Journal of Mathematical Psychology 52 (2008) 304–310

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

Combining additive representations on subsets into an overall representation

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ARTICLE INFO

Article history:
Received 18 May 2007
Received in revised form
1 April 2008
Available online 4 June 2008

Keywords:
Rank-dependence
Rank-additive utility
Coalescing
Joint independence
Tradeoff consistency

ABSTRACT

Many traditional conjoint representations of binary preferences are additively decomposable, or additive for short. An important generalization arises under rank-dependence, when additivity is restricted to cones with a fixed ranking of components from best to worst (comonotonicity), leading to configural weighting, rank-dependent utility, and rank- and sign-dependent utility (prospect theory). This paper provides a general result showing how additive representations on an arbitrary collection of comonotonic cones can be combined into one overall representation that applies to the union of all cones considered. The result is applied to a new paradigm for decision under uncertainty developed by Duncan Luce and others, which allows for violations of basic rationality properties such as the coalescing of events and other framing conditions. Through our result, a complete preference foundation of a number of new models by Luce and others can be obtained. We also show how additive representations on different full product sets can be combined into a representation on the union of these different product sets.

1. Introduction

Many modern preference models start from additively decomposable evaluations on subsets of full product sets that have a fixed rank-ordering of components from best to worst (comonotonic cones). Examples include configural weighting theories (Birnbaum, 1974), Choquet expected utility (Gilboa, 1987; Schmeidler, 1989), rank-dependent utility (Luce, 1988; Quiggin, 1982), and rank- and sign-dependent theories including prospect theory (Luce & Fishburn, 1991; Tversky & Kahneman, 1992). In each of these theories, additive representations on cones as above are combined into an overall representation on the union of all cones. This paper provides a general technique for obtaining such combinations. Our study was motivated by recent models developed by Duncan Luce jointly with Tony Marley and several other co-authors. These authors will be referred to as Luce et al. LM will refer to Luce and Marley (2005), the most important paper for our analysis.

Luce et al. developed an innovative paradigm for decision under uncertainty that is more general than Savage's (1954) commonly used paradigm, and that generalizes the aforementioned theories in several respects (Krantz, Luce, Suppes, and Tversky (1971), Section 8.2.1; Luce (1990, 2000)). It provides sophisticated models that can account for basic violations of rationality, the desirability of which was little understood in the 1970s but has become increasingly understood since the 1990s. The RAM and TAX models of Birnbaum and his colleagues (Birnbaum (2007) and the references therein) are also based on the general representation of LM (Definition 4.1). These models of Luce and Birnbaum are so general that techniques for combining representations on comonotonic cones into overall representations as used in the aforementioned papers cannot be used. Thus, the preference foundation of the rank-additive (RAU) model, a model upon which many of Luce et al.'s recent models have been based (LM; (Marley & Luce, 2005; Marley, Luce, & Kocsis, 2008)), has as yet remained an open problem (LM, Section 2.1). The general technique presented in this paper allows us to provide a preference foundation of RAU. Thus, it completes the preference foundation of the recent models by Luce et al.

This paper is organized as follows. Section 2 presents some basic definitions, including Luce's theoretical concept of "experiments," and Section 3 presents some more definitions and basic results. Section 4 presents the main result of this paper, a preference foundation of rank-additive utility. Section 5 shows how the preference foundation is related to some other foundations in the literature. The comparison to Schmeidler's (1989) Choquet expected utility clarifies the mathematics of the latter model. Proofs are in Sections 6 and 7, and Appendix B.

2. Gambles and experiments

We mostly follow the notation and setup of LM. The main deviations will be described in the main text, with full details in Appendix A. X denotes a set of *consequences* or *degenerate gambles*, and $\succcurlyeq$ denotes a binary relation on X. We assume a nonempty set

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0022-2496/$ – see front matter
doi:10.1016/j.jmp.2008.04.005

Π of *experiments*. The generic notation for an experiment is π. For each π a number $n(\pi) \geq 2$ and a subset of the product set $X^{n(\pi)}$ are given. We explicitly express π in the notation, and consider $n(\pi) + 1$ tuples $(\pi, x_1, \ldots, x_{n(\pi)})$. In the main text we assume that the subset of the product set $X^{n(\pi)}$ concerns $(\pi, x_1, \ldots, x_{n(\pi)})$ with $x_1 \succcurlyeq \cdots \succcurlyeq x_{n(\pi)}$. Such tuples are called *rank-ordered*. In Theorem 4.7 we will consider the case where the rank-ordering requirement is dropped, and the preference domain contains X and all tuples of the form $(\pi, x_1, \ldots, x_{n(\pi)})$.

The tuples $(\pi, x_1, \ldots, x_{n(\pi)})$ are called *nondegenerate gambles*. We also denote such tuples as x, and call them *π-related* to express the corresponding π. $\mathcal{G}$ is the union of all degenerate and nondegenerate gambles (the latter union over all $\pi \in \Pi$), and we also use the term *gamble* for its elements. We assume that a binary preference relation is given on $\mathcal{G}$ that extends $\succcurlyeq$ from X to $\mathcal{G}$, and we use the same symbol $\succcurlyeq$ to denote it. We will later assume that for each nondegenerate gamble there exists a degenerate gamble that is equivalent. Thus, the degenerate gambles serve as a scale to calibrate the nondegenerate gambles and to compare them across different experiments. We next consider an example of our setup.

Example 2.1 (*Savage's Classical Model of Decision under Uncertainty*). A fixed universal event S is given, the *state space*. An experiment π is a finite ordered partition $(C_1, \ldots, C_{n(\pi)})$ of S. Subsets of S are called *events*. A nondegenerate gamble x is taken to be a function from S to X, assigning consequence x_j to each element of C_j (Savage, 1954). Degenerate gambles are identified with constant functions on S, yielding a consequence with certainty. Thus, $((C_1, \ldots, C_{n(\pi)}), (\alpha, \ldots, \alpha))$ is identified with α. In rank-dependent models the ranking of the consequences within a gamble is important (Gilboa, 1987; Schmeidler, 1989; Tversky & Kahneman, 1992). Additive decomposability then only holds within sets of gambles with the same rank-ordering of the states as regards favorability. Hence, it is useful to restrict the set of gambles for an experiment to gambles with the same rank-orderings (comonotonicity). For expected utility, additive decomposability holds within the set of all gambles. It may then be more convenient to combine all gambles into one experiment without any rank-ordering restriction. This case is briefly considered in Theorem 4.7.

In this example of Savage's model, the gamble sets of different experiments have overlaps. For example, gambles

$$((A, B, C), (\$10, \$10, \$0)),$$
$$((B, A, C), (\$10, \$10, \$0)), \quad \text{and}$$
$$((A \cup B, C), (\$10, \$0)) \tag{2.1}$$

concern the same function from S to X. They are, therefore, identical *by definition*. Consequently, Savage's model cannot represent preferences that distinguish between the different framings in Eq. (2.1). □

Many experimental investigations have shown that preferences depend on the way gambles are presented or "framed" (Birnbaum, 2007; Luce, 2000; Starmer & Sugden, 1989). This makes it desirable, for instance, to distinguish between the different framings used in Eq. (2.1). Luce developed a formal paradigm that allows to distinguish between such different framings in Eq. (2.1). In his model, gambles are general tuples and they are not identified with functions from S to X. When Luce originally introduced this paradigm, for instance in Krantz et al. (1971), its importance was not widely understood. Only in the 1990s, when empirical studies demonstrated the importance of the aforementioned framing effects, did it become widely understood how desirable Luce's general setup is. Our paper will follow his general approach and will not equate the tuples with functions from S to X.

Luce generalized Savage's classical model in several other respects. As did Example 2.1, Luce assumes that experiments specify a finite ordered set of mutually exclusive events. Unlike in Example 2.1, these events need not be exhaustive, i.e. they need not cover all logical possibilities. Their union need not be the fixed event S but it may depend on π, and may be a conditioning event. Krantz et al. (1971, Section 8.2.1) gave an example where one experiment concerns traveling by car and another one concerns traveling by bus (see also Luce (2000), Section 1.1.6.1). Consequences can be the various traveling times, and the events specify various causes of delays. In Savage's classical approach, defining a set S will be complex and will involve irrelevant events: There all *combinations of* causes of delays for bus trips and car trips must be specified, as well as their joint probability distributions in, for instance, expected utility. Such combinations are, however, irrelevant for the actual decisions where we are not interested in correlations between delays of bus and car trips. Another difference with Savage's approach is that Luce deliberately does not specify whether the experiment taking place is chosen by the decision maker or is in some sense chosen by chance.

Whereas we maintain Luce's term experiment, an experiment can be anything in our analysis. It need not be an n-tuple, need not specify (sub)sets, and serves only to index the rank-ordered product set. We can thus deal with any collection of preferences over any collection of (rank-ordered) product sets with variable finite dimension, as long as also the "calibration" set X of degenerate gambles is present in the domain of preference.

Many alternative interpretations of experiments are possible. Experiments could designate tuples of persons, or of time points. Then a gamble $(\pi, x_1, \ldots, x_{n(\pi)})$ designates an allocation over these persons with person $j \in \pi$ receiving the consequence x_j, or it designates the consumption profile with consequence x_j consumed on time point $j \in \pi$. The implicit assumption then is that all persons not listed in π receive a consequence "nothing" or "neutral," or that the consumption on the time points not specified in π is nothing or neutral. Such approaches with variable sets of persons considered have been widely studied in welfare theory (Blackorby, Bossert, & Donaldson, 2001). Alternatively, experiments could specify a list of properties of persons such as kindness, honesty, age, and the *xs* could designate scores regarding these properties, as for instance in Birnbaum (1974).

3. Assumptions

The notation $\succcurlyeq, \succ, \preccurlyeq, \prec$, and $\sim$ is as usual. We assume that

$\succcurlyeq$ is a *weak order*, i.e. it is *complete* ($x \succcurlyeq y$ or $y \succcurlyeq x$ for all $x, y \in \mathcal{G}$) and transitive. (3.1)

A function U *represents* $\succcurlyeq$ if $[x \succcurlyeq y \Leftrightarrow U(x) \geq U(y)]$. Weak ordering is a necessary condition for the existence of a representing function. For brevity, we will display a number of assumptions as numbered equations. The starting point of this paper is the assumption that for each fixed π we have a representation within the set of π-related gambles. More precisely, for each π there exist functions $V_1(\pi, .), \ldots, V_{n(\pi)}(\pi, .)$ from X to $\mathbb{R}$, and a function V_π on the set of π-related x such that:

The function

$$x = (\pi, x_1, \ldots, x_{n(\pi)}) \mapsto V_1(\pi, x_1) + \cdots + V_{n(\pi)}(\pi, x_{n(\pi)}) = V_\pi(x)$$

represents $\succcurlyeq$ when restricted to the π-related x, where the image of each function $V_j(\pi, .)$ is a nondegenerate interval. (3.2)

To obtain a complete preference foundation of RAU, a preference foundation of Eq. (3.2) should be added to the axioms for RAU provided later. Such a preference foundation is in Wakker (1993), with generalizations in Chateauneuf and Wakker (1993).

For brevity, we will not repeat these foundations here, but refer the reader to those works. As regards technical topological assumptions, the interval-image assumption in Eq. (3.2) implies that we have connected preference topologies. Further, it implies that, for each experiment π, preferences are continuous w.r.t. the product topology on the π-related gambles. The same assumptions were made by LM.

At this stage, V_π need not represent $\succcurlyeq$ on all of $\mathcal{G}$ because it may not compare gambles from different experiments properly (Example 5.1). Adding a condition to make V_π representing on all of $\mathcal{G}$ is the main purpose of this paper. The condition will entail that a preference midpoint operation on X, derived from preferences over nondegenerate gambles, is consistent across different experiments.

It is easy to see that we can choose any real constants $\tau_1, \ldots, \tau_{n(\pi)}$ in Eq. (3.2), and any positive $\sigma > 0$, and then replace every $V_j(\pi, .)$ by $\tau_j + \sigma V_j(\pi, .)$ without affecting the representation of preferences. It can also be proved that this is the only freedom we have for this representation, so that the functions $V_j(\pi, .)$ are unique up to level and joint unit (Wakker, 1993). The functions $V_1(\pi, .), \ldots, V_{n(\pi)}(\pi, .)$ are *joint interval scales*. The function $V_\pi(x)$ is an interval scale, being unique up to the level $\tau = \tau_1 + \cdots + \tau_{n(\pi)}$ and the unit σ. We further assume

$$\text{\textit{strong monotonicity}: } (\pi, x_1, \ldots, x_{n(\pi)}) \succcurlyeq (\pi, y_1, \ldots, y_{n(\pi)}) \text{ whenever } x_j \succcurlyeq y_j \text{ for all } j, \text{ where the former preference is strict if at least one of the latter is.} \tag{3.3}$$

In Luce et al.'s approach, strong monotonicity implies that null events are suppressed. By comparing gambles $(\pi, \alpha, \beta, \ldots, \beta)$ and $(\pi, \beta, \beta, \ldots, \beta)$, it easily follows that every function $V_1(\pi, .)$ represents $\succcurlyeq$ on X. It can similarly be demonstrated that every function $V_j(\pi, .)$ represents $\succcurlyeq$ on X. A crucial assumption in our analysis is:

$$\text{For each } x \in \mathcal{G} \text{ there exists a \textit{certainty equivalent} } \alpha \in X, \text{ defined by } \alpha \sim x. \tag{3.4}$$

Idempotence, requiring that $(\pi, \alpha, \ldots, \alpha) \sim \alpha$, is a natural assumption in some applications such as in Example 2.1, but in others it is not. It was not assumed by LM and we, therefore, do not assume it either. For example, if experiments specify mutually exclusive but not necessarily exhaustive events as in Luce's approach, and $(\pi, x_1, \ldots, x_{n(\pi)})$ designates a gamble conditional on the information or decision (these are both possible in Luce's model) that the event occurring is an element of the experiment π, then idempotence is a natural condition. If, however, $(\pi, x_1, \ldots, x_{n(\pi)})$ designates a gamble with the implicit assumption that the consequence received is 0 for all events not contained in π, then idempotence is not a natural assumption. Then the certainty equivalent of $(\pi, \alpha, \ldots, \alpha)$ will be a mix of α and 0. Also in non-averaging models such as in Birnbaum (1974) and Blackorby et al. (2001), idempotence need not hold: if $V_j(\pi, x_j) = \lambda_j x_j$ for each j in Eq. (3.2) with the λ_j's summing to 3, then the certainty equivalent of $(\pi, \alpha, \ldots, \alpha)$ will be 3α rather than α.

For each gamble x we can choose a certainty equivalent α and, for one arbitrarily chosen experiment π', define for instance $U(x) = V_1(\pi', \alpha)$, obtaining a function that represents $\succcurlyeq$ on $\mathcal{G}$. Since each V_π represents $\succcurlyeq$ on the π-related gambles, each V_π is a strictly increasing transformation $L_\pi(U(x))$ of U on the set of π-related gambles. We will assume that the aforementioned functions are continuously related in the sense of the following equation, which summarizes the assumptions just made.

$$\text{There exists a function } U: \mathcal{G} \to \mathbb{R} \text{ that represents } \succcurlyeq; \text{ the image of } U \text{ is a nondegenerate interval. For each experiment } \pi, \text{ there exists a continuous strictly increasing function } L_\pi \text{ such that } V_\pi(x) = L_\pi(U(x)) \text{ for all } \pi\text{-related } x. \tag{3.5}$$

In LM's analysis, Eq. (3.5) follows from other assumptions. For brevity, we state it directly. Eq. (3.5) implies weak ordering of Eq. (3.1), which is why we need not state the latter in our theorems.

For each consequence $\alpha \in X$ and experiment π for which there exists a π-related x such that $\alpha \sim x$, we write $V_\pi(\alpha) = V_\pi(x)$. Thus, we have extended the domain of V_π to part of X. It follows that

Eq. (3.5) continues to hold for the extension ofV_π

$$(V_\pi(\alpha) = L_\pi(U(\alpha)) \quad \text{for all relevant } \alpha).$$

The following assumption is satisfied under idempotence, but has to be added for more general cases. We also assume:

$$\text{There exists a consequence } \alpha^0 \text{ such that for each } \pi \text{ there is a } \pi\text{-related } x, \text{ nonmaximal in the } \pi\text{-related gambles, with } x \sim \alpha^0. \tag{3.6}$$

This assumption avoids cases of degeneracy and cases of different π's having no overlapping indifference classes so that their representations would be unrelated. In the papers by Luce et al., α^0 can be "no change with respect to the status quo", but α^0 can also designate other consequences. We allow α^0 to be minimal but not maximal.

4. A preference foundation of rank-additive utility

We are interested in the special case of Eq. (3.5) where the ordinal transformations L_π can be dropped:

Definition 4.1. *Rank-additive utility* (*RAU*) holds if all functions L_π in Eq. (3.5) are the identity, in which case we have for each π-related x,

$$U(x) = V_1(\pi, x_1) + \cdots + V_{n(\pi)}(\pi, x_{n(\pi)}) = V_\pi(x). \quad \square \tag{4.1}$$

We provide a preference foundation of RAU. It will imply that all functions V_π coincide on common subdomains of X.

Our preference foundation will be based on a variation of the tradeoff technique of Köbberling and Wakker (2003, 2004). A natural way to obtain a preference foundation for a decision model arises from requiring consistency of elicitations of its subjective quantities.[1] More precisely, one develops deterministic parameter-free ways to elicit these subjective quantities, and then excludes contradictions in those measurements. We will next explain how U in Eq. (4.1) can be elicited from preferences. A preparatory notation: For $x = (\pi, x_1, \ldots, x_{n(\pi)})$, and $\mu \in X$, $\mu_i x$ denotes $(\pi, x_1, \ldots, x_{n(\pi)})$ with x_i replaced by μ; it is implicit in this notation that $i \leq n(\pi)$. It is also implicit in this notation that the replacement respects rank-ordering, so that $x_{i-1} \succcurlyeq \mu \succcurlyeq x_{i+1}$. The following notation will use the tradeoff technique to observe utility midpoints.

We write $\alpha \ominus \beta \sim^* \beta \ominus \gamma$, or $\alpha\beta \sim^* \beta\gamma$ for short, if

$$\begin{aligned} &\alpha \sim \mu_i x, && \beta \sim \mu_i y, \\ &\beta \sim \nu_i x, && \gamma \sim \nu_i y. \end{aligned} \tag{4.2}$$

for consequences μ and ν, an index i, and all gambles $\mu_i x$, $\mu_i y$, $\nu_i x$, and $\nu_i y$ related to the same π. Intuitively, the left two indifferences suggest that the value difference between α and β is matched by the change from μ to ν on coordinate i, and the right two indifferences suggest the same for the value difference between β to γ. These observations suggest that β is the value midpoint between α and γ. The proof of Lemma 4.2 further illustrates these suggestions.

[1] In a subjective expected utility, such subjective quantities are probabilities and utilities. For the RAU model they concern the various functions in Eq. (4.1).

In the notation $\alpha\beta \sim^* \beta\gamma$ we deliberately "forget" the experiment π. The main point of the following discussion in fact amounts to establishing that this notation with π forgotten is useful. The next lemma and the subsequent discussion show how V_π can be measured from $\sim^*$ observations and, hence, how U on X can be measured if RAU holds so that U agrees with V_π.

Lemma 4.2. *Assume $\alpha\beta \sim^* \beta\gamma$ with π as specified following Eq. (4.2). Then $V_\pi(\alpha) - V_\pi(\beta) = V_\pi(\beta) - V_\pi(\gamma)$.*

Proof. By Eqs. (3.2) and (4.2),

$$V_\pi(\alpha) - V_\pi(\beta) = V_\pi(\mu_i x) - V_\pi(\nu_i x) = V_i(\pi, \mu) - V_i(\pi, \nu).$$
$$V_\pi(\beta) - V_\pi(\gamma) = V_\pi(\mu_i y) - V_\pi(\nu_i y) = V_i(\pi, \mu) - V_i(\pi, \nu).$$

As a result of the same right-hand sides, the lemma follows. □

For the measurement of continuous monotonic interval scales on interval domains all that we need to observe is midpoints, so that the observations of $\sim^*$ can capture the essential characteristics of U. This is demonstrated in the following example.

Example 4.3 (*Efficiently Measuring U under RAU Using the $\sim^*$ Relation*). Assume that RAU holds. We scale $U(\alpha^0) = 0$ and $U(\alpha^1) = 1$ for some arbitrary $\alpha^1 \succ \alpha^0$. Then a number of elicitations $\alpha^{z+1}\alpha^z \sim^* \alpha^z\alpha^{z-1}$ reveals $U(\alpha^z) = z$ for all integers z considered. Here each α^z is a midpoint between α^{z+1} and α^{z-1}. For example, the $m-1 \sim^*$ relations needed to reveal $U(\alpha^z) = z$ for $z = 0, \ldots, m$ can be obtained as follows.

First, $\alpha^1 \succ \alpha^0$, π, μ^0, and i are chosen for convenience by the experimenter. All following prospects should be π-related. y and x are elicited from a subject to generate the two indifferences

$$\alpha^1 \sim \mu_i^0 x, \quad \alpha^0 \sim \mu_i^0 y. \tag{4.3}$$

Here x and y, whose ith coordinates are immaterial, serve as a kind of gauge to calibrate the preference difference between α^1 and α^0. We construct another gauge for this preference difference on the ith coordinate by finding μ^1 to yield the right indifference below

$$\alpha^2 \sim \mu_i^1 x, \quad \alpha^1 \sim \mu_i^1 y. \tag{4.4}$$

We then find α^2 to satisfy the left indifference above, implying (cf. Eq. (4.2) with Eq. (4.4) put above Eq. (4.3)) $\alpha^2\alpha^1 \sim^* \alpha^1\alpha^0$. We next find μ^2 to generate the right indifference below and then α^3 to generate the left indifference.

$$\alpha^3 \sim \mu_i^2 x, \quad \alpha^2 \sim \mu_i^2 y. \tag{4.5}$$

Putting Eq. (4.5) above Eq. (4.4) and comparing to Eq. (4.2), we get $\alpha^3\alpha^2 \sim^* \alpha^2\alpha^1$. We continue until we elicited

$$\alpha^j \sim \mu_i^{j-1} x, \quad \alpha^{j-1} \sim \mu_i^{j-1} y \tag{4.6}$$

for some value $j = m$ to conclude that $U(\alpha^z) = z$ for $z = 0, \ldots, m$. All prospects should be π-related, so that for instance $\mu^j \preccurlyeq x_{i-1}$ and $\mu^j \preccurlyeq y_{i-1}$ for all j. If such μ^j cannot be found because x_{i-1} or y_{i-1} is too low in preference, then the process must stop or different starting values must be chosen.

More refined measurements result from a number of elicitations $\beta^{z+1}\beta^z \sim^* \beta^z\beta^{z-1}$ with $\beta^0 = \alpha^0$ and $\beta^m = \alpha^1$, which implies that $U(\beta^z) = z/m$ for all integers z. Such β's exist because of continuity. □

RAU is obviously violated if measurements of U run into contradictions. If, for example, one experiment π were to reveal $\alpha\beta \sim^* \beta\gamma$, and another experiment π' were to reveal $\alpha'\beta \sim^* \beta\gamma$ for an $\alpha' \succ \alpha$, then the implied $U(\alpha') - U(\beta) = U(\beta) - U(\gamma) = U(\alpha) - U(\beta)$ contradicts $U(\alpha') > U(\alpha)$, and RAU would be violated (in a deterministic model).[2] A necessary condition for RAU is, consequently, that such violations be excluded. Similarly, we should not be able to improve one of β or γ above without breaking the relationship.

Definition 4.4. *RAU-tradeoff consistency* (or, briefly, *tradeoff consistency*) holds if strictly improving α, β, or γ in any relationship $\alpha\beta \sim^* \beta\gamma$ breaks that relationship. □

Tradeoff consistency implies that standard sequences such as the α^j and β^j above will be consistent across different experiments π. It is similar to the standard sequence invariance condition of Krantz et al. (1971, Section 6.11.2). The following example illustrates the empirical testing of this condition.

Example 4.5. Assume that an experimenter has observed indifferences as in Eqs. (4.3)–(4.6). So as to do cross-checking (von Winterfeldt & Edwards, 1986) of the measurements, he next observes two indifferences

$$\alpha'^1 \sim \mu_k'^0 x', \quad \alpha'^0 \sim \mu_k'^0 y' \tag{4.7}$$

where these gambles are π' related. Here $\alpha'^0 = \alpha^0, \alpha'^1 = \alpha^1$, and π', μ'^0, and k are chosen for convenience by the experimenter and may be different from π, μ^0, and i in Eq. (4.3). Next y' and x' are elicited from a subject to generate the two indifferences. From there on, the experimenter elicits, for each j, μ'^j, and then α'^{j+1} such that

$$\alpha'^{j+1} \sim \mu_k'^j x', \quad \alpha'^j \sim \mu_k'^j y'. \tag{4.8}$$

By tradeoff consistency, $\alpha'^j \sim \alpha^j$ should hold for all j. The first j for which the equality is violated (if that were to happen), implies that $\alpha'^j\alpha^{j-1} \sim^* \alpha^{j-1}\alpha^{j-2}$ which, together with $\alpha^j\alpha^{j-1} \sim^* \alpha^{j-1}\alpha^{j-2}$ as established before, entails a violation of tradeoff consistency. □

In the above example, the empirical measurement of RAU and its axiomatic testing went hand in hand. The following theorem shows that tradeoff consistency in a way entails a critical test of RAU. That is, whenever RAU is violated, it should be possible to observe this violation through violations of tradeoff consistency, given the other assumptions.

Theorem 4.6. *Assume Eqs. (3.2)–(3.6). Then RAU holds if and only if tradeoff consistency holds.* □

A similar result holds if we have additive representations not only on rank-ordered sets but on full product sets. All definitions of Section 2 are readily extended to this case, with the simplification that rank-ordering constraints can be dropped.

Theorem 4.7. *Let the domain of preference consist of X and all π-related tuples of the form $x = (\pi, x_1, \ldots, x_{n(\pi)})$ without the restriction that $x_1 \succcurlyeq \cdots \succcurlyeq x_{n(\pi)}$. Assume Eqs. (3.2)–(3.6). Then the overall representation in Eq. (4.1) holds if and only if tradeoff consistency holds.* □

5. Well-known special cases of RAU

The most popular descriptive models in the literature on risk and uncertainty today are the rank-dependent models, also known as Choquet expected utility. They were initiated by Gilboa (1987) and Schmeidler (1989), were discovered independently by Luce (1988), and were adopted by Luce and Fishburn

[2] The measurements underling the observations of $\sim^*$, with the "forgetting" of π, have then served to falsify RAU, and they obviously cannot be used to measure utility differences as in Lemma 4.2.

(1991) and Tversky and Kahneman (1992), who added a sign-dependent generalization. These models are also popular for welfare evaluations (Ebert (2004); Weymark (1981), Theorem 3).

Chew and Wakker (1996), henceforth CW, provided a common generalization of the aforementioned models. Let us, for simplicity, only consider a finite set $S = \{s_1, \ldots, s_n\}$ $(n \geq 3)$ called *state space*. The experiments that CW considered are permutations of S, all having n elements, and sets of π-related gambles are called comonotonic cones. CW assume the classical model of Example 2.1. Thus, with $\pi = (s_1, s_2, s_3)$ and $\pi' = (s_2, s_1, s_3)$, the gamble $(\pi, \alpha, \alpha, \beta)$ is identical to the gamble $(\pi', \alpha, \alpha, \beta)$. CW used Wakker (1993) to obtain Eq. (3.2) within comonotonic cones.

CW did not need an extra condition such as RAU tradeoff consistency to obtain an overall representation on the union of all comonotonic cones because this result is automatically implied by the other conditions. To see this point, consider the two-dimensional rank-ordered product set containing all gambles of the form $(\pi, \alpha, \alpha, \beta)$. We have two additive representations for preferences on this set: $V_1(\pi, \alpha) + V_2(\pi, \alpha) + V_3(\pi, \beta)$ and $V_1(\pi', \alpha) + V_2(\pi', \alpha) + V_3(\pi', \beta)$. After common normalization, common uniqueness results implied by CS's assumptions imply that these two additive representations must coincide, which implies in particular that $V_3(\pi, \beta) = V_3(\pi', \beta)$. By further reasonings of this kind, using the overlaps of different "adjacent" comonotonic cones, it can be demonstrated that all additive representations within comonotonic cones must coincide on common domain and that they constitute one overall function that, in addition, is representing on the union of the comonotonic cones. This automatically implies RAU tradeoff consistency.

The model of CW generalizes Green and Jullien's (1988) model and Segal's (1989) measure-model from risk to uncertainty. It also generalizes sign-dependence. The representation theorems of all aforementioned models follow the pattern of that in CW. The models of Luce et al. generalize the above models so much that sufficiently rich overlaps between the gamble sets related to different experiments, needed for CW's proof, need no longer be available. Then RAU need not hold, as illustrated by the following Example 5.1. By addition of tradeoff consistency, our paper still obtained overall RAU representations.

Example 5.1. Assume $S = \{s_1, s_2, s_3\}$ and $X = \mathbb{R}^+$. Experiments are ordered permutations of S. Preferences over the union of all gambles considered are represented by a function $U : \mathcal{G} \to \mathbb{R}$ defined as follows. On X, U is the identity. If $\pi = (s_1, s_2, s_3)$ then $U(\pi, (\alpha, \beta, \gamma)) = (((\sqrt{\alpha}) + (\sqrt{\beta}) + (\sqrt{\gamma}))/3)^2$. The latter function can be seen to be the certainty equivalent of a prospect yielding α, β, or γ, each with probability 1/3, under expected utility with a concave risk-averse square-root utility function. For all other π, $U(\pi, (\alpha, \beta, \gamma)) = (\alpha + \beta + \gamma)/3$. Eq. (3.2) is satisfied for $\pi = (s_1, s_2, s_3)$ and also for all other π. Idempotence is also satisfied: $(\pi, (\alpha, \alpha, \alpha)) \sim \alpha$ for all π. $U((s_2, s_1, s_3), (6, 6, 0)) = 4 > U((s_1, s_2, s_3), (6, 6, 0))$ shows that the two gambles are not identical and that the classical model of Example 2.1 does not hold.

RAU does not hold because tradeoff consistency is violated, as we now demonstrate. We use bold printing to indicate the role of μ and ν of Eq. (4.2). The indifferences

$$4 \sim ((s_2, s_1, s_3), (9, \mathbf{3}, 0)); \quad 3 \sim ((s_2, s_1, s_3), (6, \mathbf{3}, 0)) \quad \text{and}$$
$$5 \sim ((s_2, s_1, s_3), (9, \mathbf{6}, 0)); \quad 4 \sim ((s_2, s_1, s_3), (6, \mathbf{6}, 0))$$

imply $5 \ominus 4 \sim^* 4 \ominus 3$. However, writing $\alpha = (6 - 3\sqrt{3})^2$, we have

$$4 \sim ((s_1, s_2, s_3), (36, \mathbf{0}, 0)); \quad 3 \sim ((s_1, s_2, s_3), (27, \mathbf{0}, 0)) \quad \text{and}$$
$$5.14 \sim ((s_1, s_2, s_3), (36, \boldsymbol{\alpha}, 0)); \quad 4 \sim ((s_1, s_2, s_3), (27, \boldsymbol{\alpha}, 0)),$$

which implies $5.14 \ominus 4 \sim^* 4 \ominus 3$. Tradeoff consistency is violated because strictly improving 5 in the relationship $5 \ominus 4 \sim^* 4 \ominus 3$ into 5.14 did not break the relationship. RAU cannot hold. □

6. Proof of Theorem 4.6

We demonstrated before that tradeoff consistency is a necessary condition for the RAU model. Henceforth, we assume the condition and demonstrate that the RAU model is implied. Since the V_π's are interval scales, it will suffice to reduce the L_π functions to strictly increasing affine functions. We can then turn the L_π's into the identity by appropriately rescaling the V_π's.

For every experiment π, define X_π as the set of consequences $\{\alpha \in X$: there exists a π-related gamble x with $x \sim \alpha\}$. In other words, X_π is the domain of V_π (in its extended sense) intersected with X. Since the ranges of all functions are intervals, X_π is a preference interval in the sense that if it contains two consequences, then it contains all consequences in between. α^0 is contained in each X_π. For every experiment, we can choose the levels of the representations such that $V_\pi(\alpha^0) = 0 = V_j(\pi, \alpha^0)$ for all j because the representations in Eq. (3.2) are joint interval scales, and so we do. Therefore, from now on these functions are *ratio scales*, meaning they are unique up to a unit.

We will now construct U. Take any fixed experiment π_f. For each $\alpha \in X_{\pi_f}$, define $U(\alpha) = V_{\pi_f}(\alpha)$. Consider an arbitrary other experiment π. Since X_π and X_{π_f} both contain α^0, both contain a strictly preferred consequence, and both are preference intervals, there is a consequence $\alpha_\pi \succ \alpha^0$ contained in both sets. Since V_π is a ratio scale, we can choose its unit such that $V_\pi(\alpha_\pi) = V_{\pi_f}(\alpha_\pi)$ and so we do for each experiment π.

We now compare two experiments π and π'. As will be demonstrated in Appendix B, for each consequence λ in $X_\pi \cap X_{\pi'}$ that is neither minimal nor maximal in this set, we can find $\sigma \succ \lambda \succ \tau$ sufficiently close to λ to imply that, for all consequences α, β, γ with $\sigma \succcurlyeq \alpha \succcurlyeq \beta \succcurlyeq \gamma \succcurlyeq \tau$ and $V_\pi(\alpha) - V_\pi(\beta) = V_\pi(\beta) - V_\pi(\gamma)$ we have

$$\alpha \sim \mu_i x, \beta \sim \mu_i y,$$
$$\beta \sim \nu_i x, \gamma \sim \nu_i y \tag{6.1}$$

for properly chosen x, y, i, μ,ν with $\mu_i x$, $\mu_i y$, $\nu_i x$, $\nu_i y$ all π-related. In other words, using Lemma 4.2, in a neighborhood of λ, $[V_\pi(\alpha) - V_\pi(\beta) = V_\pi(\beta) - V_\pi(\gamma) \Leftrightarrow \alpha\beta \sim^* \beta\gamma]$ and the $\sim^*$ relation is powerful enough to detect all V_π midpoints.

We can have an analog of Eq. (6.1) for π' instead of π with $\sigma' \succ \lambda \succ \tau'$ sufficiently close to λ, and with different μ', i', x', and so on in an analogous role. Instead of σ and σ' we can take their minimum, and instead of τ and τ' we can take their maximum. That is, we can take $\sigma = \sigma'$ and $\tau = \tau'$. Then, by tradeoff consistency, for all α, β, and γ between σ and τ, if β is a V_π-midpoint of α and γ, it must also be a $V_{\pi'}$ midpoint. (Sets of midpoints are, obviously, $\sim$ indifference classes.)

V_π and $V_{\pi'}$ are interval scales such that for each nonmaximal and nonminimal element in their common domain within X there is an open preference-neighborhood within which they have the same midpoints and, hence, the same standard sequences. Consequently, the strictly increasing transformation that relates V_π and $V_{\pi'}$ on their common connected domain must be affine, which by continuity extends to the maximal and minimal consequences in their common domain. Since V_π and $V_{\pi'}$ coincide with V_f at α^0 and at points strictly preferred to but close to α^0, they agree with each other at two or more points, so that they must be identical on their common domain. In this manner, all functions V_π coincide on their common domains, and they can be written as one function U, first only on X but then, through Eq. (3.4), on all of $\mathcal{G}$. This function U obviously represents preference on X and, hence, on $\mathcal{G}$.

7. Proof of Theorem 4.7

The proof of Theorem 4.7 is virtually identical to that of Theorem 4.6 in the preceding section. The proof in Appendix B, needed there, also remains valid, although it could be simplified if only Theorem 4.7 had to be proved.

Acknowledgment

Duncan Luce, Tony Marley, and an anonymous referee made many useful comments.

Appendix A. Notational differences with Luce and Marley (2005)

LM assume a multistage setup where gambles can serve as consequences for other gambles, and then assume backward induction (folding back). This entails that a gamble, if serving as a consequence of a multistage gamble, can be replaced by its certainty equivalent without affecting the preference value of the multistage gamble. For our analysis these assumptions are not needed. Hence, to achieve greater generality, we do not commit to these assumptions and allow consequences to be general. Whereas LM often need not distinguish between consequences and gambles, we have maintained such distinctions. For this reason, we cannot use x for both gamble and consequence, so that we use Greek letters or indexed letters such as x_i for consequences. In the same spirit, LM can define some functions on gambles/consequences as transformations of the U value of the gambles/consequences where they need not distinguish between gambles and consequences, something that we cannot do.

For the results relevant to this paper, LM restrict attention to gains, that is, all consequences are weakly preferred to a minimal consequence e that reflects "no change with respect to the status quo". Our analysis does not need the presence of a neutral and/or minimal consequence e. We let functions be 0 at α^0 only for convenience, and α^0 can but need not play the role of neutral element, as it can but need not be minimal. LM also assume that there does not exist a maximal consequence (end of Section 1.1). We do not need this assumption.

As a result of the central role of variable experiments in our paper, and their general nature, we used the simpler notation π instead of LM's $(\vec{C}_n)$. Accordingly, whereas LM usually denote gambles as $(x_1, C_1; \ldots; x_n, C_n)$, we use the notation $(\pi, x_1, \ldots, x_{n(\pi)})$, with $\pi = (C_1, \ldots, C_n)$. Similarly, we often use L_π with subscript π to express dependency on the experiment considered. Thus, we prefer not to use the notaion L_i of LM (2005,Section 2.1), and we use V_i instead.

Several papers in mathematical psychology use solvability and an Archimedean axiom instead of our continuity (Krantz et al., 1971). Wakker (1988) argued that, in general, solvability and Archimedeanity axioms are preferable to continuity axioms. In the present context, however, LM assume that the images of representing functions are intervals (LM, end of Section 1.1) and so do we. Then continuity with respect to the order topology is no more a restrictive assumption and it is more conventional, which is why we have used it.

Appendix B. Derivation of Eq. (6.1)

Let λ in $X_\pi \cap X_{\pi'}$ be neither maximal nor minimal in this set. We write n for $n(\pi)$. In this proof, we write $\alpha\beta x$ for x with x_1 replaced by α and x_n replaced by β. For $\alpha\beta x$ to be contained in X_π, besides $x \in X_\pi$ also $\alpha \succcurlyeq x_2$ and $\beta \preccurlyeq x_{n-1}$ are required. We next demonstrate that we can obtain the following equation, with all gambles contained in X or π-related.

$$\begin{aligned}&\lambda \sim s \sim r = r_1 r_n s \quad \text{for } s \in X_\pi \quad \text{with } s_1 = \cdots = s_n\\ &\quad\text{and with } r_1 \succ s_1 \quad \text{and} \quad r_n \prec s_n.\end{aligned} \tag{B.1}$$

Here, s abbreviates safe and r abbreviates risky. The indifference $s \sim r = r_1 r_n s$ in Eq. (B.1) suggests that the preference difference between r_1 and s_1 is matched by that between s_n and r_n. To demonstrate that Eq. (B.1) can be obtained, we first construct $s \sim \lambda$. In the absence of idempotence, the s_j's may differ from λ.

Since $\lambda \in X_\pi$ there exists a π-related s' with $s' \sim \lambda$. We can replace s' by s with $s_1 = \cdots = s_n$ with this consequence being between s'_1 and s'_n. Since λ is neither maximal nor minimal in X_π, the s_j's are neither maximal nor minimal in X. Thus, we have constructed the desired s. We next construct r.

Take any $s'_n \prec s_n$ and any $s'_1 \succ s_1$. If $s'_1 s'_n s \succcurlyeq \lambda$ then we can find $s'_1 \succcurlyeq s''_1 \succ s_1$ such that $s''_1 s'_n s \sim \lambda$, and we define $r = s''_1 s'_n s = r_1 r_n s$. If $s'_1 s'_n s \prec \lambda$ then we can find $s'_n \prec s''_n \prec s_n$ such that $s'_1 s''_n s \sim \lambda$ and we define $r = s'_1 s''_n s = r_1 r_n s$. Eq. (B.1) has been established.

We define certainty equivalents σ and τ as follows.

$$\begin{aligned}&\sigma \sim r_1 s_n s, \quad \lambda \sim s_1 s_n s,\\ &\lambda \sim r_1 r_n s, \quad \tau \sim s_1 r_n s\end{aligned} \tag{B.2}$$

implying $\sigma\lambda \sim^* \lambda\tau$. These σ and τ are as desired. To demonstrate this, assume $\sigma \succcurlyeq \alpha \succ \beta \succ \gamma \succcurlyeq \tau$ with β the V_π midpoint between α and γ. We will arrange

$$\begin{aligned}&\beta \sim s' \sim r' = r'_1 r'_n s' \quad \text{for } s' \in X_\pi \quad \text{with } s'_1 = \cdots = s'_n\\ &\quad\text{and with } r'_1 \succ s'_1 \quad \text{and} \quad r'_n \prec s'_n\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}&\alpha \sim r'_1 s'_n s', \quad \beta \sim s'_1 s'_n s',\\ &\beta \sim r'_1 r'_n s', \quad \gamma \sim s'_1 r'_n s'\end{aligned} \tag{B.4}$$

where all gambles are contained in X or are π-related. Since $(\pi, r_1, \ldots, r_1) \succcurlyeq \beta \succcurlyeq (\pi, r_n, \ldots, r_n)$ we can indeed find $s' \sim \beta$ with $s'_1 = \cdots = s'_n$. First assume that $s' \preccurlyeq s$. Then $r_n s' \preccurlyeq r_n s \sim \tau \preccurlyeq \gamma \prec s'$ so that we can find $r'_n \prec s'_n$ such that $r'_n s' \sim \gamma$. Since α and γ are closer to each other than σ and τ, and λ and β are their preference midpoints, we have $V_\pi(\alpha) - V_\pi(\beta) \le V_\pi(\sigma) - V_\pi(\lambda) = V_\pi(r_1 s) - V_\pi(s_1 s) = V_1(\pi, r_1) - V_1(\pi, s_1) \le V_1(\pi, r_1) - V_1(\pi, s'_1) = V_\pi(r_1 s') - V_\pi(s'_1 s') = V_\pi(r_1 s') - V_\pi(\beta)$ and, hence, there exists a r'_1 with $r_1 \succcurlyeq r'_1 \succ s'_1$ and $r'_1 s' \sim \alpha$. Since $V_1(\pi, r'_1) - V_1(\pi, s'_1) = V_\pi(r'_1 s') - V_\pi(s'_1 s') = V_\pi(\alpha) - V_\pi(\beta) = V_\pi(\beta) - V_\pi(\gamma) = V_\pi(s'_n s') - V_\pi(r'_n s') = V_n(\pi, s'_n) - V_n(\pi, r'_n)$, we can also conclude that $r'_1 r'_n s' \sim s' \sim \beta$. All indifferences in Eqs. (B.3) and (B.4) have been established.

A symmetric reasoning applies to the case of $s' \succcurlyeq s$. Then $r_1 s' \succcurlyeq r_1 s \sim \sigma \succcurlyeq \alpha \succcurlyeq s'$ so that we can find r'_1 such that $r'_1 s' \sim \alpha$. Since $V_\pi(\beta) - V_\pi(\gamma) \le V_\pi(\lambda) - V_\pi(\tau) = V_\pi(s_n s) - V_\pi(r_n s) = V_n(\pi, s_n) - V_n(\pi, r_n) \le V_n(\pi, s'_n) - V_1(\pi, r_n) = V_\pi(s'_n s') - V_\pi(r_n s') = V_\pi(\beta) - V_\pi(r_n s')$, there exists a r'_n with $r_n \preccurlyeq r'_n \preccurlyeq s'_n$ and $r'_n s' \sim \gamma$. Since $V_n(\pi, s'_n) - V_n(\pi, r'_n) = V_\pi(\beta) - V_\pi(\gamma) = V_\pi(\alpha) - V_\pi(\beta) = V_1(\pi, r'_1) - V_1(\pi, s'_1)$, we can also conclude that $r'_1 r'_n s' \sim s' \sim \beta$. All indifferences in Eqs. (B.3) and (B.4) have been established again. Thus, Eq. (6.1) and $\alpha\beta \sim^* \beta\gamma$ can always be established.

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