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Reference-dependent utility with shifting reference points and incomplete preferences

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Abstract

Many empirical studies have shown that people's preferences are reference-dependent. Previous theoretical studies of reference-dependence assumed that the reference point was fixed and then imposed the usual assumptions of decision theory, in particular completeness of preferences. This paper gives preference foundations for additive reference-dependent utility when the reference point varies across decisions and is one of the options in the decision maker's opportunity set. This decision situation is common, for example because usually the retention of the status quo is an available option, but is difficult to handle axiomatically because it implies incompleteness of preferences. The results of this paper provide tools to extend existing theories of reference-dependent preferences, such as prospect theory, to new and empirically important decision contexts.

Keywords: Reference-dependence; Incompleteness; Additive utility; Tradeoff consistency; Prospect theory

1. Introduction

Many empirical studies have shown that people's preferences are *reference-dependent*, i.e. preferences over final outcomes depend on the reference point from which they are judged. Such studies include Kahneman and Tversky (1979) for reference-dependence in decision under risk, Kahneman, Knetsch, and Thaler (1990) for choice among commodity bundles, Loewenstein and Prelec (1992) for intertemporal choice, Dolan and Robinson (2001) for welfare theory, Bateman, Munro, Rhodes, Starmer, and Sugden (1997) for contingent valuation, and Bleichrodt and Pinto (2002) for multiattribute utility. To explain the commonly observed preference patterns, the reference point needs to shift across decision situations (Wakker, 2005, Observation 4.4 and Theorem 4.5). If the reference point is fixed then a simple rescaling of utility makes reference-dependent utility equivalent to the standard theory of choice. Tversky and Kahneman (1991) analyze how shifts in the reference point can account for violations of standard models of consumer theory and Bateman et al. (1997) show how shifts in the reference point can explain the difference between willingness to pay and willingness to accept valuations.

The following example from medical decision making illustrates the importance of shifts in the reference point to explain observed empirical data. A widely used method to measure the utility of impaired health is to ask people how much of their remaining life duration they are willing to give up for an improvement in health. Suppose that a patient has 40 more years to live with rheumatoid arthritis and that we ask this patient how many years in full health he considers equivalent to 40 years with rheumatoid arthritis. Let the patient's answer be 30 years. Several studies have observed that if we tell this patient instead that he has 30 more years to live in full health and ask him how many years with rheumatoid arthritis he considers equivalent to 30 years in full health, the patient typically states a number exceeding 40 years (e.g. Bleichrodt, Pinto, & Abellan-Perpiñan, 2003; Spencer, 2003). When the reference point is fixed such preferences cannot occur because transitivity of the reference-dependent preference relation implies that in both decisions we should observe the same indifference. However, when the reference point shifts from 40 years with rheumatoid arthritis in the first

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decision to 30 years in full health in the second decision then such preferences can be accommodated by loss aversion.

Previous theoretical analyses of reference-dependence took the reference point as fixed and then applied the common axioms of decision theory, such as transitivity and completeness, to the reference-dependent preference relation (Köbberling & Wakker, 2003; Luce & Fishburn, 1991; Sugden, 2003; Tversky & Kahneman, 1991, 1992; Wakker & Tversky, 1993; Zank, 2001). This approach is somewhat unsatisfactory because, as explained above, reference-dependence can only explain the observed empirical regularities when the reference point shifts across decision contexts. A notable exception is Schmidt (2003), who analyzed shifts in the reference point. Schmidt (2003) also imposed the common assumptions of decision theory, in particular completeness of reference-dependent preferences.

Because reference-dependence is a relatively new concept it entails new difficulties for theoretical studies of reference-dependence. One of these difficulties, incompleteness of preferences, is the topic of the present paper. I will argue that completeness of preferences is often not plausible when preferences are reference-dependent and that new preference foundations need to be developed that allow for incomplete preferences. The problem of incompleteness arises when the reference point is always one of the available options in the decision maker's opportunity set. This often happens, because in many decision situations the reference point is the status quo and retention of the status quo is always possible. Sometimes a status quo is not readily available, for example when the choice is between two treatments for a particular disease both involving health states the decision maker is unfamiliar with. In such situations the decision maker often takes one of the alternatives as his reference point (Robinson, Loomes, & Jones-Lee, 2001).

One example where a decision maker takes one of the alternatives in his opportunity set as his reference point was given above for health utility measurement. Empirical evidence shows that when people are asked how much of their remaining life duration they are willing to give up for an improvement in their health, they handle such tasks by taking the alternative impaired health for the rest of their life as their reference point and by trading off the gain in health quality and the loss in life duration. As another illustration, Hershey and Schoemaker (1985) observed that when people compare a sure amount of money with a risky prospect, they take the sure amount as their reference point and evaluate the outcomes of the risky prospect as gains and losses relative to this sure amount (see also Bleichrodt, Pinto, & Wakker, 2001; Johnson & Schkade, 1989; Morrison, 2000; Robinson et al., 2001; Stalmeier & Bezembinder, 1999). Perhaps it is worth mentioning here that I do not claim that the reference point is in each decision context part of the decision maker's opportunity set. My point is that there are decision contexts of considerable interest for which this is the case and, hence, that this case is important to explore.

If the reference point is always in the decision maker's opportunity set then the preference relation can no longer be taken as complete. To see this point consider two alternatives x and y that are both worse than a reference point r, which is also in the decision maker's opportunity set. Then the decision maker will always choose r over x and y and, hence, will never choose between x and y given reference point r. For example, take a decision maker who considers a choice between jobs, where r is his current job in which he earns € 80K per year and has 20 min travel time per day, x is a job in which he earns € 50K per year and has 30 min travel time per day and y is a job in which he earns € 60K per year and has 60 min travel time per day. Because retaining his current job is an option for the decision maker he will always choose r over x and y and we cannot observe the decision maker's choice between x and y. The decision maker's choices are the primitive of utility theory and it is only by observing a decision maker's choices that his preferences can be inferred.[1] Consequently, a preference between x and y *judged from* r cannot be inferred and the reference-dependent preference relation with r as the reference point must be taken incomplete. Note that the inclusion of the reference alternative r in the decision maker's opportunity set is crucial in the above argument.

Because the available characterizations of reference-dependent utility models all take reference-dependent preferences complete, they do not cover the case where the reference point is always one of the elements in the decision maker's opportunity set. New preference foundations must be developed to cover this empirically important case and this is the topic of this paper. The paper presents a preference foundation for reference-dependent utility when the reference point can vary and is one of the alternatives in the decision maker's opportunity set. This means that I have to take reference-dependent preferences over alternatives as incomplete. I will present a preference foundation for a general additive model, which underlies all the reference-dependent utility models that have been proposed in the literature. By imposing additional conditions on this general model, preference foundations for more specific cases of reference-dependent utility can be given. Hence, this paper provides the tools to characterize, for example, Tversky and Kahneman's (1991) model of constant loss aversion in consumer theory and Loewenstein and Prelec's (1992) general hyperbolic discounting model in intertemporal choice and to extend prospect theory, currently the main descriptive theory of decision under uncertainty, to the case of reference-dependence considered here.

[1]Some authors distinguish between revealed and psychological preferences (e.g. Mandler, 2005). I do not make this distinction. As is common in economics and decision theory, this paper derives preferences from binary choice.

The type of incompleteness considered implies that an additive representation of binary preferences will be derived on a subset of a Cartesian product, being a subset where the preference relation is complete. A common misunderstanding in the literature is that additive representation theory on subsets of Cartesian products does not differ from additive representation theory on full Cartesian products. That this is not so, and that the restriction of representation theorems to subsets is usually complex, has been pointed out by Falmagne (1981), Fishburn (1970, p. 74), Krantz, Luce, Suppes, and Tversky (1971, p. 276), Shapiro (1979) and Wakker (1989, Remark III.7.8). Wakker (1993) cited many misunderstandings about the nontrivial nature of extensions to subsets from economics, psychology, operations research, and functional equations and, in a complex analysis, demonstrated how additive conjoint measurement can be extended to the special class of subsets that are comonotonic cones.[2] Even though such cones are well-behaved subsets, connected with full-dimensional interior, the extra complexities are already large there.

There has recently been a revived interest in incomplete preference relations (Dubra, Maccheroni, & Ok, 2004; Eliaz & Ok, 2006; Girotto & Holzer, 2005; Maccheroni, 2004; Mandler, 2005; Masatlioglu & Ok, 2005; Ok, 2002, earlier contributions are Aumann, 1962; Bewley, 1986; Vind, 1991). The type of incompleteness that these studies examined stemmed from indecisiveness, confusion, and lack of information of the decision maker. This paper provides another argument why studying incomplete preferences is important, namely reference-dependence. Reference-dependence requires a different form of incompleteness than studied in the abovementioned works. Consequently, the methods, results, and fields of application of the present paper are different. The paper that is closest to this paper is Vind (1991) who also studied general additive utility, but did not consider reference-dependence.

In what follows, Section 2 explains notation and assumptions, Section 3 gives the main results of this paper and Section 4 concludes. All proofs are in the Appendix.

2. Notation and assumptions

2.1. Notation

Let $n > 2$. Let $X_1, \ldots, X_n$ be nonempty sets. *Alternatives* are elements of $X = X_1 \times \cdots \times X_n$, and are denoted as $x = (x_1, \ldots, x_n)$. Examples of alternatives are acts in decision under uncertainty, commodity bundles in consumer theory, multiattribute outcomes, time streams, and income distributions. Let $\alpha_j x$ denote the alternative x with x_j replaced by $\alpha_j \in X_j$, and $\alpha_i \beta_j x$ the alternative x with x_i replaced by $\alpha_i \in X_i$ and x_j replaced by $\beta_j \in X_j$, $i \neq j$.

Let $r \in X$ denote a *reference alternative*. Each alternative can serve as a reference alternative. Let $\mathscr{F}$ be the collection of all nonempty finite subsets of 2^X, i.e. $\mathscr{F}$ includes all singleton sets and is union-closed. $\mathscr{F}$ may be interpreted as the set of all choice problems and any element A of $\mathscr{F}$ is an opportunity set. For every reference alternative r, we define $\mathscr{F}_r = \{A \in \mathscr{F} : r \in A\}$. A *choice function* c_r is a mapping from $\mathscr{F}_r$ to $\mathscr{F}$ such that for all $A \in \mathscr{F}_r$, $\emptyset \neq c_r(A) \subseteq A$. Given an opportunity set $A \in \mathscr{F}_r$, the choice function specifies the alternatives that the decision maker is willing to choose from A. We derive a preference relation $\succcurlyeq_r$ on X from c_r in the following manner, where $x \succcurlyeq_r y$ means "x is weakly preferred to y when judged from reference alternative r." We write

(1) $x \succcurlyeq_r y$ if there is a $A \in \mathscr{F}_r$ such that $x \in c_r(A)$ and $y \in A$.
(2) $x \succ_r y$ if there is a $A \in \mathscr{F}_r$ such that $x \in c_r(A)$ and $y \in A \backslash c_r(A)$.
(3) $x \sim_r y$ if there is a $A \in \mathscr{F}_r$ such that $x \in c_r(A)$ and $y \in c_r(A)$.

We assume that the choice functions c_r satisfy the *weak axiom of revealed preference*, i.e. for all alternatives x, y, if $x \succcurlyeq_r y$ then not $y \succ_r x$. This ensures that for all r the preference relation $\succcurlyeq_r$ is transitive and represents the choice function c_r: for all $A \in \mathscr{F}_r$, $c_r(A) = \{x \in A : x \succcurlyeq_r y$ for all $y \in A\}$ (Wakker, 1989, Theorem I.2.5). I will denote by $\preccurlyeq_r$ and $\prec_r$ the reversed binary relations.

2.2. Preference conditions

The preference relation $\succcurlyeq_r$ need not be complete. In particular, if two alternatives x and y are both strictly less preferred than r, then the preference between x and y judged from r cannot be observed, because in that case $c_r(\{x, y, r\}) = \{r\}$. On the other hand, if at least one of x and y is at least as good as r then a preference between x and y judged from r can be observed. Hence, I consider a special type of incompleteness: loosely speaking, "above" the reference alternative, preferences are complete, but "below" the reference alternative they do not exist. Let us now formalize the above discussion.

Definition 2.1. For a given alternative r, *r-upper completeness* holds if (i) $r \sim_r r$ and (ii) for all $x, y \in X$, if $x \succcurlyeq_r r$ or $y \succcurlyeq_r r$, then either $x \succcurlyeq_r y$ or $y \succcurlyeq_r x$; if $r \succ_r x$ and $r \succ_r y$ then neither $x \succcurlyeq_r y$ nor $y \succcurlyeq_r x$.

Completeness of preferences above the reference alternative may be too strong. Indecisiveness of the decision maker may, for example, lead to some incompleteness of preferences above the reference alternative. I do not consider such incompleteness. To handle it, tools like those in Ok (2002) and Dubra et al. (2004) may have to be combined with the tools presented in this paper.

For $r \in X$, let $B_r = \{(x, y) \in X \times X : x \succcurlyeq_r r$ or $y \succcurlyeq_r r\}$. That is, B_r is the set of pairs of alternatives for which, judged from r, a preference can be observed. Coordinate j is *essential with respect to r* if there exist $(\alpha_j x, x) \in B_r$ such that $\alpha_j x \succ_r x$. I will need the assumption that there exist at

[2] For a definition of comonotonic cones see Wakker (1993).

least three coordinates that are essential with respect to r. For convenience, I will assume throughout that for all r, all coordinates are essential with respect to r.

Suppose that the preference relations $\succcurlyeq_r$ satisfy *weak separability* (Wakker, 1989) i.e. for all alternatives x, y, v, w, r and for all coordinates j, if (x_jv, y_jv), $(x_jw, y_jw) \in B_r$ then $x_jv \succcurlyeq_r y_jv$ iff $x_jw \succcurlyeq_r y_jw$. In other words, if $x_jv \succcurlyeq_r y_jv$ then changing the $n-1$ common coordinates v_i into $w_i, i = 1, \ldots, j-1, j+1, \ldots, n$, such that $(x_jw, y_jw) \in B_r$ does not change reference-dependent preferences. On each X_j a weak order $\succcurlyeq_{j,r}$ can be defined. For all $j, \alpha_j, \beta_j \in X_j$, write $\alpha_j \succcurlyeq_{j,r} \beta_j$ if there exist alternatives v and r such that $\alpha_j v \succcurlyeq_r \beta_j v$. By weak separability the $\succcurlyeq_{j,r}$ relations do not depend on v. The $\succcurlyeq_{j,r}$ relations define preference relations on the coordinates. It will follow from solvability (see Definition 2.3) that the $\succcurlyeq_{j,r}$ relations are weak orders. It is assumed that the $\succcurlyeq_{j,r}$ relations are reference-independent: for all j, $\alpha_j, \beta_j \in X_j$ and for all reference alternatives r and r', $\alpha_j \succcurlyeq_{j,r} \beta_j$ iff $\alpha_j \succcurlyeq_{j,r'} \beta_j$. Hence, I will write $\succcurlyeq_j$ for the preference relations on the coordinates in what follows. For the $\succ_j$ relations the notations $\succcurlyeq_j$, etc. are used, similar to those for $\succcurlyeq_r$.

In our setup the $\succcurlyeq_j$ relations are derived from $\succcurlyeq_r$ by weak separability. An alternative approach would be to take the $\succcurlyeq_j$ relations as primitive and to require that the overall relation $\succcurlyeq_r$ satisfies monotonicity with respect to the $\succcurlyeq_j$'s. It can be seen that taking the $\succcurlyeq_j$ relations as primitive and imposing monotonicity is equivalent to deriving the $\succcurlyeq_j$ relations from $\succcurlyeq_r$ by weak separability as above.

Definition 2.2. *Weak monotonicity* holds if for all $(x, y) \in B_r$, $x_j \succcurlyeq_j y_j$ for all coordinates j implies $x \succcurlyeq_r y$.

In other words, if the pair (x, y) belongs to B_r and if alternative x gives for each coordinate an outcome that is at least as good as the outcome given by alternative y then weak monotonicity entails that x is at least as preferred as y when judged from r.

Definition 2.3. *Solvability* holds if for all alternatives x, y, r, with $y \succcurlyeq_r r$ and for all coordinates j there exists a consequence α_j such that $\alpha_j x \sim_r y$.

Solvability is a rather strong condition, which will imply unboundedness of utility. It would be desirable to replace solvability by a restricted version as in Krantz et al. (1971, Definition 12, p. 301). I am pretty sure that this can be done, but the problems involved are quite difficult, would probably require additional axioms, and would make the already long and complicated proofs of the main results even longer and more complicated. I will not pursue this topic in this paper.

2.3. Tradeoff consistency

I next define a new relation $\sim^*_{r,j}$, which is central in the preference foundations given below. For a coordinate j and consequences α_j, β_j, γ_j, $\delta_j \in X_j$, write

$$\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta_j$$

if there exist alternatives $x, y, r \in X$ such that

$$\alpha_j x \sim_r \beta_j y \quad \text{and} \quad \gamma_j x \sim_r \delta_j y.$$

The interpretation of the $\sim^*_{r,j}$ relation is that, judged from r_j, receiving α_j instead of β_j is an equally good improvement as receiving γ_j instead of δ_j: both exactly offset the receipt of the y_is instead of the x_is. Loosely speaking, we may interpret the $\sim^*_{r,j}$ relations as measuring strength of preference. Note, however, that the $\sim^*_{r,j}$ relations are defined in terms of $\sim_r$ and do not require the introduction of new primitives. In terms of the additive representations derived below $\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta_j$ implies that $V_j(\alpha_j, r_j) - V_j(\beta_j, r_j) = V_j(\gamma_j, r_j) - V_j(\delta_j, r_j)$ where V_j is a utility function on attribute j that represents $\succcurlyeq_j$. More detailed discussions of relationships similar to the $\sim^*_{r,j}$ relations can be found in Wakker (1989) and in Köbberling and Wakker (2003, 2004). For the interpretation of the $\sim^*_{r,j}$ relation as strength of preference relations to make sense we must introduce a consistency condition. The next definition presents this consistency condition.

Definition 2.4. *Tradeoff consistency* holds if improving an outcome in any $\sim^*_{r,j}$ relationship breaks that relationship. For instance, if $\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta_j$ and $\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta'_j$ both hold then tradeoff consistency implies that $\delta_j \sim_j \delta'_j$.

The intuition behind tradeoff consistency is that if judged from r_j, receiving α_j instead of β_j is an equally good improvement as receiving γ_j instead of δ_j and also receiving α_j instead of β_j is an equally good improvement as receiving γ_j instead of δ'_j then δ_j and δ'_j must be equally good. Tradeoff consistency is a central condition in what follows and will ensure that the relations $\succcurlyeq_r$ have additive representations. An important advantage of using tradeoff consistency as a condition in preference foundations, besides its intuitive appeal, is that it is easily tested empirically. Measurements of utility by the tradeoff method (Wakker & Deneffe, 1996) provide direct tests of tradeoff consistency. The tradeoff method measures standard sequences of outcomes. Inspection of different standard sequences provides information whether tradeoff consistency is satisfied. Suppose for example that we have observed that $\alpha_j x \sim_r \beta_j y$ and $\gamma_j x \sim_r \delta_j y$. Then we can use gauge outcomes x' and y', which are different from x and y, such that $\alpha_j x' \sim_r \beta_j y'$ and $\gamma_j x' \sim_r \delta'_j y'$. A comparison between δ_j and δ'_j yields a test of tradeoff consistency. Many empirical studies have used the tradeoff method to measure utility and to test the validity of decision models in different decision contexts, showing that such measurements are feasible and easily performed (e.g. Abdellaoui, 2000; Abdellaoui, Vossmann, & Weber, 2005; Bleichrodt & Pinto, 2000, 2005; Etchart-Vincent, 2004; Fennema & van Assen, 1998).

Finally, some technical assumptions are introduced. Consider the order topologies on the X_j, which are generated by the sets $\{y_j \in X_j : y_j \succ_j x_j\}$ and $\{y_j \in X_j : y_j \prec_j x_j\}$, where $x_j \in X_j$. X is endowed with the product topology.

Definition 2.5. *Preference continuity* holds if for all alternatives x and r, the sets $\{y \in X : y \succcurlyeq_r x\}$ and $\{y \in X : y \preccurlyeq_r x\}$ are closed in X.

Functions V_j, $j = 1, \ldots, n$, are *joint ratio scales* if they can be replaced by functions $W_j, j = 1, \ldots, n$, if and only if there exists a positive σ such that $W_j = \sigma V_j$ for all j.

3. Results

3.1. One fixed reference point

I first derive an additive representation for a given reference alternative. This case is considered separately, because it is the case most commonly encountered in theoretical analyses of reference-dependence and there are decision situations in which it is descriptively realistic, in particular when retention of the status quo is an option. For example, in a comparison between risky assets, "do nothing", i.e. zero gain and zero loss, may be a plausible reference point.

Theorem 3.1. *Consider a given reference alternative $r \in X$. Let there be at least three coordinates, which are all essential with respect to r. The following two statements are equivalent for $\succcurlyeq_r$:*

1. *The order topologies on X_j are connected, $\succcurlyeq_r$ is transitive and satisfies r-upper completeness, weak monotonicity, solvability, preference continuity, and tradeoff consistency.*
2. *There exist functions $V_j : X_j \to \mathbb{R}$ such that:*

   (a) $(x, y) \in B_r$ iff $\sum_{j=1}^n V_j(x_j) \geqslant 0$ *or* $\sum_{j=1}^n V_j(y_j) \geqslant 0$;
   (b) *for all* $(x, y) \in B_r$, $x \succcurlyeq_r y$ *iff* $\sum_{j=1}^n V_j(x_j) \geqslant \sum_{j=1}^n V_j(y_j)$;
   (c) $V_j(r_j) = 0$ *for all* j;
   (d) *for all* $j \in \{1, \ldots, n\}$, V_j represents $\succcurlyeq_j$: *for all* $x_j, y_j \in X_j$, $V_j(x_j) \geqslant V_j(y_j)$ *iff* $x_j \succcurlyeq_j y_j$;
   (e) *the* V_j *are continuous and their range is* $\mathbb{R}$.

Furthermore, the V_j are joint ratio scales.

In terms of the choice function c_r, part 2 of Theorem 3.1 can be summarized as for all $A \in \mathscr{F}_r$, $c_r(A) = \arg\max_{x \in A} \{\sum_{j=1}^n V_j(x_j)\}$ with the V_j as defined in Theorem 3.1.

3.2. Variable reference points

The V_j in Theorem 3.1 obviously depend on the given reference alternative r. I will now consider the case where the reference alternative can vary. As mentioned before, this case is important for reference-dependent theories to be able to explain empirical regularities. Because we now consider variable reference points, the $V_j(x_j)$ in Theorem 3.1. will be replaced by $V_{j,r}(x_j, r_j)$ to make explicit the dependency on the reference alternative. Without further restrictions the functions $V_{j,r}$ are too general to be tractable. Intuitively it seems plausible that when two different reference alternatives r and r' yield the same outcome for some coordinate j, i.e. $r_j = r'_j$, then the functions $V_{j,r}$ and $V_{j,r'}$ are identical, $V_{j,r}(x_j) = V_{j,r'}(x_j) = V_j(x_j)$ for all x_j. This alignment of the different reference-dependent representations greatly facilitates the elicitation of the model and, hence, I will derive the following representation.

Definition 3.1. *General reference-dependent utility* (GRU) holds if there exist functions $V_j : X_j \times X_j \to \mathbb{R}$ such that

(a) $(x, y) \in B_r$ iff $\sum_{j=1}^n V_j(x_j, r_j) \geqslant 0$ or $\sum_{j=1}^n V_j(y_j, r_j) \geqslant 0$.
(b) For all $(x, y) \in B_r$, $x \succcurlyeq_r y$ iff $\sum_{j=1}^n V_j(x_j, r_j) \geqslant \sum_{j=1}^n V_j(y_j, r_j)$.
(c) $V_j(r_j, r_j) = 0$ for all j.
(d) For all $j \in \{1, \ldots, n\}$, V_j is increasing in its first argument (i.e. V_j represents $\succcurlyeq_j$): for all x_j, $y_j \in X_j$, $V_j(x_j, r_j) \geqslant V_j(y_j, r_j)$ iff $x_j \succcurlyeq_j y_j$.
(e) For all $j \in \{1, \ldots, n\}$, V_j is decreasing in its second argument: for all x_j, y_j, $z_j \in X_j$, $V_j(x_j, z_j) \geqslant V_j(x_j, y_j)$ iff $z_j \preccurlyeq_j y_j$.
(f) The V_j are continuous in their first argument and their range is $\mathbb{R}$.

Furthermore, the V_j are joint ratio scales.

In terms of the choice function c_r, GRU amounts to for all $A \in \mathscr{F}_r$, $c_r(A) = \arg\max_{x \in A} \{\sum_{j=1}^n V_j(x_j, r_j)\}$ with the V_j as defined in Definition 3.1.

To derive GRU some new definitions must be introduced. The first definition extends r-upper completeness to all reference alternatives r.

Definition 3.2. *Upper completeness* holds if r-upper completeness holds for all r.

The functions $V_j(x_j, r_j)$ are increasing in x_j. It also makes sense that the functions V_j are decreasing in their second argument, the reference level r_j. The less attractive the reference level r_j, the more attractive appear all other alternatives relative to the reference alternative. The following condition formalizes this intuition.

Definition 3.3. *Reference monotonicity* holds when for all alternatives x and r and for all outcomes α_j, β_j, $\gamma_j \in X_j$, $\beta_j \preccurlyeq_j \gamma_j$ iff $\alpha_j x \sim_{\gamma_j r} \gamma_j r$ implies $\alpha_j x \succcurlyeq_{\beta_j r} \beta_j r$.

The next condition is central in the derivation of Theorem 3.2, because it implies that functions $V_j(x_j, r_j)$ can be chosen identical for different reference alternatives r and r' for which $r_j = r'_j$.

Definition 3.4. *Neutral independence* holds if for all α_j, $\beta_j \in X_j$ and for all alternatives x, y, r, $\alpha_j x \sim_{\alpha_j r} \alpha_j y$ implies $\beta_j x \sim_{\beta_j r} \beta_j y$.

In other words, neutral independence says that an indifference with all j-coordinates, both of the alternatives compared and of the reference alternative, being the same is not affected if this common coordinate is changed. An example may illustrate. Consider a medical decision problem where a patient has symptoms that indicate one of three possible states of the world: either he has disease A or he has disease B or he has disease C. The reference treatment means that he lives 8 more years if he turns out to have disease A, 10 more years if he turns out to have disease B, and 2 more years if he turns out to have disease C, denoted (8,10,2). There exist two alternative treatments, treatment 1 gives (8,8,4) and treatment 2 gives (8,6,6). Suppose that the patient prefers both treatments to the reference treatment and is indifferent between treatments 1 and 2. Then neutral independence says that he should also be indifferent between (10,8,4) and (10,6,6) if the reference treatment is (10,10,2) because these new treatments result from the original ones by replacing the common outcome 8 years under disease A by the new common outcome 10 years under disease A.

We are now in a position to state the extension of Theorem 3.1 to $\{\succcurlyeq_r : r \in X\}$, i.e. the case where the reference point can vary. Theorem 3.2 is the central result of this paper.

Theorem 3.2. *Let there be at least three coordinates, which are all essential with respect to every $r \in X$. The following two statements are equivalent for $\{\succcurlyeq_r : r \in X\}$:*

1. *The order topologies on X_j are connected, $\succcurlyeq_r$ is transitive and satisfies upper completeness, weak monotonicity, solvability, preference continuity, tradeoff consistency, reference monotonicity, and neutral independence.*
2. *GRU holds.*

4. Conclusion

The central point of this paper is that there are important decision contexts in which the reference alternative is part of the decision maker's opportunity set and that reference-dependent preference relations must be incomplete in such decision contexts. The paper has derived additive representations for such incomplete preference relations. The main model of this paper, GRU, can be used as a building block for more specific reference-dependent utility models such as prospect theory by imposing additional assumptions. Pursuing such extensions is, however, beyond the scope of this paper.

Reference-dependence is an important explanation for deviations from the standard models of decision theory. This paper has shown that reference-dependence also creates new theoretical problems. By addressing one of these problems, the fact that reference-dependence often leads to incompleteness of preference data, I hope that this paper has contributed to the applicability of the concept of reference-dependence.

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Appendix A

Proof of Theorem 3.1. First assume that statement (2) of Theorem 3.1 holds. For all j, the set of $\sim_j$ equivalence classes of X_j is homeomorphic to the range of V_j, hence it is connected. If $x \succcurlyeq_r y$ and $y \succcurlyeq_r z$ then by parts (a) and (c) and the definition of B_r, $(x, z) \in B_r$ and by part (b) $x \succcurlyeq_r z$, which establishes transitivity. r-Upper completeness follows from parts (a) and (c). For $(x, y) \in B_r$, if $x_j \succcurlyeq_j y_j$ for all j, then because the V_j represent preferences over outcomes, $V_j(x_j) \geqslant V_j(y_j)$ for all j and thus $\sum_{j=1}^n V_j(x_j) \geqslant \sum_{j=1}^n V_j(y_j)$. Because $(x, y) \in B_r$, part (b) implies that $x \succcurlyeq_r y$ and thus weak monotonicity follows. Solvability holds because $V_j(X_j) = \mathbb{R}$ for all j.

Continuity of the V_j implies continuity of $\sum_{j=1}^n V_j(x_j)$. Suppose $r \succ_r x$. Then $\{y \in X : y \preccurlyeq_r x\} = \emptyset$, which is closed. Because $V = \sum_{j=1}^n V_j(x_j)$ is continuous, the inverse of V of the closed subset $[0, \to)$ in $\mathbb{R}$ is closed. This inverse is $\{y \in X : y \succcurlyeq_r r\}$. If $x \succ_r r$ and $V(x) = \sum_{j=1}^n V_j(x_j) = c > 0$, then by continuity of V, the inverses of V of the closed subsets $[0, c]$ and $[c, \to)$ in $\mathbb{R}$ are both closed. These inverses are $\{y \in X : y \preccurlyeq_r x\}$ and $\{y \in X : y \succcurlyeq_r x\}$. Preference continuity follows.

Finally, tradeoff consistency is established. If $\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta_j$ then there exist alternatives x, y such that $\alpha_j x \sim_r \beta_j y$ and $\gamma_j x \sim_r \delta_j y$. Hence, $V_j(\alpha_j) + \sum_{i \neq j} V_i(x_i) = V_j(\beta_j) + \sum_{i \neq j} V_i(y_i)$ and $V_j(\gamma_j) + \sum_{i \neq j} V_i(x_i) = V_j(\delta_j) + \sum_{i \neq j} V_i(y_i)$, which gives $V_j(\alpha_j) - V_j(\beta_j) = V_j(\gamma_j) - V_j(\delta_j)$. The relation $\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta'_j$ implies that there exist alternatives x', y' such that $\alpha_j x' \sim_r \beta_j y'$ and $\gamma_j x \sim_r \delta'_j y$. Hence, by a similar reasoning as above, $V_j(\alpha_j) - V_j(\beta_j) = V_j(\gamma_j) - V_j(\delta'_j)$. Therefore, $V_j(\delta_j) = V_j(\delta'_j)$, and because the V_j represent preferences over outcomes, it follows that $\delta_j \sim \delta'_j$. Statement (1) has been derived.

Next assume that statement (1) of Theorem 3.1 holds. Let $r \in X$. The preference relation satisfies *strong monotonicity* if for all $(x, y) \in B_r$, for all coordinates j, $x_j \succcurlyeq_j y_j$ and for at least one coordinate i, $x_i \succ_i y_i$ then $x \succ_r y$. It is easily verified that strong monotonicity implies weak separability.

Lemma 1. *If $\succcurlyeq_r$ satisfies transitivity, r-upper completeness, and weak monotonicity, then tradeoff consistency implies strong monotonicity.*

Proof. Let $(x, y) \in B_r$ be such that for all coordinates j, $x_j \succcurlyeq_j y_j$ and for at least one i, $x_i \succ_i y_i$. If $x \succcurlyeq_r r$ and $r \succ_r y$ then the result follows by transitivity. So let $y \succcurlyeq_r r$. By weak monotonicity and r-upper completeness, $x \succcurlyeq_r y$. Define $x' = (x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)$. By weak monotonicity, $x \succcurlyeq_r x' \succcurlyeq_r (y_i)_i x' \succcurlyeq_r y$. By transitivity it suffices to show that $x' \succ_r (y_i)_i x'$. Suppose that $x' = (x_i)_i x' \sim_r (y_i)_i x'$. By r-upper completeness,$(x_i)_i x' \sim_r (x_i)_i x'$ and, thus, $x_i y_i \sim^*_{r,j} x_i x_i$. The indifferences $(x_i)_i x' \sim_r (x_i)_i x'$ and $(x_i)_i x' \sim_r (x_i)_i x'$ imply that $x_i x_i \sim^*_{r,j} x_i x_i$. Tradeoff consistency implies that $y_i \sim_i x_i$, which contradicts the assumed $x_i \succ_i y_i$. □

Lemma 2. *Let $\alpha_j y \succcurlyeq_r r$, $\beta_j y \succcurlyeq_r r$. Then $\alpha_j x \succcurlyeq_r \alpha_j y$ iff $\beta_j x \succcurlyeq_r \beta_j y$.*

Proof. Let $\alpha_j y \succcurlyeq_r r$, $\beta_j y \succcurlyeq_r r$ and let $\alpha_j x \succcurlyeq_r \alpha_j y$. Because $n > 2$, there exists, by solvability, a $\gamma_i \in X_i$, $i \neq j$, such that $\alpha_j x \sim_r \gamma_i \alpha_j y$. By strong monotonicity, $\gamma_i \succcurlyeq_i y_i$ and $\gamma_i \beta_j y \succcurlyeq_r r$. From $\gamma_i \alpha_j y \sim_r \gamma_i \alpha_j y$ and $\gamma_i \beta_j y \sim_r \gamma_i \beta_j y$, we obtain $\alpha_j \alpha_j \sim^*_{r,j} \beta_j \beta_j$. Tradeoff consistency and $\alpha_j x \sim_r \gamma_i \alpha_j y$ imply $\beta_j x \sim_r \gamma_i \beta_j y$. Transitivity and weak monotonicity imply $\beta_j x \succcurlyeq_r \beta_j y$. □

Next I derive additive representations on the subset $X_2 \times \cdots \times X_n$, using tools from the proof of Theorem 2.1 in Gilboa, Schmeidler, and Wakker (2002). Start by making the $\succcurlyeq_j$ antisymmetric, i.e., if $\alpha_j \sim_j \beta_j$ then $\alpha_j = \beta_j$. For $\alpha_j \in X_j$, let $[\alpha_j] = \{\beta_j \in X_j : \beta_j \sim_j \alpha_j\}$ denote the $\sim_j$-indifference class of α_j. For any alternative x, $[x] = ([x_1], \ldots, [x_n])$. Let $[X_j]$ denote the set of indifference classes in X_j. All assumptions made remain valid if $[X_j]$ is considered. Henceforth, α_j is written instead of $[\alpha_j]$ and it is assumed that every indifference class in X_j, $j = 1, \ldots, n$, contains exactly one element. The proof of the theorem for this modified structure implies the proof for the original structure.

Because for any y, the set $\{x \in X : x \sim_r y\}$ is the intersection of the sets $\{x \in X : x \succcurlyeq_r y\}$ and $\{x \in X : x \preccurlyeq_r y\}$, the set $\{x \in X : x \sim_r y\}$ is closed. For each x' in $X_2 \times \cdots \times X_n$, there exists, by solvability, an α_1 such that $(\alpha_1, x') = (\alpha_1, x_2, \ldots, x_n) \sim_r r$. By strong monotonicity and antisymmetry, α_1 is unique and thus the map V from $X_2 \times \cdots \times X_n$ to X_1 which assigns to each x' in $X_2 \times \cdots \times X_n$ the α_1 such that $(\alpha_1, x') \sim_r r$ is well-defined. V defines a binary relation $\succcurlyeq^{(1)}$ on $X_2 \times \cdots \times X_n$ defined by: for all x', $y' \in X_2 \times \cdots \times X_n$, $x' \succcurlyeq^{(1)} y'$ iff $V(x') \preccurlyeq_1 V(y')$. The relations $\succ^{(1)}$, $\sim^{(1)}$, $\preccurlyeq^{(1)}$, and $\prec^{(1)}$ are defined as usual. Because $\succcurlyeq_1$ on X_1 is complete and transitive, $\succcurlyeq^{(1)}$ is complete and transitive.

Let $\alpha_j x'$ denote the $(n-1)$-tuple x' with x_j replaced by α_j, $j = 2, \ldots, n$.

Lemma 3. *$\succcurlyeq^{(1)}$ satisfies strong monotonicity: for all $j \in \{2, \ldots, n\}$, if $\alpha_j \succ_j \beta_j$ then $\alpha_j x' \succ^{(1)} \beta_j x'$.*

Proof. Suppose that $(\gamma_1, \alpha_j x') \sim_r (\delta_1, \beta_j x') \sim_r r$ with $\alpha_j \succ_j \beta_j$. It immediately follows from Lemma 1 that $\gamma_1 \prec_1 \delta_1$ and, thus, that $\alpha_j x' \succ^{(1)} \beta_j x'$. □

Lemma 4. *V is continuous.*

Proof. Let $\alpha_1 \in X_1$. The set $\{\beta_1 \in X_1 : \beta_1 \succ_1 \alpha_1\}$ is open in the order topology on X_1. Consider the set $V^{-1}\{\beta_1 \in X_1 : \beta_1 \succ_1 \alpha_1\}$, and let $(x_2, \ldots, x_n)$ be an element thereof. Then $(\gamma_1, x_2, \ldots, x_n) \sim_r r$ for some $\gamma_1 \succ_1 \alpha_1$, and, by strong monotonicity, $(\gamma_1, x_2, \ldots, x_n) \succ_r (\alpha_1, x_2, \ldots, x_n)$. An element $x_j \in X_j$ is *maximal* if for all $y_j \in X_j$, $x_j \succcurlyeq_j y_j$. By solvability, there exists a y_j such that $(\alpha_1, x_2, \ldots, x_{j-1}, y_j, x_j, \ldots, x_n) \sim_r r$. By strong monotonicity, $y_j \succ_j x_j$. Hence, none of the x_j is maximal. By preference continuity and connectedness, there is an $x'_2 \succ_2 x_2$ such that still $r \succ_r (\alpha_1, x'_2, x_3, \ldots, x_n)$. Similarly, by preference continuity and connectedness, there is an $x'_3 \succ_3 x_3$ such that still $r \succ_r (\alpha_1, x'_2, x'_3, \ldots, x_n)$. We end up with an inductively defined neighborhood of $(x_2, \ldots, x_n)$ in $V^{-1}\{\beta_1 \in X_1 : \beta_1 \succ_1 \alpha_1\}$ of the form $B_2 \times \cdots \times B_n$ where for each j, $B_j = \{\delta_j \in X_j : \delta_j \prec_j x'_j\}$ for an $x'_j \succ_j x_j$. For every element of $V^{-1}\{\beta_1 \in X_1 : \beta_1 \succ_1 \alpha_1\}$ a neighborhood within $V^{-1}\{\beta_1 \in X_1 : \beta_1 \succ_1 \alpha_1\}$ can be constructed, so that the latter set must be open for each α_1. Similarly, $V^{-1}\{\beta_1 \in X_1 : \beta_1 \prec_1 \alpha_1\}$ is open for each α_1. Continuity of V follows. □

From Lemma 4 it immediately follows that $\succcurlyeq^{(1)}$ is continuous.

The relation $\succcurlyeq^{(1)}$ satisfies the *generalized Reidemeister condition* if for any coordinate $j \in \{2, \ldots, n\}$, $\alpha_j x' \sim^{(1)} \gamma_j y'$, $\beta_j x' \sim^{(1)} \delta_j y'$, and $\alpha_j v' \sim^{(1)} \gamma_j w'$ imply $\beta_j v' \sim^{(1)} \delta_j w'$.

Lemma 5. *$\succcurlyeq^{(1)}$ satisfies the generalized Reidemeister condition.*

Proof. Let $j \in \{2, \ldots, n\}$ and let $\alpha_j x' \sim^{(1)} \gamma_j y'$, $\beta_j x' \sim^{(1)} \delta_j y'$, and $\alpha_j v' \sim^{(1)} \gamma_j w'$. The three $\sim^{(1)}$ indifferences imply by antisymmetry that there exist consequences σ_1, τ_1, and μ_1 such that $(\sigma_1, \alpha_j x') \sim_r (\sigma_1, \gamma_j y') \sim_r (\tau_1, \beta_j x') \sim_r (\tau_1, \delta_j y') \sim_r (\mu_1, \alpha_j v') \sim_r (\mu_1, \gamma_j w') \sim_r r$. For $i = 2, \ldots, n$, $i \neq j$, define $z'_i = x'_i$ if $x'_i \succcurlyeq_i y'_i$ and $z'_i = y'_i$ if $y'_i \succ_i x'_i$, $z'_j = \alpha_j$ if $\alpha_j \succcurlyeq_j \beta_j$ and $z'_j = \beta_j$ if $\beta_j \succ_j \alpha_j$ and let $z' = (z'_2, \ldots, z'_n)$. By Lemma 2 and strong monotonicity, $(\sigma_1, \alpha_j z') \sim_r (\tau_1, \beta_j z') \succcurlyeq_r r$ and $(\sigma_1, \gamma_j z') \sim_r (\tau_1, \delta_j z') \succcurlyeq_r r$. This means that $\alpha_j \beta_j \sim^*_{r,j} \gamma_j \delta_j$. By solvability, ζ_1 and δ'_j can be found such that $(\zeta_1, \beta_j v') \sim_r (\zeta_1, \delta'_j w') \sim_r r$. For $i = 2, \ldots, n$, $i \neq j$, define $z''_i = v'_i$ if $v'_i \succcurlyeq_i w'_i$ and $z''_i = w'_i$ if $w'_i \succ_i v'_i$, $z''_j = \alpha_j$ if $\alpha_j \succcurlyeq_j \beta_j$ and $z''_j = \beta_j$ if $\beta_j \succ_j \alpha_j$, and let $z'' = (z''_2, \ldots, z''_n)$. By Lemma 2 and strong monotonicity, $(\mu_1, \alpha_j z') \sim_r (\zeta_1, \beta_j z') \succcurlyeq_r r$ and $(\mu_1, \gamma_j z') \sim_r (\zeta_1, \delta'_j z') \succcurlyeq_r r$. Hence, $\alpha_j \beta_j \sim^*_{r,j} \gamma_j \delta'_j$. By tradeoff consistency, $\delta_j \sim_j \delta'_j$, and by antisymmetry $\delta_j = \delta'_j$. Hence, $(\zeta_1, \delta_j w') \sim_r r$ and $\beta_j v' \sim^{(1)} \delta_j w'$. □

Lemma 6. *There exist continuous functions $V^1_j : X_j \to \mathbb{R}$, $j = 2, \ldots, n$, such that $\succcurlyeq^{(1)}$ is represented by $\sum_{j=2}^{n} V^1_j(x_j)$ and $V^1_j(r_j) = 0$ for all j. The V^1_j's are joint ratio scales that represent the $\succcurlyeq_j$ relations.*

Proof. This follows from Lemma 6 in Gilboa et al. (2002), who used several results from Wakker (1989). The only difference is that now $V^1_j(r_j) = 0$ for all j. □

The superscript 1 in V^1_j serves as a reminder that the first coordinate was used to define the preference relation $\succcurlyeq^{(1)}$. Any other coordinate $i, i \in \{2, \ldots, n\}$ could also have been used to define a preference relation $\succcurlyeq^{(i)}$ over $\prod_{j\neq 1} X_j$. By the above argument there exists a representation $\sum_{j\neq 1}^{n} V_j^i$, with $V_j^i(r_j)=0$ for all $j\neq i$, for $\succcurlyeq^{(i)}$. I will now demonstrate that for each given j the V_j^i can be chosen identical for different i.

Define induced preference relations $\succcurlyeq^{(A)}$ on subsets A of $\{1,\ldots,n\}$ by keeping outcomes for coordinates not in A fixed. By repeated application of Lemma 2, the $\succcurlyeq^{(A)}$, when defined, do not depend on the level at which the outcomes for coordinates not in A are kept fixed.

Lemma 7. *Let the $V_j^i : X_j \to \mathbb{R}$, $i,j=1,\ldots,n$, $i\neq j$, be as defined above. Then every V_j^i can be chosen identical to every other V_j^k, $k=1,\ldots,n$, $k\neq i,j$.*

Proof. First assume $n=3$. Let i,j, and k be distinct coordinates. First it is shown that for each given j the V_j^i can be chosen proportional for different i. Let α_j, β_j, and γ_j be arbitrary outcomes in X_j with $\alpha_j \succ_j \gamma_j$. By solvability, there exists an $x\in X$ such that $r_i\gamma_j x \sim_r r$. By strong monotonicity (Lemma 1), $r_i\alpha_j x \succ_r r$. By solvability, there also exist $y\in X$ and $\delta_j\in X_j$ such that $r_i\alpha_j x\sim_r r_i\beta_j y$ and $r_i\gamma_j x\sim_r r_i\delta_j y$. Hence, $\alpha_j\beta_j \sim^*_{r,j} \gamma_j\delta_j$.

By solvability, there exist $\alpha_i, \beta_i \in X_i$ such that $\alpha_i\alpha_j x\sim_r \beta_i\beta_j y \sim_r r$. Because $r_i\alpha_j x\sim_r r_i\beta_j y$, Lemma 2 and antisymmetry imply that $\alpha_i=\beta_i$. By Lemma 6,

$$V_j^i(\alpha_j)+V_k^i(x_k)=V_j^i(\beta_j)+V_k^i(y_k)$$

or

$$V_j^i(\alpha_j)-V_j^i(\beta_j)=V_k^i(y_k)-V_k^i(x_k).$$

Similarly, the indifference $r_i\gamma_j x\sim_r r_i\delta_j y$ implies that

$$V_j^i(\gamma_j)-V_j^i(\delta_j)=V_k^i(y_k)-V_k^i(x_k).$$

Hence, $V_j^i(\alpha_j)-V_j^i(\beta_j)=V_j^i(\gamma_j)-V_j^i(\delta_j)$.

By solvability, there also exist x' and y' in X such that $\gamma_j r_k x' \sim_r \delta_j x_k y' \sim_r r$. By strong monotonicity, $\alpha_j r_k x' \succ_r$. It follows from tradeoff consistency that $\alpha_j r_k x' \sim_r \beta_j r_k y'$. For suppose that $\alpha_j r_k x' \prec_r \beta_j r_k y'$. Then, by solvability, there exists a $\beta_j'\in X_j$ such that $\alpha_j r_k x'\sim_r \beta_j' r_k y'$. By strong monotonicity, $\beta_j \succ_j \beta_j'$. But then $\alpha_j\beta_j \sim^*_{r,j}\gamma_j\delta_j$ and $\alpha_j\beta_j'\sim^*_{r,j}\gamma_j\delta_j$, but $\beta_j \nsim_j \beta_j'$,which contradicts tradeoff consistency. The case $\alpha_j r_k x'\succ_r \beta_j r_k y'$ is excluded by a similar line of argument.

The indifferences $\alpha_j r_k x'\sim_r \beta_j r_k y'$ and $\gamma_j r_k x'\sim_r \delta_j r_k y'$ imply by Lemmas 2 and 6 and antisymmetry that

$$V_j^k(\alpha_j)-V_j^k(\beta_j)=V_j^k(\gamma_j)-V_j^k(\delta_j).$$

Thus, $V_j^i(\alpha_j)-V_j^i(\beta_j)=V_j^i(\gamma_j)-V_j^i(\delta_j)$ implies $V_j^k(\alpha_j)-V_j^k(\beta_j)=V_j^k(\gamma_j)-V_j^k(\delta_j)$. The functions V_j^i and V_j^k being continuous on a connected domain implies that they are related by a positive linear transformation. They are 0 at r_j, and hence, they are related by a scale factor. The scale factor is positive by strong monotonicity.

Having established that for each given j the V_j^i are proportional for different i, I now show that they can be chosen identical. By solvability and strong monotonicity, there exist alternatives $x\sim_r r$, $x'\sim_r r$ with $x_j'=x_j$, and $x''\sim_r r$ with $x''=x_i'$ and $x_k''=x_k$.

Because for each given j the V_j^i are proportional for different i, we can write $V_j^i=\lambda_j^k V_j^k$. Define $V_i=V_i^j$ and $V_k=V_k^j$, i.e. $\lambda_i^j=\lambda_k^j=1$. Define $V_j=V_j^k$, i.e. $\lambda_j^k=1$. By Lemma 6, we are free to scale V_i^k such that $\lambda_i^k=1$. Similarly, by Lemma 6 V_j^i can be scaled such that $\lambda_j^i=1$. It remains to show that $\lambda_k^i=1$.

Because $x_j=x_j'$, a comparison between x and x' shows that

$$V_i(x_i)+V_k(x_k)=V_i(x_i')+V_k(x_k'). \tag{A.1}$$

Because $x_k=x_k''$, a comparison between x and x'' shows that

$$V_i(x_i)+V_j(x_j)=V_i(x_i')+V_j(x_j''), \tag{A.2}$$

and, because $x_i'=x_i''$, a comparison between x' and x'' shows that

$$V_j(x_j)+\lambda_k^i V_k(x_k')=V_j(x_j'')+\lambda_k^i V_k(x_k). \tag{A.3}$$

From (A.2) and (A.1),

$$V_j(x_j)-V_j(x_j'')=V_i(x_i')-V_i(x_i)=V_k(x_k)-V_k(x_k').$$

From (A.3),

$$V_j(x_j)-V_j(x_j'')=\lambda_k^i(V_k(x_k)-V_k(x_k')).$$

Hence, $\lambda_k^i=1$ and this establishes that for each given j the functions V_j^i can be chosen identical for different i.

Now consider the case $n>3$. Let i,j,k, and m be distinct coordinates. By solvability, there exist alternatives $x_jx_kr\sim_r y_jy_kr\sim_r r$. These indifferences imply both

$$V_j^i(x_j)+V_k^i(x_k)=V_j^i(y_j)+V_k^i(y_k)$$

and

$$V_j^m(x_j)+V_k^m(x_k)=V_j^m(y_j)+V_k^m(y_k).$$

Because $\{V_j^i, V_k^i\}$ and $\{V_j^m, V_k^m\}$ are both additive representations of $\succcurlyeq^{\{j,k\}}$, they must, by the uniqueness properties of additive representations and the fact that $V_j^i(r_j)=V_j^m(r_j)=V_k^i(r_k)=V_k^m(r_k)=0$, be related by a common scale factor, which is positive by strong monotonicity. Because j and k were chosen arbitrarily, $V_j^i=\sigma V_j^m$ for all i,j,m in $\{1,\ldots,n\}, i\neq j\neq m$. Rescaling establishes that for each given j the functions V_j^i can be chosen identical for different i. $\square$

Because of Lemma 7, I will henceforth drop the superscript i in V_j^i and simply write V_j, $j=1,\ldots,n$.

Lemma 8. *If $x\succcurlyeq_r y$ then $\sum_{j=1}^n V_j(x_j)\geqslant\sum_{j=1}^n V_j(y_j)$.*

Proof. Suppose that $x\sim_r y\succcurlyeq_r r$. If $x_j\sim y_j$ for all j then we are done because all the V_j represent preferences over outcomes. Assume therefore that x and y do not yield equivalent outcomes for each coordinate. Assume, without loss of generality, that $x_1\succ_1 y_1$. By solvability, there exist $\alpha_1,\beta_1\in X_1$ such that $\alpha_1 x\sim_r \beta_1 y\sim_r r$. If $x'\succcurlyeq^{(1)} y'$ then $\alpha_1\preccurlyeq_1\beta_1$.

Hence, by weak monotonicity, $\beta_1 x \succcurlyeq_r \beta_1 y$. By Lemma 2, $y_1 x \succcurlyeq_r y$ and, by strong monotonicity, $x \succ_r y$, contrary to assumption. Hence, $x' \prec^{(1)} y'$. By strong monotonicity, also $x_1 y \succ_r y \sim_r x$. By solvability, there exists a $y_2' \in X_2$ such that $x \sim_r x_1 y_2' y$. Because $\alpha_1 x \sim_r r$, Lemma 2 implies that $\alpha_1 y_2' y \sim_r r$. Hence, $x' \sim^{(1)} y_2' y'$. By Lemma 6

$$\sum_{j=2}^{n} V_j(x_j) = V_2(y_2') + \sum_{j=3}^{n} V_j(y_j). \tag{A.4}$$

The indifferences $x \sim_r x_1 y_2' y \sim_r y$ imply, by Lemma 2, that $(x_1, y_2', y_4, \ldots, y_n) \sim^{(3)} (y_1, y_2, y_4, \ldots, y_n)$. Hence, from Lemma 6 applied to $\succcurlyeq^{(3)}$,

$$V_1(x_1) + V_2(y_2') + \sum_{j=4}^{n} V_j(y_j) = V_1(y_1) + V_2(y_2) + \sum_{j=4}^{n} V_j(y_j). \tag{A.5}$$

Adding (A.4) and (A.5) and deleting the common terms $V_2(y_2')$ and $\sum_{j=4}^{n} V_j(y_j)$ gives $\sum_{j=1}^{n} V_j(x_j) = \sum_{j=1}^{n} V_j(y_j)$.

If $x \succ_r y$ then there exists, by solvability, $\alpha_1 \in X_1$ such that $x \sim_r \alpha_1 y$. By strong monotonicity, $\alpha_1 \succ_1 y_1$, and hence, $V_1(\alpha_1) > V_1(y_1)$. By what was established above, $\sum_{j=1}^{n} V_j(x_j) = V_1(\alpha_1) + \sum_{j=2}^{n} V_j(y_j) > \sum_{j=1}^{n} V_j(y_j)$. $\square$

Lemma 9. *If* $\sum_{j=1}^{n} V_j(x_j) \geqslant 0$ *then* $x \succcurlyeq_r r$.

Proof. Suppose that $\sum_{j=1}^{n} V_j(x_j) = 0 = \sum_{j=1}^{n} V_j(r_j)$. If $V_j(x_j) = 0$ for all j then it follows from the fact that the V_j are representing, r-upper completeness, and antisymmetry that $x \sim_r r$. So let at least two $V_j(x_j)$ be nonzero and assume, without loss of generality, that $V_1(x_1) > 0$, i.e. $x_1 \succ_1 r_1$, because, by Lemma 6, V_1 represents $\succ_1$. Then $\sum_{j=2}^{n} V_j(x_j) < 0$. By continuity and unboundedness of V_2 (see Lemma 6), there exists an outcome $\alpha_2 \in X_2$ such that $\sum_{j=2}^{n} V_j(x_j) = V_2(\alpha_2)$. Hence, by Lemma 6, $x' \sim^{(1)} \alpha_2 r'$. By Lemma 2, $x \sim_r x_1 \alpha_2 r$. Because $\sum_{j=1}^{n} V_j(x_j) = \sum_{j=1}^{n} V_j(r_j)$, $\sum_{j=1}^{n} V_j(r_j) = V_1(x_1) + V_2(\alpha_2)$. Hence, by Lemma 6, $(x_1, \alpha_2, r_4, \ldots, r_n) \sim^{(3)} (r_1, r_2, r_4, \ldots, r_n)$. That is, $(x_1, \alpha_2, r_3, r_4, \ldots, r_n) \sim_r r$. Hence, by Lemma 2, $x_1 \alpha_2 r \sim_r r$. Transitivity gives $x \sim_r r$.

Suppose that $\sum_{j=1}^{n} V_j(x_j) > 0$. By continuity and unboundedness of V_1, there exists an outcome $\alpha_1 \in X_1$ such that $V_1(\alpha_1) + \sum_{j=2}^{n} V_j(x_j) = 0$. Clearly, $V_1(\alpha_1) < V_1(x_1)$, and hence, because, by Lemma 6, V_1 is representing $\alpha_1 \prec_1 x_1$. By what was proved above, $\alpha_1 x \sim_r r$. By strong monotonicity, $x \succ_r r$. $\square$

Lemma 10. *If* $(x, y) \in B_r$, *then* $\sum_{j=1}^{n} V_j(x_j) \geqslant \sum_{j=1}^{n} V_j(y_j) \Rightarrow x \succcurlyeq_r y$.

Proof. The proof resembles the proof of Lemma 9. Let $(x, y) \in B_r$ and suppose first that $\sum_{j=1}^{n} V_j(x_j) = \sum_{j=1}^{n} V_j(y_j)$. If $V_j(x_j) = V_j(y_j)$ for all j then because the V_j are representing, it follows from weak monotonicity that $x \sim_r y$. So let $V_1(x_1) > V_1(y_1)$. Then $\sum_{j=2}^{n} V_j(x_j) < \sum_{j=2}^{n} V_j(y_j)$. By continuity and unboundedness of V_2, there exists an outcome $\alpha_2 \in X_2$ such that $\sum_{j=2}^{n} V_j(x_j) = V_2(\alpha_2) + \sum_{j=3}^{n} V_j(y_j)$. By Lemma 6, $x' \sim^{(1)} \alpha_2 y'$. That is, there exists an outcome $\alpha_1 \in X_1$ such that $\alpha_1 x \sim_r \alpha_1 \alpha_2 y \sim_r r$. By Lemma 2, $x \sim_r x_1 \alpha_2 y$. Because $\sum_{j=1}^{n} V_j(x_j) = \sum_{j=1}^{n} V_j(y_j)$, $\sum_{j=1}^{n} V_j(y_j) = V_1(x_1) + V_2(\alpha_2) + \sum_{j=3}^{n} V_j(y_j)$.

Hence, by Lemma 6, $(x_1, \alpha_2, y_4, \ldots, y_n) \sim^{(3)} (y_1, y_2, y_4, \ldots, y_n)$. That is, there is an outcome $\alpha_3 \in X_3$ such that $(x_1, \alpha_2, \alpha_3, y_4, \ldots, y_n) \sim_r (y_1, y_2, \alpha_3, y_4, \ldots, y_n) \sim_r r$. By Lemma 2, $x_1 \alpha_2 y \sim_r y$. Transitivity gives $x \sim_r y$.

Suppose that $\sum_{j=1}^{n} V_j(x_j) > \sum_{j=1}^{n} V_j(y_j)$. By continuity and unboundedness of V_1, there exists an outcome α_1 such that $\sum_{j=1}^{n} V_j(x_j) = V_1(\alpha_1) + \sum_{j=1}^{n} V_j(y_j)$. Clearly, $V_1(\alpha_1) > V_1(y_1)$, and hence, because V_1 is representing $\alpha_1 \succ_1 y_1$. By what was proved above, $x \sim_r \alpha_1 y$. By strong monotonicity, $x \succ_r y$. $\square$

This completes the proof that statement (1) of Theorem 3.1 implies statement (2). To summarize, (2a) follows from Lemmas 6, 7, and 9 and the definition of B_r, (2b) follows from Lemmas 8 and 10, and (2c), (2d), and (2e) follow from Lemmas 6 and 7. $\square$

Proof of Theorem 3.2. Assume that statement (2) holds. For transitivity, upper completeness, weak monotonicity, solvability, preference continuity, and tradeoff consistency the proof is the same as in Theorem 3.1. Reference monotonicity follows because the V_j are decreasing in their second argument. If $\alpha_j x \succcurlyeq_{\alpha_j r} \alpha_j y$ then $V_j(\alpha_j, \alpha_j) + \sum_{i \neq j} V_i(x_i, r_i) \geqslant V_j(\alpha_j, \alpha_j) + \sum_{i \neq j} V_i(y_i, r_i)$. Because $V_j(\alpha_j, \alpha_j) = V_j(\beta_j, \beta_j) = 0$, also $V_j(\beta_j, \beta_j) + \sum_{i \neq j} V_i(x_i, r_i) \geqslant V_j(\beta_j, \beta_j) + \sum_{i \neq j} V_i(y_i, r_i)$ and, thus, $\beta_j x \succcurlyeq_{\beta_j r} \beta_j y$, which establishes neutral independence.

Assume next statement (1). As in the proof of Theorem 3.1, the $\succcurlyeq_j$ are made antisymmetric. As before, for any $\alpha_j \in X_j$, $[\alpha_j] = \{\beta_j \in X_j : \beta_j \sim_j \alpha_j\}$ and for any alternative $x, [x] = ([x_1], \ldots, [x_n])$. Define preference relations $\succcurlyeq_{[x]}$ by choosing any element $r \in [x]$ and defining $\succcurlyeq_{[x]} = \succcurlyeq_{[r]}$. By reference monotonicity, the definition does not depend on the choice of r.

The proof of Theorem 3.1 can now be applied to obtain continuous functions $V_j^r : X_j \to \mathbb{R}$, $j = 1, \ldots, n$ with range $\mathbb{R}$, such that $(x, y) \in B_r$ if $\sum_{j=1}^{n} V_j^r(x_j) \geqslant 0$ or $\sum_{j=1}^{n} V_j^r(y_j) \geqslant 0$, for all $(x, y) \in B_r$, $x \succcurlyeq_r y$ iff $\sum_{j=1}^{n} V_j^r(x_j) \geqslant \sum_{j=1}^{n} V_j^r(y_j)$, and for all $j \in \{1, \ldots, n\}$, $V_j^r(r_j) = 0$ and V_j^r represents $\succcurlyeq_j$. It remains to prove that if $r_j = r_j'$, then every V_j^r and $V_j^{r'}$ can be chosen identical. This is established by the next lemma.

Lemma 11. *Let the* $V_j^r : X_j \to \mathbb{R}, j = 1, \ldots, n$, *be as defined above. For all* j *and* r, r', *if* $r_j = r_j'$, *then* V_j^r *can be chosen identical to* $V_j^{r'}$.

Proof. By solvability and strong monotonicity, there exist nonindifferent outcomes $\alpha_i, \beta_i, \gamma_j, \delta_j$ such that $\alpha_i \gamma_j r \sim_r \beta_i \delta_j r \sim_r r$. For $k \neq i, j$, let $r_k' \neq r_k$ and for all $j \neq k$, $r_j' = r_j$. By neutral independence, $\alpha_i \gamma_j r' \sim_{r'} \beta_i \delta_j r'$. Writing out these two indifferences, canceling common terms, gives by

Lemma 8:

$$V_i^r(\alpha_i) + V_j^r(\gamma_j) = V_i^r(\beta_i) + V_j^r(\delta_j) \tag{A.6}$$

and

$$V_i^{r'}(\alpha_i) + V_j^{r'}(\gamma_j) = V_i^{r'}(\beta_i) + V_j^{r'}(\delta_j). \tag{A.7}$$

The functions V_i^r and $V_i^{r'}$ and V_j^r and $V_j^{r'}$ are continuous on a connected domain and, hence (A.6) and (A.7) imply by the uniqueness properties of additive representations (Wakker, 1989) that V_i^r and $V_i^{r'}$ and V_j^r and $V_j^{r'}$ are related by positive linear transformations with a common unit. V_i^r and $V_i^{r'}$ are 0 at $r_i = r_i'$ and V_j^r and $V_j^{r'}$ are 0 at $r_j = r_j'$. Hence, they are related by a common scale factor. The scale factor is positive by strong monotonicity. It follows that $V_i^r = \lambda_{ij} V_i^{r'}$ and $V_j^r = \lambda_{ij} V_j^{r'}$.

The above argument can be repeated, interchanging the roles of j and k, to find $V_i^r = \lambda_{ik} V_i^{r'}$ and $V_k^r = \lambda_{ik} V_k^{r'}$. Rescale the λs to 1 to give $V_j^r = V_j^{r'}$ for $r_j = r_j'$. This scaling does not interfere with the scaling selected in Lemma 7 of Theorem 3.1. For example, take $\theta_1 \succ_1 r_1, \theta_1 \succ_1 r_1'$ and set $V_1^r(\theta_1) = V_1^{r'}(\theta_1) = 1$. □

Because of Lemma 11, $V_j^r(\alpha_j)$ is independent of r_i for $i \neq j$, and we can write $V_i(\alpha_i, r_i)$ instead of $V_i^r(\alpha_i)$. By reference monotonicity the $V_i(\alpha_i, r_i)$ are decreasing in r_i. This completes the proof that statement (1) implies statement (2) and, hence, the proof of Theorem 3.2. □

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