemaker's (1985) data suggested that people take €100 as their reference point in a comparison between €100 for sure and a prospect giving €200 with some probability p and nothing otherwise. Consequently, a preference between €100 and €200 with probability p judged from any other reference point than €100 cannot be observed and preferences must be incomplete. It is hoped that the tools developed in this paper and in Bleichrodt (2007) will also prove useful in modeling the endogeneity of reference points.

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Appendix. Proofs

Proof of Theorem 1. To prove that statement (2) implies statement (1) we only have to show that tradeoff consistency holds. That the other conditions hold follows from the proof of Theorem 3.1 in Bleichrodt (2007). If $\alpha\beta \sim^*_\mu \gamma\delta$ then there exist acts f and g and a state j with $r_j = \mu$ such that $\alpha_j f \sim_r \beta_j g$ and $\gamma_j f \sim_r \delta_j g$. Hence, $p_j U(\alpha, \mu) + \sum_{i\neq j} p_i U(f_i, r_i) = p_j U(\beta, \mu) + \sum_{i\neq j} p_i U(g_i, r_i)$ and $p_j U(\gamma, \mu) + \sum_{i\neq j} p_i U(f_i, r_i) = p_j U(\delta, \mu) + \sum_{i\neq j} p_i U(g_i, r_i)$, which together with $p_j > 0$ give $U(\alpha, \mu) - U(\beta, \mu) = U(\gamma, \mu) - U(\delta, \mu)$. Suppose we also have $\alpha\beta \sim^*_\mu \gamma\delta'$. Then there exist acts f' and g' and a state k with $r_k = \mu$ such that $\alpha_k f' \sim_r \beta_k g'$ and $\gamma_k f' \sim_r \delta'_j g'$. By a similar line of argument as above we obtain $U(\alpha, \mu) - U(\beta, \mu) = U(\gamma, \mu) - U(\delta', \mu)$. It follows that $U(\delta', \mu) = U(\delta, \mu)$ and, because U represents $\succeq$, $\delta' \sim \delta$, which establishes tradeoff consistency.

We now show that statement (1) implies statement (2). Let $r \in \mathcal{F}$. By setting $X_j = \mathcal{C}$ for all j, it is easily verified that all conditions in statement 1 of Theorem 3.1 in Bleichrodt (2007) are satisfied. Our tradeoff consistency condition is a strengthening of the tradeoff consistency condition that is used there. It follows that there exist joint ratio scales $V_j : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$ that satisfy Definition 1 in the main text. It remains to show that the V_j can be taken to be proportional. To prove this we need the following lemma, which was proved in Bleichrodt (2007). The preference relation satisfies *strong monotonicity* if for all $(f, g) \in B_r$, for all states of nature j, $f_j \succeq g_j$ and for at least one state of nature i, $f_i \succ g_i$ then $f \succ_r g$.

Lemma 1. *If $\succeq_r$ satisfies transitivity, upper completeness, and weak monotonicity, then tradeoff consistency implies strong monotonicity.*

Lemma 2. *Assume that the V_j: $\mathcal{C} \times \mathcal{C} \to \mathbb{R}$, $j = 1, \ldots, n$, are as defined above. Then every V_i is proportional to every other V_j.*

Proof. Assume that the reference act is such that $r_i = r_j = \mu$. Let α, β, and γ be arbitrary outcomes with $\alpha \succ \gamma$. By solvability and because there are at least three states which are all nonnull, we can find an act f such that $\gamma_i f \sim_r r$. By strong monotonicity (Lemma 1) $\alpha_i f \succ_r r$. By solvability and because there are at least three states of nature, we can find an outcome δ and an act g such that $\alpha_i f \sim_r \beta_i g$ and $\gamma_i f \sim_r \delta_i g$. Hence, $\alpha\beta \sim^*_\mu \gamma\delta$. By strong monotonicity and transitivity it follows that $\beta \succ \delta$.

The indifference $\alpha_i f \sim_r \beta_i g$ implies that

$$V_i(\alpha, \mu) + V_j(f_j, r_j) = V_i(\beta, \mu) + V_j(g_j, r_j)$$

or

$$V_i(\alpha, \mu) - V_i(\beta, \mu) = V_j(g_j, r_j) - V_j(f_j, r_j).$$

Similarly, the indifference $\gamma_i f \sim_r \delta_i g$ implies that

$$V_i(\gamma, \mu) - V_i(\delta, \mu) = V_j(g_j, r_j) - V_j(f_j, r_j).$$

Hence, $V_i(\alpha, \mu) - V_i(\beta, \mu) = V_i(\gamma, \mu) - V_i(\delta, \mu)$.

By solvability and because there are other states than i, we can find acts f' and g' such that $\gamma_j f' \sim_r \delta_j g' \sim_r r$. By strong monotonicity, $\alpha_j f' \succ_r r$ and $\beta_j g' \succ_r r$. By tradeoff consistency and strong monotonicity, it follows that $\alpha_j f' \sim_r \beta_j g'$. For suppose that $\alpha_j f' \prec_r \beta_j g'$. By solvability there exists a β' such that $\alpha_j f' \sim_r \beta'_j g'$. By transitivity, $\beta'_j g' \prec_r \beta_j g'$, and, hence, by weak monotonicity $\beta' \prec \beta$. But then we have $\alpha\beta \sim^*_\mu \gamma\delta$ and $\alpha\beta' \sim^*_\mu \gamma\delta$ but $\beta' \nsim \beta$ which violates tradeoff consistency. $\alpha_j f' \succ_r \beta_j g'$ is excluded by a similar argument.

The indifferences $\alpha_j f' \sim_r \beta_j g'$ and $\gamma_j f' \sim_r \delta_j g'$ imply that

$$V_j(\alpha, \mu) - V_j(\beta, \mu) = V_j(\gamma, \mu) - V_j(\delta, \mu).$$

Hence, $V_i(\alpha, \mu) - V_i(\beta, \mu) = V_i(\gamma, \mu) - V_i(\delta, \mu)$ implies that $V_j(\alpha, \mu) - V_j(\beta, \mu) = V_j(\gamma, \mu) - V_j(\delta, \mu)$ and, V_i and V_j give rise to the same ordering of utility equalities. Because V_i and V_j both represent $\succeq$ and have range $\mathbb{R}$, they must also give rise to the same ordering of utility differences. This and the fact that the functions V_i and V_j are continuous in their first argument on a connected domain imply that V_i and V_j are related by a positive linear transformation. They are 0 at μ; hence they are related by a scale factor. The scale factor is positive by strong monotonicity. □

By Lemma 2, we have $V_j = \sigma_j V_i$ with $\sigma_j > 0$ because i and j are nonnull. Because i and j were arbitrary it follows that all V_j are proportional. Define $U = V_i$ for some state i and define $p_j = \sigma_j/(\sum_{j=1}^n \sigma_j)$. Because $\sigma_i = 1$, the denominator exceeds zero. Uniqueness of the p_j follows from the uniqueness properties of the V_j. Theorem 1 has been established. □

Proof of Theorem 2. This follows from Theorem 1 and the proof of Theorem 3.2 in Bleichrodt (2007). □

Proof of Theorem 3. Throughout the proof we will use the assumption that each act can serve as a reference act without further mentioning. That statement (2) implies statement (1) is easily verified. Assume that statement (1) holds. Let U be as in Theorem 2. Define $\succeq'$ on $\mathcal{C} \times \mathcal{C}$ by $\alpha\beta \succeq' \gamma\delta$ if $U(\alpha, \beta) \geq U(\gamma, \delta)$. $\succeq'$ is clearly transitive. Let $\alpha, \beta \in \mathcal{C}$. Let r be an arbitrary reference act with $r_j = \beta$. By solvability there exists an act f such that $\alpha_j f \sim_r r$. Hence, $U(\alpha, \beta)$ is defined for each pair $\alpha\beta$ and it follows that $\succeq'$ is complete. From the definition of U it is further immediately verified that for all $\alpha, \beta \in \mathcal{C}$, $\alpha\alpha \sim' \beta\beta$. We can define a preference relation over $\mathcal{C}$ by $\alpha \succeq \beta$ if $\alpha\beta \succeq' \beta\beta$. Since this implies that $U(\alpha, \beta) \geq U(\beta, \beta) = 0$ and, by Theorem 2, U is representing over outcomes, it follows that this derived preference relation over outcomes is identical to the preference relation over outcomes defined in the main text.

Lemma 3. *For all $\alpha, \beta, \gamma, \delta \in \mathcal{C}$, if $\alpha\gamma \succeq' \beta\gamma$ then $\alpha\delta \succeq' \beta\delta$, and if $\gamma\alpha \succeq' \gamma\beta$ then $\delta\alpha \succeq' \delta\beta$.*

Proof. This follows immediately from the facts that $U(\cdot, r_j)$ represents $\succeq$ over outcomes for every r_j and that U is decreasing in its second argument. □

Lemma 4. *For all $\alpha, \beta, \gamma, \kappa, \lambda, \mu \in \mathcal{C}$, if $\alpha\beta \sim' \kappa\lambda$ and $\beta\gamma \sim' \lambda\mu$ then $\alpha\gamma \sim' \kappa\mu$.*

Proof. Suppose that $\alpha\beta \sim' \kappa\lambda$ and $\beta\gamma \sim' \lambda\mu$. Hence, $U(\alpha, \beta) = U(\kappa, \lambda)$ and $U(\beta, \gamma) = U(\lambda, \mu)$. Let $r \in \mathcal{F}$. By solvability and the fact that there are at least three states of nature, we can find acts f, g, and h such that $\alpha_j f \sim_{\beta_j r} \beta_j r$, $\beta_j g \sim_{\gamma_j r} \gamma_j r$, and $\alpha_j h \sim_{\gamma_j r} \gamma_j r$. Because $U(\alpha, \beta) = U(\kappa, \lambda)$ and $U(\beta, \gamma) = U(\lambda, \mu)$, it follows that $\kappa_j f \sim_{\lambda_j r} \lambda_j r$ and that $\lambda_j g \sim_{\mu_j r} \mu_j r$. By the concatenation condition, it follows that $\kappa_j h \sim_{\mu_j r} \mu_j r$. Hence, $U(\alpha, \gamma) = U(\kappa, \mu)$ and thus $\alpha\gamma \sim' \kappa\mu$. □

Because $U(\alpha, \beta)$ is continuous in each of its variables and because $U(\alpha, \beta)$ is representing preferences over outcomes and is decreasing in its second variable, $U(\alpha, \beta)$ is continuous and, consequently, $\succeq'$ is continuous. The following lemma follows from the continuity of U and the fact that $\mathcal{C}$ is connected. For a proof see Krantz, Luce, Suppes, and Tversky (1971, pp. 309–310) or Wakker (1989, Lemma III.3.3).

Lemma 5. *For all $\alpha, \beta, \gamma, \kappa, \mu \in \mathcal{C}$, $\kappa\gamma \succeq' \alpha\beta \succeq' \mu\gamma$ implies the existence of a λ with $\lambda\gamma \sim' \alpha\beta$ and $\gamma\kappa \succeq' \alpha\beta \succeq' \gamma\mu$ implies the existence of a λ with $\gamma\lambda \sim' \alpha\beta$.*

Let α^1 and α^0 be two outcomes such that $\alpha^1 \nsim \alpha^0$. Let $r \in \mathcal{F}$. By solvability and the fact that there are states $i \neq j$, we can find an act $f \in \mathcal{F}$ such that $(\alpha^1)_j f \sim_{(\alpha^0)_j r} (\alpha^0)_j r$. We then proceed inductively to define α^{k+1} as $(\alpha^{k+1})_j f \sim_{(\alpha^k)_j r} (\alpha^k)_j r$, $k = 1, 2, \ldots$. By solvability,

α^{k+1} exists. By Theorem 2 we have $U(\alpha^{k+1}, \alpha^k) = U(\alpha^1, \alpha^0)$ and thus $\alpha^{k+1}\alpha^k \sim' \alpha^1\alpha^0$, for all $k = 1, \ldots$. The sequence $\alpha^0, \alpha^1, \ldots, \alpha^k, \ldots$ is a *standard sequence*.

If $\alpha^1 \succ \alpha^0$ then $U(\alpha^1, \alpha^0) > 0$. Because $U(\alpha^1, \alpha^0) = U(\alpha^{k+1}, \alpha^k)$, $U(\alpha^{k+1}, \alpha^k) > 0$ and $\alpha^{k+1} \succ \alpha^k$. Hence, if $\alpha^1 \succ \alpha^0$ then the standard sequence is *increasing*: $\alpha^{k+1} \succ \alpha^k$, for all nonnegative integers k. Similarly, if $\alpha^1 \prec \alpha^0$ then the standard sequence is *decreasing*: $\alpha^{k+1} \prec \alpha^k$, for all nonnegative integers k. A standard sequence is *bounded* if there exist $\beta, \gamma \in \mathcal{C}$ with $\beta \succeq \alpha^k \succeq \gamma$ for all α^k in the standard sequence. The following lemma is proved in Krantz et al. (1971, pp. 309–310).

Lemma 6. *Every bounded standard sequence is finite.*

The above results establish that $\succeq'$ satisfies all the conditions in Theorem 1 in Köbberling (2006). Her definition of solvability follows from Lemma 5, Archimedeanity follows from Lemma 6, her definition of weak separability follows from Lemma 3, and weak ordering of $\succeq'$ and neutrality were established in the text above Lemma 3. Her definition of the concatenation condition follows from Lemma 4. Theorem 1 in Köbberling (2006) ensures that there exists an u: $\mathcal{C} \to \mathbb{R}$ such that for all $\alpha, \beta, \gamma, \delta \in \mathcal{C}$, $\alpha\beta \succeq' \gamma\delta$ iff $u(\alpha) - u(\beta) \geq u(\gamma) - u(\delta)$, with u unique up to positive linear transformations. Hence, by the definition of $\succeq'$, $U(\alpha, \beta) \geq U(\gamma, \delta)$ iff $u(\alpha) - u(\beta) \geq u(\gamma) - u(\delta)$. Because $\mathcal{C}$ is connected and $\succeq'$ is continuous, we obtain that u is continuous by the proof of Theorem 5.3 in Wakker (1988).

Because there are nonnull states of nature, it follows that $\Delta[u] = \{u(\alpha) - u(\beta) : \alpha, \beta \in \mathcal{C}\}$ is nondegenerate. For all $\alpha, \beta \in \mathcal{C}$, define F on $\Delta[u]$ by $F(u(\alpha) - u(\beta)) = U(\alpha, \beta)$. F is clearly strictly increasing. Because u is continuous, $u(\alpha) - u(\beta)$ is continuous both in α and in β and represents preferences over outcomes, $u(\alpha) - u(\beta)$ is continuous. Because $u(\alpha) - u(\beta)$ and U are continuous real-valued functions on connected topological spaces and F is strictly increasing, we obtain by Theorem 2.1 in Wakker (1991) that F is continuous. Because $U(\alpha, \alpha) = 0$, it follows that $F(0) = 0$. The uniqueness properties of F follow from the uniqueness properties of u and U. This completes the proof of Theorem 3.

We finally show that Sugden's (2003) conditions imply the conditions of Theorem 3. For ease of comparison with Sugden (2003) we express the derivations in terms of U. Reference act continuity follows from Sugden's condition S2 (consequence-space continuity). Suppose that $\beta \preceq \alpha$. Because U represents preferences over outcomes, for all $\delta \in \mathcal{C}$, $U(\alpha, \delta) \geq U(\beta, \delta)$. By Sugden's S3 (gain/loss symmetry), $U(\delta, \beta) \geq U(\delta, \alpha)$. If $U(\delta, \beta) \geq U(\delta, \alpha)$ then by S3, $U(\alpha, \delta) \geq U(\beta, \delta)$ and, hence, $\beta \preceq \alpha$ because U represents preferences over outcomes.

Sugden's condition S4 (gain/loss additivity) is equivalent to the strong crossover condition in difference measurement (Suppes & Winet, 1955; Scott & Suppes, 1958; Debreu, 1958; Pfanzagl, 1968). Köbberling (2006) showed that the strong crossover condition implies the concatenation condition. She also showed that the reverse implication does not hold (p. 390). □

Proof of Theorem 4. That statement (2) implies statement (1) is easily verified. Assume that statement (1) holds. An element $\alpha \in \mathcal{C}$ is *maximal* if for all $\beta \in \mathcal{C}, \alpha \succeq \beta$. An element $\alpha \in \mathcal{C}$ is *minimal* if for all $\beta \in \mathcal{C}, \alpha \preceq \beta$. It follows immediately from U being unbounded that:

Lemma 7. *$\mathcal{C}$ contains no maximal or minimal elements.*

Let $r \in \mathcal{F}$. Let $f_i \succ r_i, f_j \succ r_j, f_k = r_k, k \neq i, j$. Such an act f can be constructed by Lemma 7. By strong monotonicity, $f \succ_r r$. Select γ such that $f_i \succ \gamma \succ r_i$. Such a γ exists by connectedness of the order topology. By solvability, there exists an outcome δ such that $\gamma_i\delta_j r \sim_r f$. By strong monotonicity $\delta \succ f_j$. Let $g = \gamma_i\delta_j r$.

Let $r'_j = r_j$ for all $j \neq i$, and let $r'_i \prec r_i$. By Lemma 7, r' can be constructed. Because $f_i \succ g_i = \gamma \succ r_i \succ r'_i$, by reference-independence for gains we obtain $f \sim_{r'} g$. The indifference $f \sim_r g$ gives by Theorem 3 and cancellation of common terms:

$$p_iF(u(f_i) - u(r_i)) + p_jF(u(f_j) - u(r_j)) = p_iF(u(\gamma) - u(r_i)) + p_jF(u(\delta) - u(r_j)).$$

The indifference $f \sim_{r'} g$ gives by Theorem 3 and cancellation of common terms:

$$p_iF(u(f_i) - u(r'_i)) + p_jF(u(f_j) - u(r_j)) = p_iF(u(\gamma) - u(r'_i)) + p_jF(u(\delta) - u(r_j)).$$

Let $x = u(f_i) - u(r_i)$, $y = u(\gamma) - u(r_i)$, and $\varepsilon = u(r_i) - u(r'_i)$. Then the above two equalities yield after some rearranging: $F(x) - F(y) = F(x + \varepsilon) - F(y + \varepsilon)$. Because $u(\mathcal{C}) = \mathbb{R}$ and because the above preferences can be constructed for all x, y, and ε thanks to solvability, $F(x) - F(y) = F(x + \varepsilon) - F(y + \varepsilon)$ holds for all x, y, and ε and, hence, F must be linear: $F(x) = ax + b$. Since F is increasing, $a > 0$ and since $F(0) = 0$, $b = 0$.

A similar line of argument shows that $F = bx$ for losses, i.e. $x < 0$. We can rescale utility such that $a = 1$. It then follows that $\lambda = \frac{b}{a}$ and constant loss aversion results. □

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