Decision-Making under Uncertainty

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Overview

Lecture 1: Uncertainty and Preferences, Arbitrage and Expected Value

Lecture 2: Expected Value, Additivity, Arbitrage

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Lecture 11: Probability weighting under risk

Lecture 12: Probability weighting under risk (2)

Lecture 13: Prospect theory under risk

Lecture 14: Ambiguity preferences

Housekeeping

Book: Peter Wakker, 2010: *Prospect Theory - for Risk and Ambiguity*, Cambridge University Press 2010.

See Moodle course page for slides and other material. Sign up please.

Class exercises: Held weekly by David Danz, danz@wzb.eu, Fri 14-16

Lecture 1: Preferences, Arbitrage and Expected Value

State space

S — (finite or infinite) set of possible states. One and only one state $s \in S$ is true—unbeknownst to the decision-maker.

Example 1.1.1: Deciding what merchandise to take, with three weather states.

	no rain (s_1)	**some rain (s_2)**	**all rain (s_3)**
x ("ice cream")	400	100	-400
y ("hot dogs")	-400	100	400
0 ("neither")	0	0	0
x+y ("both")	0	200	0

Financial market example

S — (finite or infinite) set of possible states.

Example 1.1.2: Betting on the copper price next month

	price $\geq$ 2.53	**2.53 $>$ price $\geq$ 2.47**	**2.47 $>$ price**
x	50K	-30K	-30K
y	-30K	-30K	50K
0 ("neither")	0	0	0
x+y ("both")	20K	-60K	20K

Events

$E \subset S$ — Subsets of the state space, empty or containing one or multiple $s \in S$.

Example 1.1.2: E.g. All copper prices next month that lie above 2.53

	price $\geq$ 2.53 **(E_1)**	2.53 $>$ price $\geq$ 2.47 **(E_2)**	2.47 $>$ price **(E_3)**
x	50K	-30K	-30K
y	-30K	-30K	50K
0 (“neither”)	0	0	0
x+y (“both”)	20K	-60K	20K

Prospects

$\mathbb{R}$ — Outcome space.

Prospects — Mappings from S to $\mathbb{R}$.

	no rain (s_1)	some rain (s_2)	all rain (s_3)
x (“ice cream”)	**400**	**100**	**-400**
y (“hot dogs”)	-400	100	400
0 (“neither”)	0	0	0
x+y (“both”)	0	200	0

Think of $x, y, 0$ more generally as investments in a retail context.

Note: No probabilities defined

Prospects (2)

$\mathbb{R}$ — Outcome space.

Prospects — Mappings from S to $\mathbb{R}$.

	price $\geq$ 2.53(E_1)	2.53 $>$ price $\geq$ 2.47(E_2)	2.47 $>$ price (E_3)
x	50K	-30K	-30K
y	-30K	-30K	50K
0 ("neither")	0	0	0
x+y ("both")	20K	-60K	20K

Financial markets often allow investors to make bets on all possible events.

Prospects and preference

Notation

Prospect — $x = (E_1 : x_1, ..., E_n : x_n)$ for events $\{E_1, ...E_n\} = S$
Complementary event — $E^c = S - E$
Binary prospect — $\alpha_E \beta \equiv (E : \alpha, E^c : \beta)$
Constant prospect — $\alpha = (E_1 : \alpha, ..., E_n : \alpha)$

Domain of preference — All prospects that take on finitely many values.
Preference relation $\succsim$ — A binary relation on the set of all prospects in the domain

Preference

$x \succsim y$ — You are willing to choose x from $\{x, y\}$.
$\rightarrow$ Preference is a binary choice.

$x \succ y$ — Strict preference: $x \succsim y$ and not $y \succsim x$
$x \sim y$ — Indifference: $x \succsim y$ and $y \succsim x$

$\alpha = CE(x)$ — Certainty equivalent: You are preference equivalent (indifferent) between outcome α and prospect x.

A function $V(\cdot)$ represents $\succsim$ — For all x, y, $x \succsim y$ if and only if $V(x) \geq V(y)$.

Basic properties of $\succsim$

Weak order — $\succsim$ is complete and transitive
Reflexivity — $x \sim x$, for all x
Monotonicity —
(i) If $x(s) \geq y(s)$ for all $s \in S$, then $x \succsim y$,
and
(ii) If $x(s) > y(s)$ for all $s \in S$, then $x \succ y$

Notice that the properties require the statement for the entire domain of preference.

A first representation result

Exercise 1.2.5:
(a) Assume weak order and monotonicity. Then:

$$[\alpha \succsim \beta \Leftrightarrow \alpha \geq \beta]$$

(Proof on the board.)

(b) Assume weak order, monotonicity, and that a certainty equivalent $CE(x)$ exists for all x. Then: $CE(\cdot)$ represents $\succsim$.

Proof: Since $CE(x) \sim x$ and $CE(y) \sim y$ hold for certainty equivalents, we know from transitivity that $x \succsim y$ iff $CE(x) \succsim CE(y)$.
By part (a), $CE(x) \succsim CE(y)$ holds iff $CE(x) \geq CE(y)$ and hence $CE(\cdot)$ represents $\succsim$. $\square$

Collecting assumptions

Nondegeneracy — There exists an event E and outcomes γ, β such that $\gamma_E\gamma \succ \gamma_E\beta \succ \beta_E\beta$.

Structural Assumption 1.2.1 ("Decision under Uncertainty"): S is a finite or infinite state space and $\mathbb{R}$ is the outcome set. Prospects map states to outcomes, taking only finitely many values. $\succsim$ is a preference relation on the set of prospects, i.e. on all such maps. Nondegeneracy holds.

Lecture 2: Expected Value, Additivity, Arbitrage

Collecting assumptions

Recall:

Structural Assumption 1.2.1 ("Decision under Uncertainty"): S is a finite or infinite state space and $\mathbb{R}$ is the outcome set. Prospects map states to outcomes, taking only finitely many values. $\succsim$ is a preference relation on the set of prospects, i.e. on all such maps. Nondegeneracy holds.

Expected value

(Not least) in the interest of the researcher: Can the representing function be a weighted average of outcomes?

Probabilities — For a given state space S, a set of probabilities over the possible events is a collection $\{P(E_i)\}_i$ for all events E_i in the state space, satisfying $P(S) = 1, P(\varnothing) = 0$ and $P(E_i \cup E_j) = P(E_i) + P(E_j)$ for disjoint E_i, E_j.

Expected Value — Under Structural Assumption 1.2.1, *expected value (EV) holds* if there exist probabilities $P(E_i)$ for all events E_i in the state space, such that

$$x = (E_1 x_1 \dots E_n x_n) \to \sum_{i=1}^{n} P(E_i) x_i \equiv EV(x)$$

represents $\succsim$.

Deriving decisions

Exercise 1.3.1

Discussion of EV

- Very convenient for analysis
- Degrees of freedom: "subjective probabilities" $P(E_i)$
- Note: $P(E_i)$ is derived from $\succsim$.
- Will see shortly: Maximization is equivalent to consistency of $\succsim$.
- Black box: an as-if construction. Realism?
- Normatively useful?

Eliciting subjective parameters

Exercises 1.3.5 & 1.3.4

Additivity

A related concept that is defined directly on the preference:
Additivity — $[x \succsim y \Rightarrow x + z \succsim y + z]$ for all prospects x, y, z

Tables 1.5.1 & 1.5.3

Tables 1.5.2 & 1.5.4

EV implies additivity

(Part of Exercise 1.6.4.:) Assume that EV holds. Then: $\succsim$ is a weak order, for each prospect there exists a certainty equivalent, and additivity and monotonicity are satisfied.

(Proof on the board, taking as given that CE is additive under EV: $CE(x + y) = CE(x) + CE(y)$.)

Discussion of additivity

- A strong consistency requirement that is easy to grasp
- Rules out diminishing sensitivity
- Rules out considerations of correlation
- Normatively appealing for small outcomes.

Freedom from arbitrage

Additivity refers to sums of outcomes that are combined. A property of combined choice is freedom from arbitrage.

Dutch book — Fix the preference $\succsim$. Arbitrage, or a Dutch book, is a collection of pairs of prospects (x^j, y^j), with $j = 1...m$, such that the $\{x^j\}_j$ are the preferred prospects but when combined they yield strictly less than the $\{y^j\}_j$:

$x^j \succsim y^j$ for all $j = 1...m$,
and
$\sum_{j=1}^{m} x^j(s) < \sum_{j=1}^{m} y^j(s)$ for all $s \in S$.

"Freedom from arbitrage": No Dutch book exists, i.e. the decision-maker's preference does not allow the construction.

Discussion of freedom from arbitrage

- Normatively appealing
- Important concept in finance

De Finetti's theorem

Theorem 1.6.1 — Under Structural Assumption 1.2.1, the following three statements are equivalent.

(i) Expected Value holds.

(ii) $\succsim$ is a weak order, for each prospect there exists a certainty equivalent, and no arbitrage (Dutch book) is possible.

(iii) $\succsim$ is a weak order, for each prospect there exists a certainty equivalent, and additivity and monotonicity are satisfied.

Discussion

- Modulo weak/technical constraints, we have equivalence of EV, freedom from arbitrage, and additivity.
- (iii) $\Rightarrow$ (i). *Only* EV satisfies additivity.
- (i) or (iii) $\Rightarrow$ (ii). EV and additivity both avoid Dutch books
- (ii) $\Rightarrow$ (i). *Only* EV / additivity avoids Dutch books.
- A representation theorem: (i) $\Leftrightarrow$ (iii)
- You will prove the theorem in class. We only proved (i) $\Rightarrow$ (iii).
- EV now more appealing?
- Freedom from arbitrage seems very weak. But it relates to choice between x^j versus y^j that is not combined with other choice.

Finance example

Assignment 1.6.11

Lecture 3: Risk versus uncertainty

Probability-contingent prospects

We continue to use:
Structural Assumption 1.2.1 ("Decision under Uncertainty"): S is a finite or infinite state space and $\mathbb{R}$ is the outcome set. Prospects map states to outcomes, taking only finitely many values. $\succsim$ is a preference relation on the set of prospects, i.e. on all such maps. Nondegeneracy holds.

Additional assumption: There exists an objective probability measure P over events $E \subset S$.
Let $p_j = P(E_j)$ denote the probability of event E_j. A prospect $x = (E_1 : x_1, ..., E_n : x_n)$ has a probability distribution $(p_1 : x_1, ..., p_n : x_n)$ (a "lottery").
$\succsim$ is defined over probability-contingent prospects or lotteries, which are probability distributions with finitely many outcomes values.

Example

Example 2.1.1

Decision under risk

Assumption 2.1.2 ("Decision under Risk"): Structural Assumption 1.2.1 holds. In addition, an objective probability measure P is given on the state space, assigning to each event E its probability $P(E)$. Different event-contingent prospects that generate the same probability-contingent prospect are preference equivalent.

Construction of probabilities

It is unclear where the values of objective probabilities should come from.
Empirical evidence? Deduction?
Notice the difference from subjective probabilities in EV, which are derived from $\succsim$.

$\succsim$ obeys p: Different event-contingent prospects that generate the same probability-contingent prospect are preference equivalent. A strong assumption for any given p.

But at least it is within our previous assumptions: for given P and preference $\succsim$, we can construct an appropriate S so that Structural Assumption 1.2.1 holds. (Next slides.)

Risk as a special case of uncertainty

The aim is to represent any given lottery $(p_1 : x_1, ..., p_n : x_n)$ as an event-contingent prospect. Let $S = [0, 1)$ be the unit interval. We assign n events by partitioning $S = \{[0, q_1), [q_1, q_2), ..., [q_{n-1}, 1)\}$. Let a random number be drawn from S, and let this number be the true state.

E.g., event 1 is that the true state lies in $[0, q_1)$.

Mapping the n possible events into the outcome space $\mathbb{R}$ yields an event-contingent prospect

$\{[0, q_1) : x_1, [q_1, q_2) : x_2, ..., [q_{n-1}, 1) : x_n\}$ like in Structural Assumption 1.2.1.

To generate probabilities, we take the uniform probability measure ("Lebesgue measure") $p_j = q_j - q_{j-1}$. (With $q_0 = 0$ and $q_n = 1$.) We are free to choose the $\{q_j\}_j$, and hence any given lottery can be expressed in such a way, as generated by an event-contingent prospect. A lottery is a prospect, but with the additional information about the probabilities of events. All previous results apply to the case where probabilities are known.

Risk as a special case of uncertainty (2)

But notice that multiple event-contingent prospects can generate the same probability-contingent prospect: E.g.

$$\{[0, \frac{1}{2}) : \$0, [\frac{1}{2}, 1) : \$100\}$$

and

$$\{[0, \frac{1}{2}) : \$100, [\frac{1}{2}, 1) : \$0\}$$

both yield the lottery $(\frac{1}{2} : \$0, \frac{1}{2} : \$100)$.

Getting used to it

Exercises 2.4.1, 2.4.2

Full-domain assumption

Assumption 2.2.1 ("Richness for decision under risk"): Every possible distribution over the outcomes that takes on finitely many values is available in the preference domain.

Summarizing the previous assumptions:

Structural Assumption 2.5.2 ("Decision under risk and richness"): $\succsim$ is a preference relation over the set of all probability-contingent prospects, i.e. over all finite probability distributions over $\mathbb{R}$.

Risk preferences as behavioral assumptions

With objective probabilities, the expected value of prospects (and all their other moments) are defined without reference to $\succsim$.
We can ask how $\succsim$ relates to the moments.
For example, we say that $\succsim$ exhibits risk aversion if every lottery is weakly less preferred than its expected value.

Risk Aversion — $E[x] \succsim x$, for all x in the domain of preference.

Risk Neutrality — $E[x] \sim x$, for all x.

Risk Seeking — $x \succsim E[x]$, for all x.

Risk preferences and EV

Under Structural Assumption 2.5.2, "EV holds" if and only if $\succsim$ exhibits risk neutrality.

Note the argument: Under Str. Ass. 2.5.2, preferences obey the objective measure p. "EV holds" means that the EV function represents $\succsim$. With objective p, the EV function is given by E[x]. Hence a lottery x is preferred to a lottery y iff E[x] $\geq$ E[y].

EV may surprise

Example 2.5.1

Expected utility

Bernoulli's invention: When probabilities are known, the value of the outcome may still be flexible.

Expected Utility — Under Structural Assumption 2.5.2, *expected utility (EU) holds* if there exists a strictly increasing function $U : \mathbb{R} \to \mathbb{R}$, mapping an outcome into a utility value, such that the expected utility function

$$x = (p_1 : x_1 \ldots p_n : x_n) \to \sum_{i=1}^{n} p_i U(x_i) \equiv EU(x)$$

represents $\succsim$.

Lecture 4: Expected utility under risk

Recall the earlier definitions

Structural Assumption 2.5.2 ("Decision under risk and richness"): $\succsim$ is a preference relation over the set of all probability-contingent prospects, i.e. over all finite probability distributions over $\mathbb{R}$.

Expected Utility — Under Structural Assumption 2.5.2, *expected utility (EU) holds* if there exists a strictly increasing function $U : \mathbb{R} \to \mathbb{R}$, mapping an outcome into a utility value, such that the expected utility function

$$x = (p_1 : x_1 ... p_n : x_n) \to \sum_{i=1}^{n} p_i U(x_i) \equiv EU(x)$$

represents $\succsim$.

Deriving decisions under EU

Exercise 2.5.1 & Example 2.5.4

Eliciting utilities under EU

Exercise 2.5.3

(Note the notation $100_{0.58}0$ referring to the first outcome occurring with probability 0.58.)

Behavioral foundation of EU

Wherever possible, we use simple binary lotteries.

Standard gamble — $(p : M, 1 - p : m)$, for some $M > m$ and p.

Standard gamble solvability

Standard gamble solvability—For all outcomes $M > \alpha > m$ there exists a "standard gamble probability" $p \in (0, 1)$ satisfying

$$\alpha \sim (p : M, 1 - p : m)$$

If EU holds, we can normalize $U(M) = 1$ and $U(m) = 0$ (to be shown in class). Consider $M > \alpha > m$. Under EU, $U(M) > U(\alpha) > U(m)$ holds, and there exists $p \in (0, 1)$ such that

$$U(\alpha) = pU(M) + (1 - p)U(m)$$

$\Rightarrow$ EU implies SG solvability.

SG solvability makes utilities and probabilities commensurable.

Standard gamble dominance

Standard gamble dominance — For all outcomes $M > m$ and probabilities $p > q$,

$$(p : M, 1 - p : m) \succ (q : M, 1 - q : m)$$

SG dominance corresponds to monotonicity.

EU implies SG dominance. (Board.)

Linearity of EU

EU is tractable especially when dealing with complicated lotteries.

Probabilistic mixture — For a pair of lotteries x, y and a probability $\lambda \in [0, 1]$, let $x_\lambda y$ denote the probabilistic mixture of x and y: a lottery that assigns to each outcome α a probability of λ times α's probability under x plus $1 - \lambda$ times α's probability under y. (See Example 2.6.1.)

Proposition: (Exercise 2.6.6.) *EU* is linear in probability:

$$EU(x_\lambda y) = \lambda EU(x) + (1 - \lambda) EU(y)$$

(Proof on board.)

Standard gamble consistency

A weak form of linearity:

Standard gamble consistency — For all outcomes α, M, m, all probabilities p, λ, and all lotteries C, it holds that

$$\alpha \sim (p : M, 1-p : m)$$

implies

$$\alpha_\lambda C \sim (p : M, 1-p : m)_\lambda C$$

where the last term denotes a probabilistic mixture between $(p : M, 1-p : m)$ and C.

Note: Under EU, SG holds:
$EU(\alpha_\lambda C) = \lambda EU(\alpha) + (1-\lambda)EU(C)$
$EU((p : M, 1-p : m)_\lambda C) = \lambda EU(p : M, 1-p : m) + (1-\lambda)EU(C)$

Realism? Normative appeal?

von Neumann-Morgenstern's EU representation theorem

Theorem 2.6.3 — Under Structural Assumption 2.5.2, the following two statements are equivalent:

1. EU holds.

2. $\succsim$ satisfies weak ordering, SG solvability, SG dominance and SG consistency.

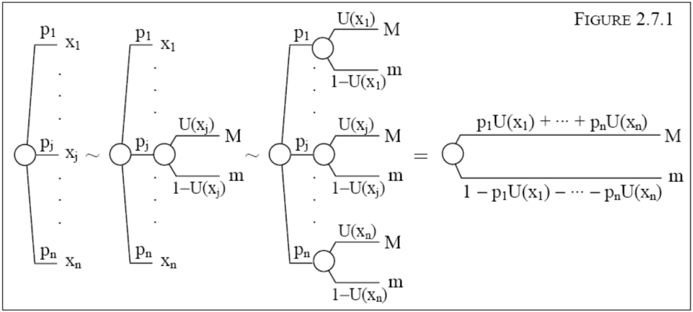
Proof of vNM's theorem

We saw that EU implies weak ordering, SG solvability, SG dominance and SG consistency. It remains to show the reverse.

Fix two prospects $x = (p_1 : x_1 ... p_n : x_n)$ and $y = (q_1 : y_1 ... q_n : x_n)$. Let M be the largest outcome and m the smallest outcome of all outcomes in x or y, and normalize $U(M) = 1$ and $U(m) = 0$. For each outcome x_j define $U(x_j)$ to be the SG probability for the standard gamble with outcomes (M, m). $U(x_j)$ exists due to SG solvability.
Consider the indifferences in the Figure 2.9.1. The first indifference holds due to the application of SG consistency. The second equivalence uses the repeated application of SG consistency, for all outcomes. The equality is by construction, and Assumption 2.1.2 implies preference equivalence between the two equal prospects.

Proof of vNM's theorem (2)

Figure 2.9.1 (not 2.7.1)



Proof of vNM's theorem (3)

Summing up Figure 2.9.1: The prospect x is preference equivalent to the binary lottery that yields M with probability $\sum_{i=1}^{n} p_i U(x_i)$ and yields m otherwise.
Define $EU(x) \equiv \sum_{i=1}^{n} p_i U(x_i)$ as our candidate EU function, and $U(\cdot)$ as the candidate utility function.
$\sum_{i=1}^{n} p_i U(x_i)$ is a probability, but has the "right" structure of $EU(x)$: it is a function that maps lotteries into $\mathbb{R}$, and it is linear in the outcome probabilities p_i.

Need to check that (i) $\succsim$ can be represented by this particular function $EU(x)$, and that (ii) $U(x_j)$ is strictly increasing in x_j.

Proof of vNM's theorem (4)

Need to check that (i) $\succsim$ can be represented by $EU(x)$, and (ii) $U(x_j)$ is strictly increasing in x_j.

(i) Consider x and y. x is preference equivalent to a binary lottery assigning probability $EU(x)$ to M and the remaining probability to m, and y is preference equivalent to a binary lottery assigning $EU(y)$ to M and $1 - EU(y)$ to m. By SG dominance, all binary lotteries with M and m as outcomes are ordered by the probability of receiving M. Hence, by transitivity, $EU(x) \geq EU(y)$ is equivalent to $x \succsim y$.

(ii) Consider two outcomes $x_k > x_j$. Apply SG dominance with a different selection of $M, m : M = x_k, m = x_j, p = 1, q = 0$ to find that $x_k \succ x_j$. Since we already saw that $\sum_{i=1}^{n} p_i U(x_i)$ represents $\succsim$, it holds equivalently that $U(x_k) > U(x_j)$. ■

EU as decision aid

Figures 3.1.1 and 3.1.2

Lecture 5: Expected utility and stochastic dominance

Alternative formulation of EU axioms

Def: $\succsim$ satisfies continuity if for all lotteries $x \succ y \succ z$ there exists a $p \in (0,1)$ satisfying the indifference

$$y \sim (p : x, 1-p : z).$$

Def: $\succsim$ satifies independence if for all probabilities λ and all lotteries x, y, C, it holds that

$$x \succsim y$$

implies

$$(\lambda : x, 1-\lambda : C) \succsim (\lambda : y, 1-\lambda : C).$$

Alternative formulation of EU axioms (2)

Proposition: Under Structural Assumption 2.5.2, the following two are equivalent:

1. EU holds, but with U not necessarily strictly increasing.
2. $\succsim$ satisfies weak ordering, continuity, and independence.

(First-order) stochastic dominance

x first-order stochastically dominates y — x can be generated from y by shifting probability mass from an outcome to a preferred outcome (once or in multiple instances).

$\succsim$ satisfies stochastic dominance — Whenever x first-order stochastically dominates y, it holds that $x \succsim y$.

Exercise 2.7.1

Counterexamples to EU

Exercise 2.8.1

Counterexamples to EU (2)

Problem 1. Which of the following options do you prefer?
C1. A sure gain of 1 million Euros.
C2. An 80% chance to gain 5 million Euros and a 20% chance to gain nothing.

Problem 2. Which of the following options do you prefer?
D1. A 5% chance to gain 1 million Euros and a 95% chance to gain nothing.
D2. A 4% chance to gain 5 million Euros and a 96% chance to gain nothing.

Counterexamples to EU (3)

Figure 2.4.1, (g) and (h)

Stochastic dominance for continuous distributions

For convenience: Also consider lotteries F_x with a bounded continuum of outcomes: $\alpha \in [x_{\min}, x_{\max}]$ and $\exists$ a density for all α.

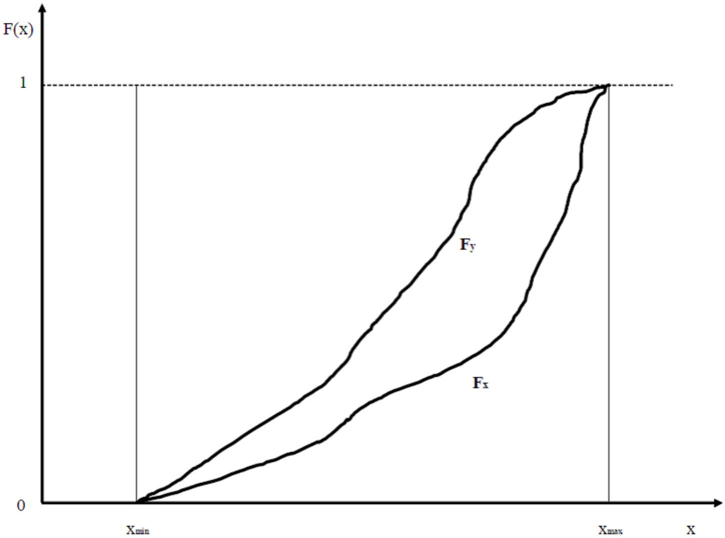
Notice that we can approximate any pair of such lotteries (F_x, F_y) by two lotteries (x, y) with finitely many outcomes (satisfying Structural Assumption 2.5.2).

We formulate some properties/ideas for continuous lotteries but apply the results to finite lotteries.

x first-order stochastically dominates y —

$$F_x(\alpha) \leq F_y(\alpha), \text{ for all } \alpha \in [x_{\min}, x_{\max}].$$

Stochastic dominance for continuous distributions (2)



Equivalence of EU and stochastic dominance

Proposition: Under Structural Assumption 2.5.2, the following two statements about lotteries x, y are equivalent:

1. x first-order-stochastically dominates y.

2. All EU-representable preferences prescribe $x \succsim y$.

Equivalence of EU and stochastic dominance (2)

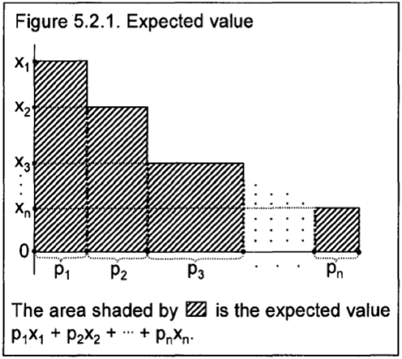
We already showed 1. $\Rightarrow$ 2.

2. $\Rightarrow$ 1. is unlikely to be applied in practice, since 2. is usually not known when 1. is not yet known. But the statement [2. $\Rightarrow$ 1.] implies that we know exactly what the EU assumptions buy us: we restrict attention to all preferences that do not violate stochastic dominance.

Suppose that we assumed EU and that neither x nor y first-order stochastically dominates the other. Then, [*not* 1. $\Rightarrow$ *not* 2.] implies that we ruled out neither $x \succsim y$ nor $y \succsim x$.

Proof of the proposition

Consider lottery $x = (p_1 : x_1, ..., p_n : x_n)$, where $x_1 > ... > x_n$. The expected value is the summed area inside the rectangles.



$$EV(x) = \sum_i p_i x_i$$

Proof of the proposition (2)

Anticipating the case of a continuous outcome range, we take the lottery with n outcomes to be equi-distant (wlog), as an approximation of a continuous distribution.
Notation caveat: x is used for both the lottery and as an outcome value.

Look at the figure "row by row" from left to right, and note that we can determine the area differently, by multiplying two things for each outcome: how much better is the outcome than the next-worse outcome $(= x_i - x_{i+1})$ and what is the probability of receiving at least x_i, which is $p_i + p_{i-1} + ... + p_1 = \sum_{j=1}^{i} p_j$.

$$EV(x) = \sum_{i=1}^{n}(\sum_{j=1}^{i} p_j)(x_i - x_{i+1}).$$

Proof of the proposition (3)

To approximate the continuous case, we let $n \to \infty$.

$$(\sum_{j=1}^{i} p_j) = \Pr(outcome \geq x_i) = 1 - \Pr(outcome < x_i)$$

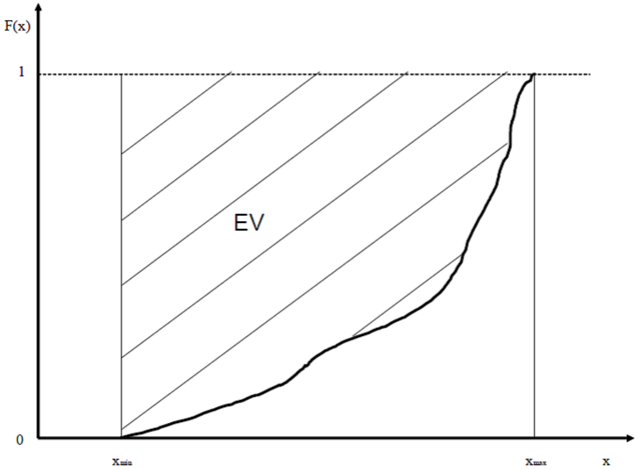
$$\to 1 - \Pr(outcome \leq x_i) = 1 - F(x_i)$$

Drop the subscript i and replace $(x_i - x_{i+1})$ by its infinitesimal analog, the differential dx. The summation over $(x_i - x_{i+1})$ becomes an integral over dx:

$$EV(x) = \int_{x_{\min}}^{x_{\max}} (1 - F(x))dx.$$

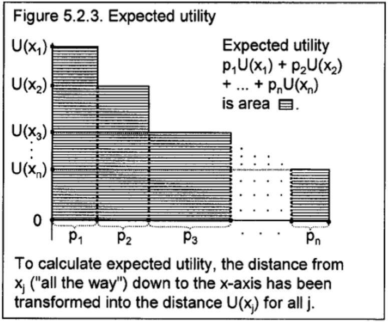
This is the area above the cdf, or the "epigraph".

Proof of the proposition (4)



Proof of the proposition (5)

Transform the x axis in the first graph (note: vertical axis) to measure U—replace each x_i in the graph by $U(x_i)$.



Proof of the proposition (6)

Analogously to the derivation of EV:

$$EU(x) = \sum_i (\sum_{j=1}^{i} p_j)(U(x_i) - U(x_{i+1}))$$

x_i's marginal contribution to U is relevant.

Again, let $n \to \infty$. The marginal U-contribution becomes dU. Since the derivative is $U'(x) = \frac{dU}{dx}$, we have $dU = U'(x)dx$.

$$EU(x) = \int_{x_{\min}}^{x_{\max}} U'(x)(1 - F(x))dx. \qquad (1)$$

The normalized EU of the distribution is the area above the cdf, but weighted according to the U-contribution of x. (For EV, think of an equal weight of 1.)

Proof of the proposition (7)

To show that 2. $\Rightarrow$ 1. : We show the counterpositive, i.e. that not 1. implies not 2. Assume that y is not first-order-stochastically dominated by x. Then there exists a value $\widetilde{\alpha}$ with $1 - F_y(\widetilde{\alpha}) > 1 - F_x(\widetilde{\alpha})$. Consider a "pseudo-EU-maximizer" who maximizes EU with a weakly increasing step function:

$$\widetilde{U}(\alpha) = 1 \text{ if } \alpha > \widetilde{\alpha}$$
$$\widetilde{U}(\alpha) = 0 \text{ otherwise}$$

Using expression (1), she chooses y over x (strictly). Moreover, one can find strictly increasing functions that are arbitrarily close to $\widetilde{U}$, and hence have the same property. That is, there exist an EU agent who chooses y and hence, 2. does not hold. ■

Lecture 6: Choosing not to choose

See presentation slides flipping_coins_slides_2013_05_31.ppt

Lecture 7: Risk preferences under expexted utility

Collecting assumptions

Structural Assumption 3.0.1 ("Decision under risk and EU"): $\succsim$ is a preference relation over the set of all probability-contingent prospects, which is the set of all finite probability distributions over the outcome set $\mathbb{R}$. Expected utility holds with a utility function U that is continuous and strictly increasing.

Risk aversion and concavity

Recall:

Risk Aversion — $E[x] \succsim x$, for all x in the domain of preference.

Risk Neutrality — $E[x] \sim x$, for all x.

Risk Seeking — $x \succsim E[x]$, for all x.

Notice that these assumptions on $\succsim$ can stand alone, e.g. without assuming EU.

Risk aversion and concavity (2)

Recall also:

$f : X \to \mathbb{R}$ is concave — $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$

$f : X \to \mathbb{R}$ is linear — $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$

$f : X \to \mathbb{R}$ is convex — $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

for all $x, y \in X$ and all $\lambda \in [0, 1]$.

Theorem 3.2.1 — Under Structural Assumption 3.0.1,

$$\text{risk aversion} \Leftrightarrow U \text{ concave}$$
$$\text{risk neutrality} \Leftrightarrow U \text{ linear}$$
$$\text{risk loving} \Leftrightarrow U \text{ convex.}$$

Figure 3.2.1

Experiment

1. Choose between

$(1 : \text{EUR } 4000)$ and $(0.5 : \text{EUR } 0, 0.5 : \text{EUR } 10000)$

2. Choose between

(1 : EUR 84000) and (0.5 : EUR 80000, 0.5 : EUR 90000)

3. Are you typically a risk averter? I.e. do you reject gambles in favor of their expected value?

Note that the answer to 3. does not determine 1. and 2.

$\rightarrow$ Need measure of risk aversion

Comparative risk aversion

4 comparisons of two preference relations $\succsim_1$ and $\succsim_2$, under Structural Assumption 3.0.1:

1. $\succsim_2$is more risk averse than $\succsim_1$ — $\alpha \sim_1 x$ implies that $\alpha \succsim_2 x$ for all lotteries x and all outcomes α

2. Risk premium of a lottery x — the distance between the lottery's expected value and the lottery's certainty equivalent, $\pi(x) = EV(x) - CE(x)$.

Exercise 3.2.3

Note: Structural Assumption 3.0.1 not required for Definitions 1. and 2.

Comparative risk aversion (2)

Let U_1 and U_2 be utility functions that represent $\succsim_1$ and $\succsim_2$ in the EU sense.

Let $\phi(u) = U_2(U_1^{-1}(u))$ describe the utility that agent $\succsim_2$ derives from the sure amount that gives agent $\succsim_1$ a utility level of u.

3. U_2 is a concave transformation of U_1 — ϕ is concave.

4. Arrow-Pratt degree of absolute risk aversion — $r_{AP}(x) = -\frac{U''(x)}{U'(x)}$.

Comparative risk aversion (3)

Proposition (see Thm 3.4.1 and Ex 3.4.1)**:** The following four statements are equivalent.

- $\succsim_2$ is more risk averse than $\succsim_1$.
- $\succsim_2$ has a higher risk premium: for all lotteries x, $\pi_2(x) \geq \pi_1(x)$.
- $\phi(u) = U_2(U_1^{-1}(u))$ is a concave transformation of U_1.
- $\succsim_2$ has a higher degree of absolute risk aversion: for all outcomes α, $r_{AP,2}(\alpha) \geq r_{AP,1}(\alpha)$.

Mean-preserving spread

(On the board.)

Constant Absolute Risk Aversion

$$U(x) = 1 - \exp(-rx)$$
$$x \in \mathbb{R}, r > 0.$$

See Figure 3.5.2.

More generally (allowing for convex functions):

$$U(x) = 1 - \exp(-rx) \text{ for } r > 0$$
$$U(x) = x \text{ for } r = 0$$
$$U(x) = \exp(-rx) - 1 \text{ for } r < 0$$

(Sometimes rescaled as $U(x) = \frac{1-\exp(-rx)}{r}$.)

As suggested by the name, it has a constant (independent of x) Arrow-Pratt degree of risk aversion:
$r_{AP}(x) = -\frac{U''(x)}{U'(x)} = -\frac{-r^2 \exp(-rx)}{r \exp(-rx)} = r$

Constant Absolute Risk Aversion (2)

Proposition: Assume Structural Assumption 3.0.1 and that the utility function is differentiable. The following are equivalent:

- $\succsim$ is represented by CARA utility.
- The preference between two lotteries (x, y) is not affected if μ is added to both lotteries, for all $\mu \in \mathbb{R}$ and all (x, y).

(Board.)

Decreasing Absolute Risk Aversion

Economists often assume that the degree of absolute risk aversion decreases with the outcome size. This has also been measured in experiments.

Preferences exhibit decreasing absolute risk aversion (DARA) if the risk premium $\pi(x)$ for any given lottery x weakly decreases if a sure payment $\mu \geq 0$ is added to the lottery, i.e. $\frac{\partial \pi(x+\mu)}{\partial \mu} \leq 0$.

We get the following characterization.
Proposition: Under EU, preferences are DARA if and only if the Arrow-Pratt degree $r_{AP}(\alpha) = -\frac{U''(\alpha)}{U'(\alpha)}$ weakly decreases in α.

Constant Relative Risk Aversion

$$U(x) = x^r$$
$$x \in \mathbb{R}^+, r \neq 0.$$

More generally:

$$U(x) = x^r \text{ for } r > 0$$
$$U(x) = \ln x \text{ for } r = 0$$
$$U(x) = -x^r \text{ for } r < 0$$

(The *ln* curve is the unique function between the cases $r > 0$ and $r < 0$.)

The function is often written as $U(x) = x^{1-\gamma}$ or $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

Lecture 8: Multiattribute utility

Multiattribute utility

Outcome spaces may be more general than $\mathbb{R}$ and/or multi-dimensional:

Multiattribute outcome set — $X = X^1 \times X^2 \times ... \times X^m$, where X^i is the ith attribute set, which may be a general set.

(Multiattribute) outcome — $\alpha = (\alpha^1, ..., \alpha^m) \in X$

Health example

Example 3.7.1, $EU(Q, T)$, momontonicity in life duration, zero condition, SG invariance, Observation 3.7.2

Probabilistic multiattribute outcomes

Probability-contingent prospects over the elements of X are defined as before:

Prospect — $x = (p_1 : x_1, ..., p_n : x_n)$, where the jth outcome is $x_j = (x_j^1, ..., x_j^m)$

EU is defined analogously, too. See Figure 3.7.2.

Marginal prospect — $(p_1 : x_1^i, ..., p_n : x_n^i)$, the probability distribution over attribute set X^i generated by x.
See Figure 3.7.3 (and note the typo: the right panel should not depict a lottery between the marginals; it should depict just the marginals).

Multiattribute risk attitudes

Consider a sure attribute $\gamma^i \in X^i$ and let $\gamma^i\alpha$ denote outcome α but with its ith attribute replaced by γ^i.

Consider attributes γ^i, δ^i and outcomes α, β with $\gamma^i\beta \succsim \gamma^i\alpha$, $\delta^i\beta \succsim \gamma^i\beta$ and $\delta^i\alpha \succsim \gamma^i\alpha$.

Consider a choice between prospects:

$$\delta^i\alpha_{0.5}\gamma^i\beta \text{ and } \gamma^i\alpha_{0.5}\delta^i\beta$$

E.g. compare (3.7.2) and (3.7.3)

Multiattribute risk attitudes (2)

Consider a sure attribute $\gamma^i \in X^i$ and let $\gamma^i\alpha$ denote outcome α but with its ith attribute replaced by γ^i.

Consider attributes γ^i, δ^i and outcomes α, β with $\gamma^i\beta \succsim \gamma^i\alpha$, $\delta^i\beta \succsim \gamma^i\beta$ and $\delta^i\alpha \succsim \gamma^i\alpha$.

Multiattribute risk aversion — $\delta^i\alpha_{0.5}\gamma^i\beta \succsim \gamma^i\alpha_{0.5}\delta^i\beta$ for all such $i, \alpha, \beta, \gamma^i, \delta^i$

Multiattribute risk seeking — $\gamma^i\alpha_{0.5}\delta^i\beta \succsim \delta^i\alpha_{0.5}\gamma^i\beta$ for all such $i, \alpha, \beta, \gamma^i, \delta^i$

Multiattribute risk neutrality — $\gamma^i\alpha_{0.5}\delta^i\beta \sim \delta^i\alpha_{0.5}\gamma^i\beta$ for all such $i, \alpha, \beta, \gamma^i, \delta^i$

Additive decomposability

Multiattribute risk neutrality says that an improvement in one attribute i is evaluated independently of the other attributes.

Proposition (see Thm 3.7.3)**:** The following three are equivalent.

(i) Multiattribute risk neutrality

(ii) $U(\alpha^1, ..., \alpha^m) = U(\alpha^1) + ... + U(\alpha^m)$

(iii) Marginal independence: Preference over prospects (x, y) depends only on the marginal prospects generated by x and y.

Identical dimensions

Anscombe and Aumann (1963) assume $X^1 = X^2 = ... = X^m = C$ (set of prizes).

Let all but the ith attribute be fixed and consider prospects over the remaining attribute i. Preference over such prospects over C will depend on i and on the level at which the other attributes are fixed.

A & A monotonicity — All preference relations over prospects over C that do not prescribe equivalence everywhere are the same.

Theorem 3.7.6 — Assume EU and $X^1 = X^2 = ... = X^m = C$. Consider the additive decomposition

$$U(\alpha^1, ..., \alpha^m) = q^1 u(\alpha^1) + ... + q^m u(\alpha^m)$$

where $u : C \to \mathbb{R}$ and $\sum_{i=1}^{m} q^i = 1$. This additive decomposition holds if and only if: (i) marginal independence and (ii) A & A monotonicity both hold.

Identical dimensions (2)

Theorem 3.7.6 — Assume EU and $X^1 = X^2 = ... = X^m = C$. Consider the additive decomposition

$$U(\alpha^1, ..., \alpha^m) = q^1 u(\alpha^1) + ... + q^m u(\alpha^m)$$

where $u : C \to \mathbb{R}$ and $\sum_{i=1}^{m} q^i = 1$. This additive decomposition holds if and only if: (i) marginal independence and (ii) A & A monotonicity both hold.

Anscombe and Aumann's interpretation of attributes: Outcome $\alpha = (\alpha^1, ..., \alpha^m)$ is a gamble on a horse race with m horses, where α^i is the prize won if horse i wins. Prospects over outcomes are prospects over gambles. q^i is the subjective probability of horse i winning.
Note that under this interpretation, marginal independence is a consistency property, similar to SG consistency.

Lecture 9: Expected utility under uncertainty

Choice experiments

Figure 4.1.1 with $cand_1 =$ Steinbrück, $cand_2 =$ Merkel, in units of EUR 1,000.00

Figure 4.1.2 with g given by the integer nearest to α^4

Figure 4.1.3

Figure 4.1.4

Figure 4.1.5

Recall previous concepts

Structural Assumption 1.2.1 ("Decision under Uncertainty"): S is a finite or infinite state space and $\mathbb{R}$ is the outcome set. Prospects map states to outcomes, taking only finitely many values. $\succsim$ is a preference relation on the set of prospects, i.e. on all such maps. Nondegeneracy holds.

Continuity — For every partition $\{E_i\}_{i=1}^n$ of S and for all prospects $y \in \mathbb{R}^n$, $y = (E_1 : y_1, ..., E_n : y_n)$, the better-than-set and worse-than-set, $\{x \in \mathbb{R}^n | x \succsim y\}$ and $\{x \in \mathbb{R}^n | y \succsim x\}$, are closed in $\mathbb{R}^n$.

EV under Structural Assumption 1.2.1: Utility known, probabilities flexible

EU under Structural Assumption 2.5.2: Utility flexible, probabilities known

Definition of EU

Expected Utility — Under Structural Assumption 1.2.1, *expected utility (EU) holds* if there exist probabilities $P(E_i)$ for all events E_i in the state space and there exists a strictly increasing function $U : \mathbb{R} \to \mathbb{R}$ that depends only on outcomes, such that

$$E_1 x_1 ... E_n x_n \to \sum_{i=1}^{n} P(E_i) U(x_i) \equiv EU(x)$$

represents $\succsim$.

(The assumption is often referred to as *Subjective Expected Utility*.)

Discussion of EU (— just like EV)

- Very convenient for analysis
- Degrees of freedom: "subjective probabilities and utilities" $P(E_i)$ and U
- Note: $P(E_i)$ and U are derived from $\succsim$.
- Will see shortly: Maximization is equivalent to consistency of $\succsim$.
- Black box: an as-if construction. Realism?
- Normatively useful?

Predicting choices

Exercise 4.2.1

Eliciting subjective parameters

Exercise 4.2.3

Getting used to EU

Exercise 4.2.5

$\alpha_E x$ — A prospect that yields α if $s \in E$ and yields $x(s)$ otherwise.

E is null — $\alpha_E x \sim \beta_E x$ for all prospects x and all outcomes α, β
— E is *nonnull* otherwise.

Exercise 4.2.6

Exercise 4.2.7

Using your experimental choices

Excercise 4.3.1: Consider Figure 4.1.1, with $\alpha^0 = 10$. Show that the assumption of EU implies that $U(\alpha^k) - U(\alpha^{k-1})$ is constant in k.

Figure 4.3.1, Figure 4.3.2

Note that we can measure U precisely with this method, hence also measure P, e.g. using standard gambles: for given E, select M, m, α such that

$$U(\alpha) = P(E)U(M) + (1 - P(E))U(m)$$
$$P(E) = U(\alpha)$$

Consistency under EU

Excercise 4.3.2

Exercise 4.3.3

Note: The predictions hold under more general assumptions than EU.

Exercise 4.3.4:
Do not assume EU but only weak ordering and strong monotonicity ($x \succ y$ if $x \geq y$ and $\exists s$ with $x(s) > y(s)$).

Exercise 4.3.5

Choices in Figure 4.1.1 — in slow motion

Consider Figure 4.1.1 (a) and (d)

$$\alpha_E^1 1 \sim \alpha_E^0 8$$
$$\alpha_E^4 1 \sim \alpha_E^3 8$$

$8 \ominus 1$ — "Receiving 8 instead of 1"

Conditional an some event (here, E^c), $8 \ominus 1$ reflects the preference value of receiving the right prospect. This value depends on the utility difference between 8 and 1 and on the likelihood of E^c.

$\alpha^1 \ominus \alpha^0$ contingent on E exactly offsets $8 \ominus 1$ contingent on E^c.

$\alpha^4 \ominus \alpha^3$ contingent on E exactly offsets $8 \ominus 1$ contingent on E^c.

We write this as

$$\alpha^1 \ominus \alpha^0 \sim_t \alpha^4 \ominus \alpha^3$$

Definition of t-indifference

Consider general prospects x, y, events E and outcomes $\alpha, \beta, \gamma, \delta$, and indifferences:

$$\alpha_E x \sim \beta_E y$$

and

$$\gamma_E x \sim \delta_E y$$

$\alpha \ominus \beta \sim^t \gamma \ominus \delta$ ("t-indifference" for $\alpha, \beta, \gamma, \delta$) — There exist prospects x, y and a nonnull event E such that the two above-listed indifferences hold.

Figure 4.5.1, Example 4.5.2

Lecture 10: Expected utility under uncertainty (2)

Definition of t-indifference

Consider general prospects x, y, events E and outcomes $\alpha, \beta, \gamma, \delta$, and indifferences:

$$\alpha_E x \sim \beta_E y$$

and

$$\gamma_E x \sim \delta_E y$$

$\alpha \ominus \beta \sim^t \gamma \ominus \delta$ ("t-indifference" for $\alpha, \beta, \gamma, \delta$) — There exist prospects x, y and a nonnull event E such that the two above-listed indifferences hold.

t-indifference and EU

Exercise 4.5.3: Show that under EU,

$$\alpha \ominus \beta \sim^t \gamma \ominus \delta \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$$

Proof: Under EU, the indifferences are

$$\alpha_E x \sim \beta_E y$$
$$\gamma_E x \sim \delta_E y$$

$$P(E)U(\alpha) + \sum_{s_j \notin E} P(s_j)U(x_j) = P(E)U(\beta) + \sum_{s_j \notin E} P(s_j)U(y_j)$$
$$P(E)U(\gamma) + \sum_{s_j \notin E} P(s_j)U(x_j) = P(E)U(\delta) + \sum_{s_j \notin E} P(s_j)U(y_j)$$

$$U(\alpha) - U(\beta) = \frac{1}{P(E)} \sum_{s_j \notin E} P(s_j)(U(y_j) - U(x_j))$$
$$U(\gamma) - U(\delta) = \frac{1}{P(E)} \sum_{s_j \notin E} P(s_j)(U(y_j) - U(x_j)) \blacksquare$$

t-indifference and EU (2)

Suppose that in addition to $\alpha \ominus \beta \sim^t \gamma \ominus \delta$ we observe that $\alpha' \ominus \beta \sim^t \gamma \ominus \delta$ with $\alpha > \alpha'$ (see Example 4.6.1).

Not under EU (by the result of Exercise 4.5.3).

Tradeoff consistency — Strictly improving an outcome in any t-indifference breaks that indifference.

EU representation theorem ($\sim$ Savage)

Theorem 4.6.4 — Under Structural Assumption 1.2.1, the following two statements are equivalent.

1. EU holds with continuous and strictly increasing $U(\cdot)$.

2. $\succsim$ satisfies weak ordering, monotonicity, continuity, and tradeoff consistency.

Observation 4.6.4': Moreover: in (2), the probabilities P are uniquely determined and utility U is unique up to positive affine transformations.

Discussion of Theorem 4.6.4

- Notice the same nature as in vN-M's theorem: Consistency and some technical conditions are equivalent to EU.
- Surprising that we need no stronger conditions in (2) to obtain a consistent P measure in (1).
- The case of decision under risk is included in Structural Assumption 1.2.1. The "tradeoff method" of stating consistency works also for this domain of preferences. See Figure 4.7.1.
- Note again the degrees of freedom—e.g. allowing for arbitrary beliefs.

Proof of the theorem

We saw (or will see on PS) that 1. (EU) implies 2. For (a sketch of) the proof of 2.⇒1., we restrict to some simplifying assumptions:

– a two-element state space $S = \{E_1, E_2\}$ ($\rightarrow (x_1, x_2)$ denotes general prospects)
– a particular but arbitrary preference ratio, as specified below ($\alpha^1 - \alpha^0 = \beta^3 - \beta^0$)
– For all $(x_1, x_2) \succ (y_1, y_2)$, there exists a large enough y_1' such that $(x_1, x_2) \sim (y_1', y_2)$, and analogously for y_2.
– Strong monotonicity: If $(x_1, x_2) \geq (y_1, y_2)$ and $(x_1, x_2) \neq (y_1, y_2)$, then $(x_1, x_2) \succ (y_1, y_2)$.

Proof of the theorem (2)

Assume that property 2. holds, and construct the EU function as follows.
Fix a small outcome $\alpha^0 = \beta^0$ and a larger outcome α^1. Define outcome β^1 by requiring

$$(\alpha^1, \beta^0) \sim (\alpha^0, \beta^1).$$

Now fix β^0, β^1 and define $\{\alpha^{i+1}\}_{i=1}^{\infty}$ recursively by

$$(\alpha^{i+1}, \beta^0) \sim (\alpha^i, \beta^1).$$

Likewise, fix α^0, α^1 and define $\{\beta^{j+1}\}_{j=1}^{\infty}$

$$(\alpha^1, \beta^j) \sim (\alpha^0, \beta^{j+1}).$$

[See Figure 4.15.1]

Proof of the theorem (3)

We constructed sequences such that

$$\alpha^{i+1} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \quad (1.)$$

and

$$\beta^{j+1} \ominus \beta^{j} \sim^{t} \beta^{1} \ominus \beta^{0} \quad (1'.)$$

for all i, j. For arbitrary $i, j > 0$, consider the indifference

$$(\alpha^{1}, \beta^{j}) \sim (\alpha^{0}, \beta^{j+1}) \quad (2.)$$

and modify the RHS by replacing α^{0} by $\alpha^{i} > \alpha^{0}$. Strong monotonicity implies

$$(\alpha^{1}, \beta^{j}) \prec (\alpha^{i}, \beta^{j+1}).$$

But for some large enough α^{*} we have

$$(\alpha^{*}, \beta^{j}) \sim (\alpha^{i}, \beta^{j+1}). \quad (3.)$$

(2.) and (3.) together imply the t-indifference:

$$\alpha^{*} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \quad (4.)$$

Proof of the theorem (4)

$$\alpha^{i+1} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \tag{1.}$$

$$(\alpha^{*}, \beta^{j}) \sim (\alpha^{i}, \beta^{j+1}). \tag{3.}$$

$$\alpha^{*} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \tag{4.}$$

(1.) and (4.) imply, by tradeoff consistency, that $\alpha^{*} = \alpha^{i+1}$. Using (3.) we therefore know

$$(\alpha^{i+1}, \beta^{j}) \sim (\alpha^{i}, \beta^{j+1}).$$

Decreasing one superscript by 1 and increasing the other by 1 does not change the preference value of a prospect (α^{i}, β^{j}). Repeated application shows that decreasing one superscript by any $k \in \mathbb{N}$ and increasing the other by k does not change the preference value.

Proof of the theorem (5)

Therefore, defining $V_1(\alpha^i) = i$ and $V_2(\beta^j) = j$, the function

$$V_1(\alpha^i) + V_2(\beta^j)$$

represents $\succsim$ over all constructed prospects (α^i, β^j). Now consider a small stepsize: choosing $\alpha^0 = \beta^0$ sufficiently small and α^1 sufficiently close to α^0 ensures that an arbitrarily dense and wide set of prospects on $\{E_1, E_2\}$ can be covered. Continuity expands the representation to all prospects on $\{E_1, E_2\}$.

But this representation does not yet have the right form. To arrive at EU representation, we need to find subjective probabilities $P(E_1)$ and $P(E_2)$ and a function $U : \mathbb{R} \to \mathbb{R}$ such that

$$V_1(\alpha^i) = P(E_1)U(\alpha^i)$$

and

$$V_2(\beta^j) = P(E_2)U(\beta^j).$$

Proof of the theorem (6)

The size ratio of the α's and β's determines the probabilities. Assume (arbitrarily) that $\alpha^1 - \alpha^0 = \beta^3 - \beta^0$.

Since $V_1(\alpha^i) + V_2(\beta^j) = i + j$ represents preference, we see that starting from (α^0, β^0) a step of size $\alpha^1 - \alpha^0$ in β-direction increases utility $(V_1 + V_2)$ by three times as much as a step of the same size in α-direction.

This suggests that E_2 is three times as likely as E_1.

Tradeoff consistency ensure that this reasoning is true (i.e. leads to the uniquely possible probabilities)—see next slides.

Proof of the theorem (7)

From $(\alpha^0, \beta^6) \sim (\alpha^3, \beta^3)$ and $(\alpha^0, \beta^3) \sim (\alpha^3, \beta^0)$ we obtain

$$\beta^6 \ominus \beta^3 \sim^t \beta^3 \ominus \beta^0.$$

Substituting $\alpha^1 = \beta^3$ and $\alpha^0 = \beta^0$,

$$\beta^6 \ominus \alpha^1 \sim^t \alpha^1 \ominus \alpha^0.$$

Because also

$$\alpha^2 \ominus \alpha^1 \sim^t \alpha^1 \ominus \alpha^0,$$

tradeoff consistency implies $\beta^6 = \alpha^2$.

Applying the same argument recursively gives $\beta^{3i} = \alpha^i$ for all i.

Proof of the theorem (8)

Once again, observe that because $V_1 + V_2 = i + j$ represents $\succsim$, a step of any size $(\alpha^i - \alpha^0)$ in β-direction increases utility by three times as much as a step of the same size in α-direction. That is,

$$3V_1(\alpha^i) = V_2(\alpha^i) \text{ or, equivalently, } V_1(\alpha^i) = \frac{1}{3}V_2(\alpha^i).$$

Preferences over (α^i, β^j) are thus represented by:

$$\frac{1}{3}V_2(\alpha^i) + V_2(\beta^j)$$

Compare this to the EU function:

$$EU = P(E_1)U(\alpha^i) + P(E_2)U(\beta^j)$$

Both are weighted sums. But we need more, namely that EU represents the same preferences, i.e. for all $(\alpha^i_1, \beta^j_1), (\alpha^i_2, \beta^j_2)$,

$$P(E_1)U(\alpha^i_1) + P(E_2)U(\beta^j_1) \geq P(E_1)U(\alpha^i_2) + P(E_2)U(\beta^j_2)$$

$$\Leftrightarrow$$

$$\frac{1}{3}V_2(\alpha^i_1) + V_2(\beta^j_1) \geq \frac{1}{3}V_2(\alpha^i_2) + V_2(\beta^j_2)$$

Proof of the theorem (9)

There exists exactly one possibility to achieve this, namely the combination of (i) and (ii) as follows. (i) The weights have to be identical

$$P(E_1) = \frac{1}{4} \text{ and } P(E_2) = \frac{3}{4},$$

(otherwise one can find two pairs $(\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$ that are differently ranked by the two functions), and (ii) U provides the same ordering of $\mathbb{R}$ as V_2, i.e.

$$U(\cdot) = \frac{4}{3} V_2(\cdot)$$

or positive affine transformations thereof. (Again because otherwise $\exists (\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$ that are differently ranked by the two functions.) Overall we have shown that the function

$$EU = \frac{1}{4}\frac{4}{3} V_2(x_1) + \frac{3}{4}\frac{4}{3} V_2(x_2)$$

represents $\succsim$ over prospects on (E_1, E_2) and that only positive affine transformations $U(\cdot) = \frac{4}{3} V_2(\cdot)$ preserve the EU form. ■

Hybrid case I

In many choice contexts, we have objective probabilities for some events R but not for general events E.

Structural Assumption 4.9.1 ("Uncertainty plus EU-for-risk"): Structural Assumption 1.2.1 (decision under uncertainty) holds. In addition, for some of the events, notated as *probabilized events* R, a probability $P(R)$ is given. If, for an event-contingent prospect $R_1 : x_1, ..., R_n : x_n$, all outcome events are probabilized with $P(R_j) = p_j$, then this prospect generates a probability distribution $p_1 : x_1, ..., p_n : x_n$ (a probability-contingent prospect) over the outcomes. All event-contingent prospects that generate the same probability-contingent prospect are preference equivalent. Preferences over probability-contingent prospects satisfy EU.

Hybrid case I (2)

To make the probabilized events comparable to the others, look for a suitable $P(R)$:

Matching probability of E — q is a probability such that $1_E 0 \sim 1_q 0$.

Matching probabilities may or may not exist under Structural Assumption 4.9.1.

Existence and additivity of matching probabilities — For all disjoint events E_1, E_2, there exist matching probabilities q_1, q_2 that further satisfy $1_{E_1 \cup E_2} 0 \sim 1_{q_1 + q_2} 0$.

(Note the different property name "addivity" on p. 120.) See Figure 4.9.2.

Hybrid case I (3)

Another consistency, relating to complex prospects:

Probabilistic matching — For each partition $E_1, \ldots, E_n$, the indifference

$$E_1 : x_1, \ldots, E_n : x_n \sim q_1 : x_1, \ldots, q_n : x_n$$

holds for all outcomes x_j whenever $\{q_j\}_j$ are the matching probabilities of events $\{E_j\}_j$.

See Figure 4.9.3.

Theorem 4.9.4 — Under Structural Assumption 4.9.1, the following two statements are equivalent.

1. EU holds.

2. $\succsim$ satisfies weak ordering, existence and additivity of matching probabilities, and probabilistic matching.

Lecture 11: Probability weighting under risk

Motivation

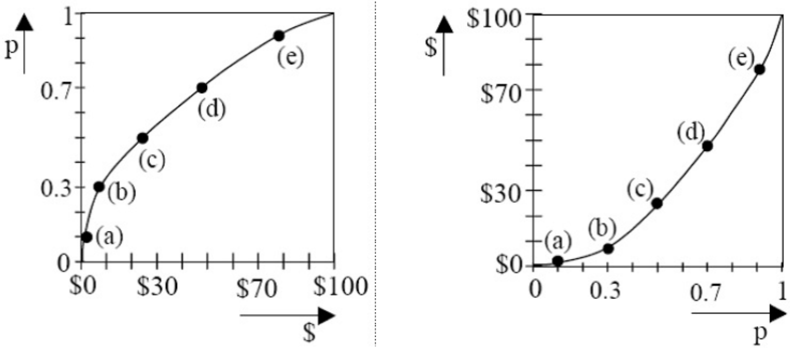
Back to Structural Assumption 2.5.2 (risk).

Is EU's linearity in probabilitites is a reasonable way to organize choices?

- Why should attitudes towards lotteries be determined solely through attitudes towards sure outcomes?
- Decision-makers often pay extra attention to small probabilities.
- For small gambles, a smooth U is close to linear, contradicting risk aversion vis-a-vis small gambles.

Motivation (2): Example

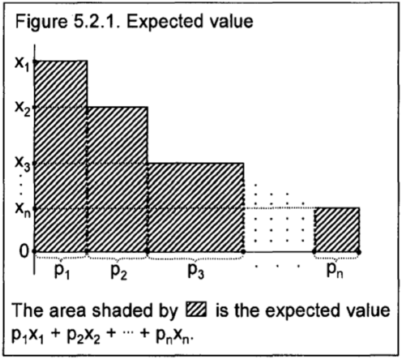
Non-linearity of U was a modelling choice that we made. Consider preferences in Figure 5.1.1.



The convex shape is akin to arguing that the decisionmaker dislikes lotteries: each probability p of receiving the high outcome lies below the p-weighted average of receiving the sure outcomes.

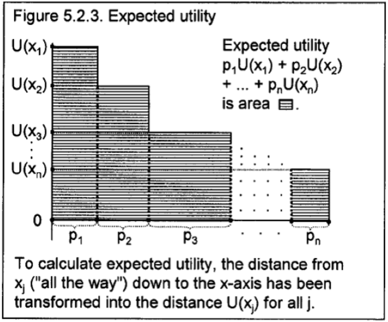
Transforming the probability axis

For lotteries with $n > 2$ outcomes, we have to be careful how to transform probabilities.
Recall the transformation from EV to EU:



Transforming the outcome axis, the height of each column in the integral was changed according to $U : x \to U(x)$—see next slide.

Transforming the probability axis (2)

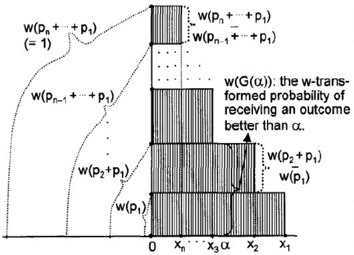


$$EU(x) = \sum_{i=1}^{n} (\sum_{j=1}^{i} p_j)(U(x_i) - U(x_{i+1}))$$

Now, instead transform the probability axis: change the length of each "row" in the integral, and swap axes.

Transforming the probability axis (3)

Figure 5.4.1. Rank-dependent utility with linear utility



Distances of endpoints of layers ("all the way") down to the x-axis are transformed, similar to Figure 5.2.3. The endpoint of the last layer now remains at a distance 1 from the x-axis, reflecting normalization of the bounded probability scale.

Figure 5.5.2 in the book (not 5.4.1). The transformation assigned non-constant weights to cumulative probabilities.

Rank of outcome x_i — The probability of receiving strictly more than x_i: $p_{i-1} + \ldots + p_1 = \sum_{j=1}^{i-1} p_j$, for $x_1 \geq x_2 \geq \ldots \geq x_n$.

A formulaeic analogue to EU

Consider again the transformation from EV to EU.

$$EV(x) = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} (\sum_{j=1}^{i} p_j)(x_i - x_{i+1}).$$

Consider $\sum_{i=1}^{n} p_i x_i$ as a summation from the worst outcome (n) to the best outcome (1). Stepping from $i+1$ and i, we ask: 'What does outcome i add to the sum?' (It adds a column in Figure 5.2.1.)

Notice that p_i and x_i have 'different roles' in this change of indices:absolute (x_i measures the distance from 0) versus marginal (p_i).

A formulaeic analogue to EU (2)

Now consider the equivalent expression $\sum_{i=1}^{n}(\sum_{j=1}^{i} p_j)(x_i - x_{i+1})$. Here, $(x_i - x_{i+1})$ is the marginal increase in outcome, and $(\sum_{j=1}^{i} p_j)$ is the (absolute) rank of outcome $i+1$.

We saw in Lecture 5, when transforming $x \to U(x)$: It is equivalent to apply the EU transformation $U : x \to U(x)$ to the absolute value x_i, or to replace the marginal x contribution of outcome i by its marginal U contribution.

$$
\begin{aligned}
EU(x) &= \sum_{i=1}^{n} p_i U(x_i) \\
&= \sum_{i=1}^{n}(\sum_{j=1}^{i} p_j)(U(x_i) - U(x_{i+1}))
\end{aligned}
$$

To transform the probability axis, we do the same but in reverse roles.

A formulaeic analogue to EU (3)

$w : [0,1] \to [0,1]$ is a probability weighting function — w is strictly increasing and satisfies $w(0) = 0$ and $w(1) = 1$.

We apply w to transform ranks:

$$w : \sum_{j=1}^{i-1} p_j \to w(\sum_{j=1}^{i-1} p_j).$$

If $w(p) = p$, then $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j) = p_i$.

Now construct an expression where the w axis has the marginal role. The marginal w contribution of outcome i is $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)$. (See Figure 5.5.2.)

A formulaeic analogue to EU (4)

Rank-dependent preferences with linear utility — Preferences are represented by

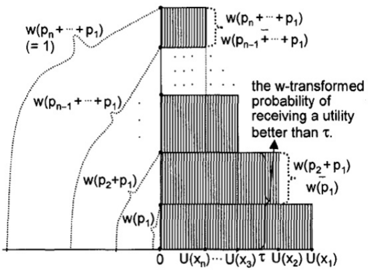
$$RDLU(x) = \sum_{i=1}^{n} [w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)] x_i$$

Another reason that we do not simply transform p, but rather transform ranks, is that the model with transformed p violates first-order stochastic dominance.

A formulaeic analogue to EU (5)

Final step: both transformations at once, of outcomes and ranks.

Figure 5.4.2. Rank-dependent utility with general utility



For points on the y-axis ("endpoints of layers"), their distances down to the x-axis are transformed using w. For points on the x-axis ("endpoints of columns"), their distances leftwards to the y-axis are transformed using U.

A formulaeic analogue to EU (6)

Rank-dependent utility — Under Structural Assumption 2.5.2, *rank-dependent utility* (RDU) holds if there exist a strictly increasing utility function $U : \mathbb{R} \to \mathbb{R}$ and a probability weighting function w such that preferences over lotteries $(p_1 : x_1, ...p_n : x_n)$ with rank-ordered outcomes $x_1 \geq ... \geq x_n$ are represented by

$$RDU(x) = \sum_{i=1}^{n} [w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)] U(x_i).$$

Remarks on RDU

- *RDU* is sometimes written as

$$\begin{aligned} RDU(x) &= \sum_{i=1}^{n} \pi_i U(x_i) \text{ where} \\ \pi_i &= w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j). \end{aligned}$$

  Importantly, note that the "decision weight" π_i is a function of all $p_j, j = 1...i$.

- For the best outcome x_1, the formula requires that we find the expression $\sum_{j=1}^{1-1} p_j$. We use the notational convention that $\sum_{j=1}^{0} p_j = 0$.

Remarks on RDU (2)

- For the worst outcome x_n, we use the weighting function's boundary restriction $w(1) = 1$: $w(\sum_{j=1}^{n} p_j) = w(1) = 1$
- If outcomes are not rank-ordered $(x_1 \geq ... \geq x_n)$ we simply re-label them to ensure rank-ordering. Under the assumption that preferences respond only to the distribution over money (see Assumption 2.1.2) this is wlog.

Example

See Section 5.6

Lecture 12: Probability weighting under risk (2)

Recall

Ranked probability p^r — A pair (p, r) where p is the probability of an outcome and r is its rank, in a given prospect.

In RDU, the decision weight depends on both p and r:

$$RDU(x) = \sum_{i=1}^{n} \pi_i U(x_i)$$
$$= \sum_{i=1}^{n} \pi(p_i^{(p_{i-1}+...+p_1)}) U(x_i)$$
$$= \sum_{i=1}^{n} (w(p_i + ...p_1) - w(p_{i-1} + ... + p_1)) U(x_i)$$

Optimism and Pessimism

Figures 6.3.1 and 6.3.2

Pessimism — Worsening the rank increases the decision weight, i.e. $\pi(p^{r'}) \geq \pi(p^r)$ whenever $r' \geq r$.

Optimism — Improving the rank increases the decision weight, i.e. $\pi(p^{r'}) \geq \pi(p^r)$ whenever $r' \leq r$.

w is convex — $w(p + r') - w(r') \geq w(p + r) - w(r)$ whenever $r' \geq r$.
w is concave — $w(p + r') - w(r') \leq w(p + r) - w(r)$ whenever $r' \geq r$.

Observation: Under RDU, pessimism holds iff w is convex.

Proof: Plug the definition of π into the definition of optimism and optimism. ■

Typical w

Figure 6.1.1

Behavioral foundation of RDU

Consider Figure 4.1.1 again and make the assumption that Steinbrück wins with probability 0.5.

We can assume that Structural Assumption 2.5.2 holds for this example and investigate RDU's prediction.

$$\pi(0.5^0)U(\alpha^1) + \pi(0.5^{0.5})U(1) = \pi(0.5^0)U(\alpha^0) + \pi(0.5^{0.5})U(8)$$
$$\Leftrightarrow \pi(0.5^0)(U(\alpha^1) - U(\alpha^0)) = \pi(0.5^{0.5})(U(8) - U(1))$$

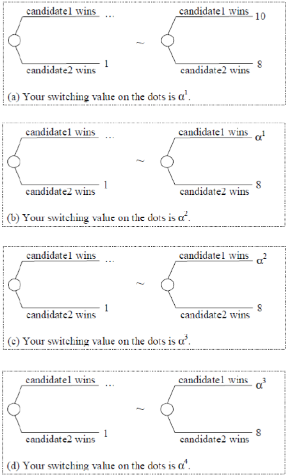
Analogously,

$$\pi(0.5^0)(U(\alpha^2) - U(\alpha^1)) = \pi(0.5^{0.5})(U(8) - U(1))$$
$$\pi(0.5^0)(U(\alpha^3) - U(\alpha^2)) = \pi(0.5^{0.5})(U(8) - U(1))$$
$$\pi(0.5^0)(U(\alpha^4) - U(\alpha^4)) = \pi(0.5^{0.5})(U(8) - U(1))$$

$\rightarrow$ U can be measured under RDU.

Behavioral foundation of RDU (2)

FIGURE 4.1.1. Eliciting $\alpha^1 \ldots \alpha^4$ for unknown probabilities



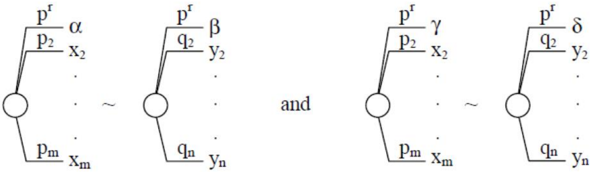
Indicate in each Fig. which outcome on the dots ··· makes the two prospects indifferent (the switching value).

Behavioral foundation of RDU (3)

$8 \ominus 1$ — "Receiving 8 instead of 1"

Conditional an some probabilized event (here, candidate 2 wins), $8 \ominus 1$ reflects the preference value of receiving the right prospect. This value depends on the utility difference between 8 and 1 and on the decision weight of the event.

$\alpha \ominus \beta \sim_c^t \gamma \ominus \delta$ — The indifferences in Figure 6.5.1 hold for some outcome probability p and some rank r and some prospects x, y.



The superscript r indicates the rank of p, which is the same for all prospects.

Behavioral foundation of RDU (4)

Observation 6.5.3: Under RDU,
$\alpha \ominus \beta \sim_c^t \gamma \ominus \delta \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$

Proof: (We consider only the case where $\alpha \neq x_i$ for all i, and similarly for β, γ, δ. See the book for the more general case.)
The two indifferences are, under RDU,

$$\pi(p^r)U(\alpha) + \sum_{i=2}^{m} \pi_i U(x_i) = \pi(p^r)U(\beta) + \sum_{j=2}^{n} \pi_j U(y_j)$$

$$\pi(p^r)U(\gamma) + \sum_{i=2}^{m} \pi_i U(x_i) = \pi(p^r)U(\delta) + \sum_{j=2}^{n} \pi_j U(y_j)$$

$$\pi(p^r)(U(\alpha) - U(\beta)) = \sum_{j=2}^{n} \pi_j U(y_j) - \sum_{i=2}^{m} \pi_i U(x_i) = \pi(p^r)(U(\gamma) - U(\delta))$$

$w' > 0 \Rightarrow \pi(p^r) > 0 \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$. ■

Behavioral foundation of RDU (5)

It could be that

$$\alpha \ominus \beta \sim_c^t \gamma \ominus \delta$$

and

$$\alpha' \ominus \beta \sim_c^t \gamma \ominus \delta$$

for $\alpha' \neq \alpha$.

Not under RDU, by observation 6.5.3.

Rank-tradeoff consistency —Improving an outcome in any $\sim_c^t$ relationship breaks the relationship.

A variant of monotonicity is also implied by RDU:

Strict stochastic dominance — Shifting positive probability mass from an outcome to a strictly preferred outcome leads to a strictly preferred outcome.

Behavioral foundation of RDU (6)

Theorem 6.5.6 — Under Structural Assumption 2.5.2, the following two statements are equivalent.

1. RDU holds with continuous and strictly increasing $U(\cdot)$.

2. $\succsim$ satisfies weak ordering, strict stochastic dominance, continuity, and rank-tradeoff consistency.

(No proof.)

Measuring w

(See also Sections 6.4, 7.1 and 7.2)

Exercise 6.5.6

(First redo Figure 4.1.1 with 50/50 probabilites, then Figure 4.1.5.)

Likelihood insensitivity

Likelihood insensitivity assings high decision weight to the tails of the outcome distribution. It may be due to cognitive rather than motivational factors.

RDU can combine it with pessimism. See Figures 7.1.2a and 7.1.2b. Also, see Figure 7.2.4 for a simple version.

w is likelihood insensitive with insensitivity region $[b_{rb}, w_{rb}]$ — The boundaries b_{rb} (best-rank boundary) and w_{rb} (worst-rank boundary) delimit an intermediate region of ranks where the decision weights are smaller than for best-ranked probabilities and worst-ranked probabilities:

$$w(p) - w(0) \geq w(p + r) - w(r) \text{ if } r + p \leq w_{rb}$$

and

$$w(1) - w(1 - p) \geq w(r + p) - w(r) \text{ if } r \geq b_{rb}$$

See Figure 7.7.1'

Loss ranks

Recall Rank of outcome x_i — The probability of receiving strictly more than x_i: $p_{i-1} + ... + p_1 = \sum_{j=1}^{i-1} p_j$, for $x_1 \geq x_2 \geq ... \geq x_n$.

Loss rank of outcome x_i — The probability of receiving strictly less than x_i: $p_{i+1} + ... + p_n = \sum_{j=i+1}^{n} p_j$.

Loss-ranked probability p_l — A pair (p, l) where p is the probability of an outcome and l is its loss-rank, in a given prospect.

Consider a weighting function z for loss ranks and decision weights

$$\pi(p_l) = z(p + l) - z(l),$$

the marginal contribution of the outcome to the loss-rank. RDU can be re-written as

$$\sum_{i=1}^{n}(z(p_i + ... + p_n) - z(p_{i+1} + ... + p_n))U(x_i),$$

Loss ranks (2)

Should/can we set $z = w$?

Only if we are willing to assume that w is symmetric:

$$w(p) = 1 - w(1 - p)$$

We may not be willing to do so. (And why should we, considering that w was defined for (gain-)ranks not loss-ranks?)

But we can use the above natural alternative notation if defining z as the dual weighting function of w:

$$z(p) = 1 - w(1 - p)$$

Lecture 13: Prospect theory under risk

Gains and losses

Figure 8.1.1a

Figure 8.1.1b

Notice that RDU or EU need to change their components if choice differs between a and b.

Asset integration versus narrow bracketing

Figure 8.1.1c

Isolation / mental accounting / narrow bracketing implies that choice in b and c are identical.
Note that this is in accordance with additivity, which in turn is equivalent to freedom from arbitrage (de Finetti).

But problems a and c are the same if things are added – asset integration. The same consumption possibilities exist iff choice is identical between a and c.

Most discussions argue for asset integration as the only rational (or normatively sound) principle. Wakker (Ch. 8.2): the problem are the risk attitudes.

Loss aversion

The reference point may be viewed as the point where risk attitudes change discontinuously.
A separate role: Utilities from sure outcomes are also evaluated differently: losses loom larger than gains—loss version.
E.g. with a reference point of 0, a given function u satisfying $u(0) = 0$, and $\lambda \in \mathbb{R}_+$:

$$U(\alpha) = u(\alpha) \text{ for } \alpha \geq 0$$
$$U(\alpha) = \lambda u(\alpha) \text{ for } \alpha < 0$$

Loss aversion — Preferences are represented by RDU with the above utility function and $\lambda > 1$.

Candidates for reference point:
(i) Status quo / initial wealth, and choice is framed as choice between changes in wealth
(ii) Expectation (See e.g. Koszegi/Rabin 2005, 2006)

Reference dependence is more than a fixed initial wealth

Figure 8.1.1 questions that the reference point is known and fixed.

Most of decision theory views the reference point as fixed, for the purpose for the present analysis.

Rabin (2000). Choose between 0 and $11_{0.5}(-10)$, for different wealth levels.
Consistently rejecting the lottery implies that that U is concave to an absurd extent.

Prospect theory — overview

Prospect theory (Tversky/Kahneman 1992) combines three elements that we studied: utility curvature (diminishing outcome sensitivity), probabilistic sensitivity and loss aversion.

For a fixed reference point (which is a gross simplification that may or may not be misleading) PT is almost the same as RDU, with the exception that it uses two weighting functions: one for gains, one for losses.

PT involves symmetry/reflection around the reference point: diminishing outcome sensitivity in gains and losses, and decision weights that depend on the reference point.

Prospect theory — formal

For a given prospect $p_1x_1 ... p_nx_n$, assign labels $1...n$ and identify k to satisfy the complete sign-ranking:

$$x_1 \geq ... \geq x_k \geq 0 \geq x_{k+1} \geq ... \geq x_n$$

Consider a weighting function w^+ that is applied only to outcomes $x_{k+1}, ..., x_n$ by weighting their gain-ranks, and another weighting function w^- that is applied to outcomes $x_1, ..., x_k$ by weighting their loss ranks. Decision weights are:

$$\pi_i = \pi(p_i^{p_{i-1}+...+p_1}) = w^+(p_i + ... + p_1) - w^+(p_{i-1} + ... + p_1)$$

for $i \leq k$, and

$$\pi_j = \pi(p_{j_{p_{j+1}+...+p_n}}) = w^-((p_j + ... + p_n) - w^-(p_{j+1} + ... + p_n)$$

for $j > k$.

Prospect theory — formal (2)

Prospect theory — Under Structural Assumption 2.5.2, *prospect theory* (PT) holds if there exist a strictly increasing utility function $U : \mathbb{R} \to \mathbb{R}$ with $U(0) = 0$ and two probability weighting functions w^+ and w^- such that preferences over lotteries $(p_1 : x_1, \ldots p_n : x_n)$ with completely sign-ranked outcomes $x_1 \geq \ldots \geq x_k \geq 0 \geq x_{k+1} \geq \ldots \geq x_n$ for some $k \in \{1, \ldots, n\}$ are represented by

$$PT(x) = \sum_{i=1}^{k} \pi(p_i^{p_{i-1}+\ldots+p_1}) U(x_i) + \sum_{j=k+1}^{n} \pi(p_{j_{p_{j+1}+\ldots+p_n}}) U(x_j),$$

where $\pi(p_i^{p_{i-1}+\ldots+p_1}) U(x_i)$ and $\pi(p_{j_{p_{j+1}+\ldots+p_n}})$ are given on the previous slide.

Calculating the prospect theory value

See pages 255-256.

Typical U, w^+ and w^-

Figures 8.4.1, 7.1.2b.
For losses, preferences are often risk seeking but closer to risk neutrality than for gains.

PT can acount for the typical pattern of (experimental) findings:

- risk averison for medium- and high-probability gains
- risk seeking for medium- and high-probability losses
- risk seeking for small-probability gains
- risk averison for small-probability losses
- $\lambda > 1$

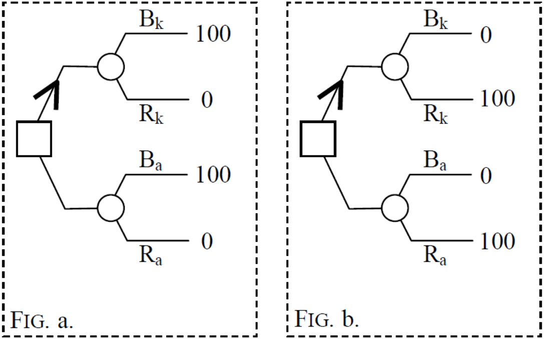
Remarks on prospect theory

- Exercise 9.3.2: For a given prospect x define x^+ as the prospect that replaces all of x's negative outcomes by 0, and x^- as the prospect that replaces all of x's positive outcomes by 0. Show that $PT(x) = PT(x^+) + PT(x^-)$.
- For $x = x^+$, PT coincides with RDU with $w(p) = w^+(p)$.
- For $x = x^-$, PT coincides with RDU with $w = 1 - w^-(1 - p)$.
- Exercise 9.3.3: The decision weights need not sum to 1.

Lecture 14: Ambiguity preferences

The Ellsberg Paradox

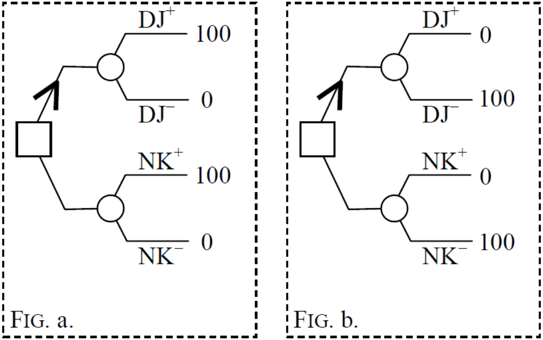
FIGURE 10.1.1. Ellsberg paradox



Arrows indicate majority preferences.

The Ellsberg Paradox (2)

FIGURE 10.1.2. Home Bias



Arrows indicate majority preferences in the United States.

The Ellsberg Paradox (3)

Source — A set of events.

Source preference — For all events A from source $\mathbb{A}$ and all events B from source $\mathbb{B}$, it may be that

$$1_A 0 \succsim 1_B 0 \text{ and } 1_{A^c} 0 \succsim 1_{B^c} 0$$

but it cannot be that

$$1_B 0 \succsim 1_A 0 \text{ and } 1_{B^c} 0 \succsim 1_{A^c} 0.$$

Probabilistic sophistication — There exists a probability measure P on S such that each event-contingent prospect is evaluated according to its corresponding probability-contingent prospect.

The Ellsberg example shows a source preference and violates probabilistic sophistication.

Overview of RDU under uncertainty

Decision weights for EU under uncertainty: events are assigned (additive) probabilites $P(E)$
Decision weights for RDU under risk: ranked probabilities are assigned $w-$transformed weights

Decision weights RDU under uncertainty: ranked events are assigned (non-additive) $W-$transformed weights.

See p. 279 on piecing together and surprising lack of surprise.

Event weights

Under Structural Assumption 1.2.1, consider a prospect $x = (E_1 : x_1, ...E_n : x_n)$, where outcomes are rank-ordered, $x_1 \geq ... \geq x_n$.

Rank of outcome x_j — The event of receiving an outcome strictly better than x_j, denoted by $R = E_{j-1} \cup ... \cup E_1$.

Ranked event — E^R, a pair (E, R) where R is event E's rank.

Event weighting function — $W : 2^S \to [0, 1]$ is a weighting function if $W(\emptyset) = 0$, $W(S) = 1$ and $[A \supset B \Rightarrow W(A) \geq W(B)]$.

Decision weight $\pi(E^R)$ — The W-contribution of event E to the rank: $\pi(E^R) = W(E \cup R) - W(R)$.

RDU under uncertainty – formal

RDU under uncertainty (Choquet expected utility) — Under Structural Assumption 1.2.1, rank-dependent utility (RDU) holds if there exist a strictly increasing continuous utility function $U : \mathbb{R} \to \mathbb{R}$ and a weighting function W such that preferences over prospects $x = (E_1 : x_1, ...E_n : x_n)$ (with $x_1 \geq ... \geq x_n$) are represented by

$$RDU(x) = \sum_i^n (W(E_i \cup ... \cup E_1) - W(E_{i-1} \cup ... \cup E_1))U(x_i)$$
$$= \sum_i^n \pi(E_i^{E_{i-1} \cup ... \cup E_1})U(x_i)$$

RDU can accommodate the Ellsberg paradox

Example 10.3.1

Note: W has many degrees of freedom – hard to use in empirical applications

Estimation of RDU

The measurements in Figure 4.1.1 and 4.1.2 are still valid: $U(\alpha^k) - U(\alpha^{k-1})$ is constant in k. See Exercise 10.5.3.

With U measured, we can find the weights:

$$\text{If } \alpha \sim 1_E 0, \text{ then } W(E) = U(\alpha)/U(1).$$

Pessimism and optimism

Pessimism — Worsening the rank increases the decision weight, i.e. $\pi(E^{R'}) \geq \pi(E^R)$ whenever $R' \supset R$.

Optimism — Improving the rank increases the decision weight, i.e. $\pi(E^{R'}) \geq \pi(E^R)$ whenever $R' \subset R$.

Exercise 10.4.2: Pessimism is equivalent to

$$W(A \cup B) \geq W(A) + W(B) - W(A \cap B)$$

Likelihood insensitivity

See Section 10.4.2 for a formulation of a likelihood insensitivity $[B_{rb}, W_{rb}]$, involving a behavioral definition of "revealed more likely than".

Example 10.4.3: As an extreme case of likelihood insensitivity, consider the weighting where $W(E) = \alpha$ for all $E \notin \{\emptyset, S\}$ and $0 \leq \alpha \leq 1$.

The weighting implies for $x = (x_1 \geq ... \geq x_n)$:

$$RDU(x) = \alpha U(x_1) + (1 - \alpha)U(x_n)$$

("α-Hurwicz criterion")

Neo-additive weighting function — There exist $(a, b) > 0$ with $a + b < 1$ and a probability measure P such that $W(\emptyset) = 0, W(S) = 1$ and $W(E) = b + aP(E)$ for all other E.

With neo-additive weighting, we have:

$$RDU(x) = b\ sup_{s \in S} U(x(s)) + aEU(x) + (1 - a - b)inf_{s \in S} U(x(s))$$

Sets of probabilities

RDU with probability intervals — There exists α and for each event E there exists an interval I_E of probabilities such that:

$$W(E) = \alpha inf(I_E) + (1 - \alpha) sup(I_E)$$

More popular, and related – but not a special case of RDU: Multiple priors (Gilboa/Schmeidler (1989)).

Maxmin expected utility — There exists a convex set C of probability measures (priors) on S, and preferences are represented by:

$$MEU(x) = inf_{P \in C} EU_p(x)$$

Behavioral foundation of RDU under uncertainty

Theorem 10.5.6 — Under Structural Assumption 1.2.1, the following two statements are equivalent.

1. RDU holds with continuous and strictly increasing $U(\cdot)$.

2. $\succsim$ satisfies weak ordering, monotonicity, continuity, and rank-tradeoff consistency.

(Essentially the same as Theorem 6.5.6 for RDU under risk, except that rank-tradeoff consistency is now defined for ranked events, not ranked probabilities.)