Fishburn's (1970) P7 Implies Savage's (1954) P7

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Savage (1954) provided the most famous and impressive preference foundation in decision theory. Hartmann (2020) provided a remarkable observation: Savage's P3 is redundant. Hartmann's Footnote 4 pointed out that his result remains valid if we replace Savage's P7 by the weaker P7 of Fishburn (1970), called P7′ here. I think that Fisburn's P7′ and its details are not important enough to deserve such mention, but Hartmann was apparently driven by an insisting referee. He had no space to prove the point. For completeness, I add such a proof here.

NOTATION: S denotes a *state space*, X a set of *consequences*, *acts* map S to X, and $\succcurlyeq$ is a *preference relation* over acts. *Events* are subsets of S. Consequences are identified with constant acts. For acts f and g, and event E, f_Eg is the act agreeing with f on E and with g on E^c. *Conditional preference*: for event E, $f \succcurlyeq_E g$ if there exists an act h such that $f_Eh \succcurlyeq g_Eh$. By P2 below, the choice of h is immaterial. Some of Savage's preference conditions follow.

P1 (*weak ordering*); $\succcurlyeq$ is transitive and complete)

P2 (*sure-thing principle*): $f_Eh \succcurlyeq g_Eh \Rightarrow f_Eh' \succcurlyeq f_Eh'$

P6 (*event continuity*):

if $f \succ g$, then for all consequences γ, β there exists a partition $E_1, \ldots, E_n$ of S such that for all i: $\beta_{E_i}f \succ g$ and $f \succ \gamma_{E_i}g$.

P7: $f \preccurlyeq_E g(s)$ for all $s \in E \;\Rightarrow\; f \preccurlyeq_E g$

and

$f \succcurlyeq_E g(s)$ for all $s \in E \;\Rightarrow\; f \succcurlyeq_E g$.

Fishburn's P7′

$f \prec_E g(s)$ for all $s \in E \;\Rightarrow\; f \preccurlyeq_E g$

and

$f \succ_E g(s)$ for all $s \in E \;\Rightarrow\; f \succcurlyeq_E g$.

Fishburn's condition is somewhat weaker than Savage's because the premise is somewhat stronger, but in the presence of other conditions they are still equivalent.

OBSERVATION 1. Under P1, P2, & P6, Fishburn's P7 implies Savage's P7; i.e., they are equivalent.

PROOF. All preferences below are conditioned on event E. Assume Savage's P7 violated. Thus, assume that we have $f(s) \succcurlyeq_E g$ for all $s \in E$, but $f \prec_E g$. The case with all preferences reversed is similar and is not discussed further. Using P6, we can worsen g a bit into g´ (changing g into f on some small nonnull event where f is worse; this involves P2 but not P3; there exists at least one such event) such that $f \prec_E g' \prec_E g$. Then we have $f(s) \succ_E g'$ for all s but $f \prec_E g'$. That is, Fishburn's P7 is violated too. □

Knowing Fishburn, I think that his main aim was not to obtain just any generalization. He had a preoccupation with strict rather than weak preference as primitive, and this is why he brought it in I think. After a life of working on preference foundations, I think that weak preferences better serve as primitive than strict. I also find Savage's P7 more appealing than Fishburn's P7´.

References

Fishburn, Peter C. (1970) "*Utility Theory for Decision Making.*" Wiley, New York.

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Savage, Leonard J. (1954) "*The Foundations of Statistics.*" Wiley, New York. (2nd edn. 1972, Dover Publications, New York.)