Decision-Making under Uncertainty

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Overview

Lecture 1: Uncertainty and Preferences, Arbitrage and Expected Value

Lecture 2: Expected Value, Additivity, Arbitrage

Lecture 3: Risk versus uncertainty

Lecture 4: Expected utility under risk

Lecture 5: Expected utility and stochastic dominance

Lecture 6: Choosing not to choose

Lecture 7: Risk preferences under EU

Lecture 8: Multiattribute utility

Lecture 9: Expected utility under uncertainty (1)

Lecture 10: Expected utility under uncertainty (2)

Lecture 11: Probability weighting under risk

Lecture 12: Probability weighting under risk (2)

Lecture 13: Prospect theory under risk

Lecture 14: Ambiguity preferences

Housekeeping

Book: Peter Wakker, 2010: *Prospect Theory - for Risk and Ambiguity*, Cambridge University Press 2010.

See Moodle course page for slides and other material. Sign up please.

Class exercises: Held weekly by David Danz, danz@wzb.eu, Fri 14-16

Lecture 1: Preferences, Arbitrage and Expected Value

State space

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x ("ice cream")	400	100	-400
y ("hot dogs")	-400	100	400
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x+y ("both")	0	200	0

Financial market example

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	price $\geq$ 2.53	**2.53 $>$ price $\geq$ 2.47**	**2.47 $>$ price**
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Note: No probabilities defined

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Financial markets often allow investors to make bets on all possible events.

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Preference relation $\succsim$ — A binary relation on the set of all prospects in the domain

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A function $V(\cdot)$ represents $\succsim$ — For all x, y, $x \succsim y$ if and only if $V(x) \geq V(y)$.

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Notice that the properties require the statement for the entire domain of preference.

A first representation result

Exercise 1.2.5:

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By part (a), $CE(x) \succsim CE(y)$ holds iff $CE(x) \geq CE(y)$ and hence $CE(\cdot)$ represents $\succsim$. $\square$

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Lecture 2: Expected Value, Additivity, Arbitrage

Collecting assumptions

Recall:

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Probabilities — For a given state space S, a set of probabilities over the possible events is a collection $\{P(E_i)\}_i$ for all events E_i in the state space, satisfying $P(S) = 1, P(\varnothing) = 0$ and $P(E_i \cup E_j) = P(E_i) + P(E_j)$ for disjoint E_i, E_j.

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Expected Value — Under Structural Assumption 1.2.1, *expected value (EV) holds* if there exist probabilities $P(E_i)$ for all events E_i in the state space, such that

$$x = (E_1 x_1 \ldots E_n x_n) \to \sum_{i=1}^{n} P(E_i) x_i \equiv EV(x)$$

represents $\succsim$.

Deriving decisions

Exercise 1.3.1

Discussion of EV

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- Normatively useful?

Eliciting subjective parameters

Exercises 1.3.5 & 1.3.4

Additivity

A related concept that is defined directly on the preference:
Additivity — $[x \succsim y \Rightarrow x + z \succsim y + z]$ for all prospects x, y, z

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Tables 1.5.2 & 1.5.4

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(Proof on the board, taking as given that CE is additive under EV: $CE(x + y) = CE(x) + CE(y)$.)

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- Rules out diminishing sensitivity
- Rules out considerations of correlation
- Normatively appealing for small outcomes.

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Dutch book — Fix the preference $\succsim$. Arbitrage, or a Dutch book, is a collection of pairs of prospects (x^j, y^j), with $j = 1...m$, such that the $\{x^j\}_j$ are the preferred prospects but when combined they yield strictly less than the $\{y^j\}_j$:

$x^j \succsim y^j$ for all $j = 1...m$,
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$\sum_{j=1}^m x^j(s) < \sum_{j=1}^m y^j(s)$ for all $s \in S$.

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"Freedom from arbitrage": No Dutch book exists, i.e. the decision-maker's preference does not allow the construction.

Discussion of freedom from arbitrage

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Discussion of freedom from arbitrage

- Normatively appealing
- Important concept in finance

De Finetti's theorem

Theorem 1.6.1 — Under Structural Assumption 1.2.1, the following three statements are equivalent.

(i) Expected Value holds.

(ii) $\succsim$ is a weak order, for each prospect there exists a certainty equivalent, and no arbitrage (Dutch book) is possible.

(iii) $\succsim$ is a weak order, for each prospect there exists a certainty equivalent, and additivity and monotonicity are satisfied.

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- EV now more appealing?
- Freedom from arbitrage seems very weak. But it relates to choice between x^j versus y^j that is not combined with other choice.

Finance example

Assignment 1.6.11

Lecture 3: Risk versus uncertainty

Probability-contingent prospects

We continue to use:
Structural Assumption 1.2.1 ("Decision under Uncertainty"): S is a finite or infinite state space and $\mathbb{R}$ is the outcome set. Prospects map states to outcomes, taking only finitely many values. $\succsim$ is a preference relation on the set of prospects, i.e. on all such maps. Nondegeneracy holds.

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Let $p_j = P(E_j)$ denote the probability of event E_j. A prospect $x = (E_1 : x_1, ..., E_n : x_n)$ has a probability distribution $(p_1 : x_1, ..., p_n : x_n)$ (a "lottery").

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$\succsim$ is defined over probability-contingent prospects or lotteries, which are probability distributions with finitely many outcomes values.

Example

Example 2.1.1

Decision under risk

Assumption 2.1.2 ("Decision under Risk"): Structural Assumption 1.2.1 holds. In addition, an objective probability measure P is given on the state space, assigning to each event E its probability $P(E)$. Different event-contingent prospects that generate the same probability-contingent prospect are preference equivalent.

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But at least it is within our previous assumptions: for given P and preference $\succsim$, we can construct an appropriate S so that Structural Assumption 1.2.1 holds. (Next slides.)

Risk as a special case of uncertainty

The aim is to represent any given lottery $(p_1 : x_1, ..., p_n : x_n)$ as an event-contingent prospect. Let $S = [0, 1)$ be the unit interval. We assign n events by partitioning $S = \{[0, q_1), [q_1, q_2), ..., [q_{n-1}, 1)\}$.

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Mapping the n possible events into the outcome space $\mathbb{R}$ yields an event-contingent prospect $\{[0, q_1) : x_1, [q_1, q_2) : x_2, ..., [q_{n-1}, 1) : x_n\}$ like in Structural Assumption 1.2.1.

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To generate probabilities, we take the uniform probability measure ("Lebesgue measure") $p_j = q_j - q_{j-1}$. (With $q_0 = 0$ and $q_n = 1$.)

Risk as a special case of uncertainty

The aim is to represent any given lottery $(p_1 : x_1, ..., p_n : x_n)$ as an event-contingent prospect. Let $S = [0, 1)$ be the unit interval. We assign n events by partitioning $S = \{[0, q_1), [q_1, q_2), ..., [q_{n-1}, 1)\}$. Let a random number be drawn from S, and let this number be the true state.

E.g., event 1 is that the true state lies in $[0, q_1)$.

Mapping the n possible events into the outcome space $\mathbb{R}$ yields an event-contingent prospect

$\{[0, q_1) : x_1, [q_1, q_2) : x_2, ..., [q_{n-1}, 1) : x_n\}$ like in Structural Assumption 1.2.1.

To generate probabilities, we take the uniform probability measure ("Lebesgue measure") $p_j = q_j - q_{j-1}$. (With $q_0 = 0$ and $q_n = 1$.) We are free to choose the $\{q_j\}_j$, and hence any given lottery can be expressed in such a way, as generated by an event-contingent prospect. A lottery is a prospect, but with the additional information about the probabilities of events. All previous results apply to the case where probabilities are known.

Risk as a special case of uncertainty (2)

But notice that multiple event-contingent prospects can generate the same probability-contingent prospect: E.g.

$$\{[0, \frac{1}{2}) : \$0, [\frac{1}{2}, 1) : \$100\}$$

and

$$\{[0, \frac{1}{2}) : \$100, [\frac{1}{2}, 1) : \$0\}$$

both yield the lottery $(\frac{1}{2} : \$0, \frac{1}{2} : \$100)$.

Getting used to it

Exercises 2.4.1, 2.4.2

Full-domain assumption

Assumption 2.2.1 ("Richness for decision under risk"): Every possible distribution over the outcomes that takes on finitely many values is available in the preference domain.

Full-domain assumption

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Summarizing the previous assumptions:

Structural Assumption 2.5.2 ("Decision under risk and richness"): $\succsim$ is a preference relation over the set of all probability-contingent prospects, i.e. over all finite probability distributions over $\mathbb{R}$.

Risk preferences as behavioral assumptions

With objective probabilities, the expected value of prospects (and all their other moments) are defined without reference to $\succsim$.

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Risk Aversion — $E[x] \succsim x$, for all x in the domain of preference.

Risk Neutrality — $E[x] \sim x$, for all x.

Risk Seeking — $x \succsim E[x]$, for all x.

Risk preferences and EV

Under Structural Assumption 2.5.2, "EV holds" if and only if $\succsim$ exhibits risk neutrality.

Risk preferences and EV

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Note the argument: Under Str. Ass. 2.5.2, preferences obey the objective measure p. "EV holds" means that the EV function represents $\succsim$. With objective p, the EV function is given by E[x]. Hence a lottery x is preferred to a lottery y iff $E[x] \geq E[y]$.

EV may surprise

Example 2.5.1

Expected utility

Bernoulli's invention: When probabilities are known, the value of the outcome may still be flexible.

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Expected Utility — Under Structural Assumption 2.5.2, *expected utility (EU) holds* if there exists a strictly increasing function $U : \mathbb{R} \to \mathbb{R}$, mapping an outcome into a utility value, such that the expected utility function

$$x = (p_1 : x_1 ... p_n : x_n) \to \sum_{i=1}^{n} p_i U(x_i) \equiv EU(x)$$

represents $\succsim$.

Lecture 4: Expected utility under risk

Recall the earlier definitions

Structural Assumption 2.5.2 ("Decision under risk and richness"): $\succsim$ is a preference relation over the set of all probability-contingent prospects, i.e. over all finite probability distributions over $\mathbb{R}$.

Expected Utility — Under Structural Assumption 2.5.2, *expected utility (EU) holds* if there exists a strictly increasing function $U : \mathbb{R} \to \mathbb{R}$, mapping an outcome into a utility value, such that the expected utility function

$$x = (p_1 : x_1 ... p_n : x_n) \to \sum_{i=1}^{n} p_i U(x_i) \equiv EU(x)$$

represents $\succsim$.

Deriving decisions under EU

Exercise 2.5.1 & Example 2.5.4

Eliciting utilities under EU

Exercise 2.5.3

(Note the notation $100_{0.58}0$ referring to the first outcome occurring with probability 0.58.)

Behavioral foundation of EU

Behavioral foundation of EU

Wherever possible, we use simple binary lotteries.

Standard gamble — $(p : M, 1 - p : m)$, for some $M > m$ and p.

Standard gamble solvability

Standard gamble solvability—For all outcomes $M > \alpha > m$ there exists a "standard gamble probability" $p \in (0, 1)$ satisfying

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If EU holds, we can normalize $U(M) = 1$ and $U(m) = 0$ (to be shown in class). Consider $M > \alpha > m$. Under EU, $U(M) > U(\alpha) > U(m)$ holds, and there exists $p \in (0,1)$ such that

$$U(\alpha) = pU(M) + (1-p)U(m)$$

$\Rightarrow$ EU implies SG solvability.

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$\Rightarrow$ EU implies SG solvability.

SG solvability makes utilities and probabilities commensurable.

Standard gamble dominance

Standard gamble dominance — For all outcomes $M > m$ and probabilities $p > q$,

$$(p : M, 1 - p : m) \succ (q : M, 1 - q : m)$$

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SG dominance corresponds to monotonicity.

EU implies SG dominance. (Board.)

Linearity of EU

EU is tractable especially when dealing with complicated lotteries.

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Probabilistic mixture — For a pair of lotteries x, y and a probability $\lambda \in [0, 1]$, let $x_\lambda y$ denote the probabilistic mixture of x and y: a lottery that assigns to each outcome α a probability of λ times α's probability under x plus $1 - \lambda$ times α's probability under y.

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Proposition: (Exercise 2.6.6.) *EU* is linear in probability:

$$EU(x_\lambda y) = \lambda EU(x) + (1 - \lambda)EU(y)$$

(Proof on board.)

Standard gamble consistency

A weak form of linearity:

Standard gamble consistency — For all outcomes α, M, m, all probabilities p, λ, and all lotteries C, it holds that

$$\alpha \sim (p : M, 1 - p : m)$$

implies

$$\alpha_\lambda C \sim (p : M, 1 - p : m)_\lambda C$$

where the last term denotes a probabilistic mixture between $(p : M, 1 - p : m)$ and C.

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Note: Under EU, SG holds:
$EU(\alpha_\lambda C) = \lambda EU(\alpha) + (1 - \lambda) EU(C)$
$EU((p : M, 1 - p : m)_\lambda C) = \lambda EU(p : M, 1 - p : m) + (1 - \lambda) EU(C)$

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Realism?

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Realism? Normative appeal?

von Neumann-Morgenstern's EU representation theorem

Theorem 2.6.3 — Under Structural Assumption 2.5.2, the following two statements are equivalent:

1. EU holds.
2. $\succsim$ satisfies weak ordering, SG solvability, SG dominance and SG consistency.

Proof of vNM's theorem

We saw that EU implies weak ordering, SG solvability, SG dominance and SG consistency. It remains to show the reverse.

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Fix two prospects $x = (p_1 : x_1 ... p_n : x_n)$ and $y = (q_1 : y_1 ... q_n : x_n)$. Let M be the largest outcome and m the smallest outcome of all outcomes in x or y, and normalize $U(M) = 1$ and $U(m) = 0$.

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Proof of vNM's theorem

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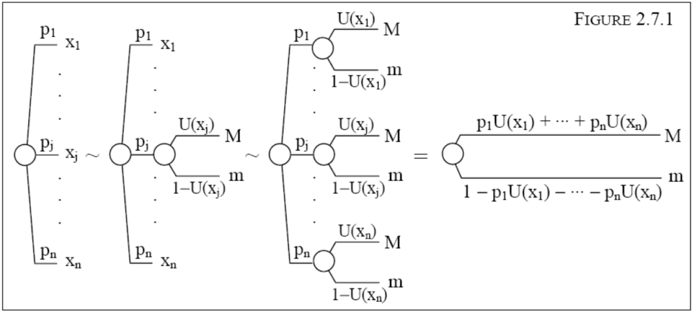
For each outcome x_j define $U(x_j)$ to be the SG probability for the standard gamble with outcomes (M, m). $U(x_j)$ exists due to SG solvability.
Consider the indifferences in the Figure 2.9.1. The first indifference holds due to the application of SG consistency. The second equivalence uses the repeated application of SG consistency, for all outcomes. The equality is by construction, and Assumption 2.1.2 implies preference equivalence between the two equal prospects.

Proof of vNM's theorem (2)

Figure 2.9.1 (not 2.7.1)



Proof of vNM's theorem (3)

Summing up Figure 2.9.1: The prospect x is preference equivalent to the binary lottery that yields M with probability $\sum_{i=1}^{n} p_i U(x_i)$ and yields m otherwise.

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$\sum_{i=1}^{n} p_i U(x_i)$ is a probability, but has the "right" structure of $EU(x)$: it is a function that maps lotteries into $\mathbb{R}$, and it is linear in the outcome probabilities p_i.

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Need to check that (i) $\succsim$ can be represented by this particular function $EU(x)$, and that (ii) $U(x_j)$ is strictly increasing in x_j.

Proof of vNM's theorem (4)

Need to check that (i) $\succsim$ can be represented by $EU(x)$, and (ii) $U(x_j)$ is strictly increasing in x_j.

(i) Consider x and y. x is preference equivalent to a binary lottery assigning probability $EU(x)$ to M and the remaining probability to m, and y is preference equivalent to a binary lottery assigning $EU(y)$ to M and $1 - EU(y)$ to m. By SG dominance, all binary lotteries with M and m as outcomes are ordered by the probability of receiving M. Hence, by transitivity, $EU(x) \geq EU(y)$ is equivalent to $x \succsim y$.

Proof of vNM's theorem (4)

Need to check that (i) $\succsim$ can be represented by $EU(x)$, and (ii) $U(x_j)$ is strictly increasing in x_j.

(i) Consider x and y. x is preference equivalent to a binary lottery assigning probability $EU(x)$ to M and the remaining probability to m, and y is preference equivalent to a binary lottery assigning $EU(y)$ to M and $1 - EU(y)$ to m. By SG dominance, all binary lotteries with M and m as outcomes are ordered by the probability of receiving M. Hence, by transitivity, $EU(x) \geq EU(y)$ is equivalent to $x \succsim y$.

(ii) Consider two outcomes $x_k > x_j$. Apply SG dominance with a different selection of $M, m : M = x_k, m = x_j, p = 1, q = 0$ to find that $x_k \succ x_j$. Since we already saw that $\sum_{i=1}^{n} p_i U(x_i)$ represents $\succsim$, it holds equivalently that $U(x_k) > U(x_j)$. ■

EU as decision aid

Figures 3.1.1 and 3.1.2

Lecture 5: Expected utility and stochastic dominance

Alternative formulation of EU axioms

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Def: $\succsim$ satisfies continuity if for all lotteries $x \succ y \succ z$ there exists a $p \in (0,1)$ satisfying the indifference

$$y \sim (p : x, 1 - p : z).$$

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Def: $\succsim$ satifies independence if for all probabilities λ and all lotteries x, y, C, it holds that

$$x \succsim y$$

implies

$$(\lambda : x, 1 - \lambda : C) \succsim (\lambda : y, 1 - \lambda : C).$$

Alternative formulation of EU axioms (2)

Proposition: Under Structural Assumption 2.5.2, the following two are equivalent:

1. EU holds, but with U not necessarily strictly increasing.
2. $\succsim$ satisfies weak ordering, continuity, and independence.

(First-order) stochastic dominance

x first-order stochastically dominates y — x can be generated from y by shifting probability mass from an outcome to a preferred outcome (once or in multiple instances).

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Exercise 2.7.1

Counterexamples to EU

Exercise 2.8.1

Counterexamples to EU (2)

Problem 1. Which of the following options do you prefer?
C1. A sure gain of 1 million Euros.
C2. An 80% chance to gain 5 million Euros and a 20% chance to gain nothing.

Counterexamples to EU (2)

Problem 1. Which of the following options do you prefer?
C1. A sure gain of 1 million Euros.
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Problem 2. Which of the following options do you prefer?
D1. A 5% chance to gain 1 million Euros and a 95% chance to gain nothing.
D2. A 4% chance to gain 5 million Euros and a 96% chance to gain nothing.

Counterexamples to EU (3)

Figure 2.4.1, (g) and (h)

Stochastic dominance for continuous distributions

For convenience: Also consider lotteries F_x with a bounded continuum of outcomes: $\alpha \in [x_{\text{min}}, x_{\text{max}}]$ and $\exists$ a density for all α.

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Stochastic dominance for continuous distributions

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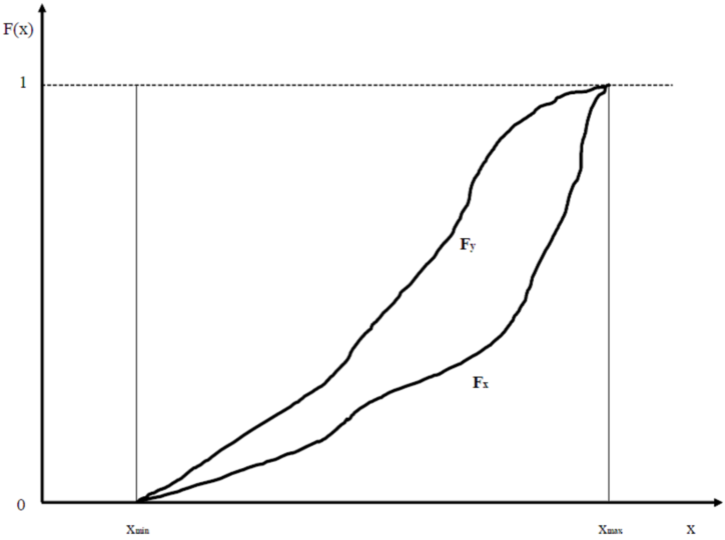
Notice that we can approximate any pair of such lotteries (F_x, F_y) by two lotteries (x, y) with finitely many outcomes (satisfying Structural Assumption 2.5.2).

We formulate some properties/ideas for continuous lotteries but apply the results to finite lotteries.

x first-order stochastically dominates y —

$$F_x(\alpha) \leq F_y(\alpha), \text{ for all } \alpha \in [x_{\min}, x_{\max}].$$

Stochastic dominance for continuous distributions (2)



Equivalence of EU and stochastic dominance

Proposition: Under Structural Assumption 2.5.2, the following two statements about lotteries x, y are equivalent:

1. x first-order-stochastically dominates y.
2. All EU-representable preferences prescribe $x \succsim y$.

Equivalence of EU and stochastic dominance (2)

We already showed 1. $\Rightarrow$ 2.

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Equivalence of EU and stochastic dominance (2)

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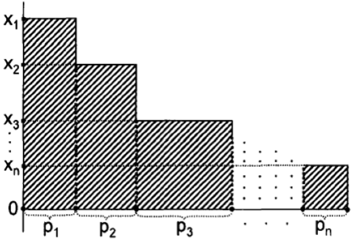
2. $\Rightarrow$ 1. is unlikely to be applied in practice, since 2. is usually not known when 1. is not yet known. But the statement [2. $\Rightarrow$ 1.] implies that we know exactly what the EU assumptions buy us: we restrict attention to all preferences that do not violate stochastic dominance.

Suppose that we assumed EU and that neither x nor y first-order stochastically dominates the other. Then, [*not* 1. $\Rightarrow$ *not* 2.] implies that we ruled out neither $x \succsim y$ nor $y \succsim x$.

Proof of the proposition

Consider lottery $x = (p_1 : x_1, ..., p_n : x_n)$, where $x_1 > ... > x_n$. The expected value is the summed area inside the rectangles.

Figure 5.2.1. Expected value



The area shaded by ▨ is the expected value $p_1x_1 + p_2x_2 + \cdots + p_nx_n$.

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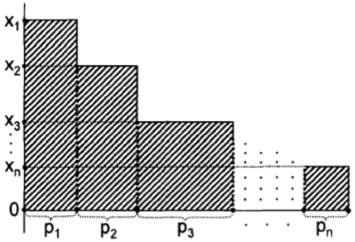

The area shaded by ▨ is the expected value $p_1x_1 + p_2x_2 + \cdots + p_nx_n$.

$$EV(x) = \sum_i p_i x_i$$

Proof of the proposition (2)

Anticipating the case of a continuous outcome range, we take the lottery with n outcomes to be equi-distant (wlog), as an approximation of a continuous distribution.

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Look at the figure "row by row" from left to right, and note that we can determine the area differently, by multiplying two things for each outcome: how much better is the outcome than the next-worse outcome $(= x_i - x_{i+1})$ and what is the probability of receiving at least x_i, which is $p_i + p_{i-1} + ... + p_1 = \sum_{j=1}^{i} p_j$.

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Look at the figure "row by row" from left to right, and note that we can determine the area differently, by multiplying two things for each outcome: how much better is the outcome than the next-worse outcome ($= x_i - x_{i+1}$) and what is the probability of receiving at least x_i, which is $p_i + p_{i-1} + ... + p_1 = \sum_{j=1}^{i} p_j$.

$$EV(x) = \sum_{i=1}^{n} (\sum_{j=1}^{i} p_j)(x_i - x_{i+1}).$$

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$$(\sum_{j=1}^{i} p_j) = \Pr(outcome \geq x_i) = 1 - \Pr(outcome < x_i)$$

$$\to 1 - \Pr(outcome \leq x_i) = 1 - F(x_i)$$

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To approximate the continuous case, we let $n \to \infty$.

$$(\sum_{j=1}^{i} p_j) = \Pr(outcome \geq x_i) = 1 - \Pr(outcome < x_i)$$
$$\to 1 - \Pr(outcome \leq x_i) = 1 - F(x_i)$$

Drop the subscript i and replace $(x_i - x_{i+1})$ by its infinitesimal analog, the differential dx.

Proof of the proposition (3)

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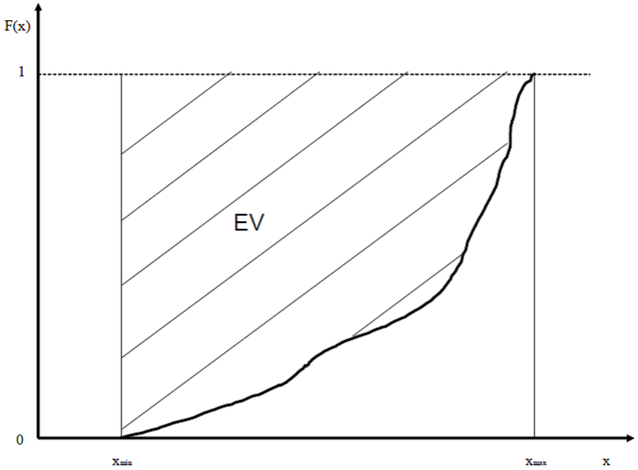
$$EV(x) = \int_{x_{\min}}^{x_{\max}} (1 - F(x))dx.$$

This is the area above the cdf, or the "epigraph".

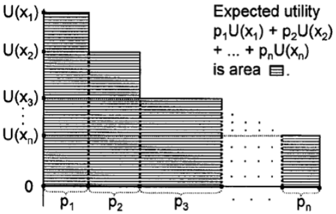
Proof of the proposition (4)



Proof of the proposition (5)

Transform the x axis in the first graph (note: vertical axis) to measure U—replace each x_i in the graph by $U(x_i)$.

Figure 5.2.3. Expected utility



To calculate expected utility, the distance from x_j ("all the way") down to the x-axis has been transformed into the distance $U(x_j)$ for all j.

Proof of the proposition (6)

Analogously to the derivation of EV:

$$EU(x) = \sum_i (\sum_{j=1}^{i} p_j)(U(x_i) - U(x_{i+1}))$$

x_i's marginal contribution to U is relevant.

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Again, let $n \to \infty$. The marginal U-contribution becomes dU. Since the derivative is $U'(x) = \frac{dU}{dx}$, we have $dU = U'(x)dx$.

$$EU(x) = \int_{x_{\min}}^{x_{\max}} U'(x)(1 - F(x))dx. \quad (1)$$

The normalized EU of the distribution is the area above the cdf, but weighted according to the U-contribution of x. (For EV, think of an equal weight of 1.)

Proof of the proposition (7)

To show that 2. $\Rightarrow$ 1. : We show the counterpositive, i.e. that not 1. implies not 2.

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$$\widetilde{U}(\alpha) = 1 \text{ if } \alpha > \widetilde{\alpha}$$
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$$\widetilde{U}(\alpha) = 1 \text{ if } \alpha > \widetilde{\alpha}$$
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Using expression (1), she chooses y over x (strictly). Moreover, one can find strictly increasing functions that are arbitrarily close to $\widetilde{U}$, and hence have the same property. That is, there exist an EU agent who chooses y and hence, 2. does not hold. ■

Lecture 6: Choosing not to choose

See presentation slides flipping_coins_slides_2013_05_31.ppt

Lecture 7: Risk preferences under expexted utility

Collecting assumptions

Structural Assumption 3.0.1 ("Decision under risk and EU"):
$\succsim$ is a preference relation over the set of all probability-contingent prospects, which is the set of all finite probability distributions over the outcome set $\mathbb{R}$. Expected utility holds with a utility function U that is continuous and strictly increasing.

Risk aversion and concavity

Recall:

Risk Aversion — $E[x] \succsim x$, for all x in the domain of preference.

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Risk Neutrality — $E[x] \sim x$, for all x.

Risk Seeking — $x \succsim E[x]$, for all x.

Notice that these assumptions on $\succsim$ can stand alone, e.g. without assuming EU.

Risk aversion and concavity (2)

Recall also:

$f : X \to \mathbb{R}$ is concave — $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$

$f : X \to \mathbb{R}$ is linear — $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$

$f : X \to \mathbb{R}$ is convex — $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

for all $x, y \in X$ and all $\lambda \in [0, 1]$.

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Theorem 3.2.1 — Under Structural Assumption 3.0.1,

$$\begin{gathered}\text{risk aversion} \Leftrightarrow U \text{ concave}\\ \text{risk neutrality} \Leftrightarrow U \text{ linear}\\ \text{risk loving} \Leftrightarrow U \text{ convex.}\end{gathered}$$

Figure 3.2.1

Experiment

1. Choose between

(1 : EUR 4000) and (0.5 : EUR $0, 0.5$: EUR 10000)

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$\rightarrow$ Need measure of risk aversion

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4 comparisons of two preference relations $\succsim_1$ and $\succsim_2$, under Structural Assumption 3.0.1:

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2. Risk premium of a lottery x — the distance between the lottery's expected value and the lottery's certainty equivalent, $\pi(x) = EV(x) - CE(x)$.

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Exercise 3.2.3

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Exercise 3.2.3

Note: Structural Assumption 3.0.1 not required for Definitions 1. and 2.

Comparative risk aversion (2)

Let U_1 and U_2 be utility functions that represent $\succsim_1$ and $\succsim_2$ in the EU sense.

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3. U_2 is a concave transformation of U_1 — ϕ is concave.

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3. U_2 is a concave transformation of U_1 — ϕ is concave.

4. Arrow-Pratt degree of absolute risk aversion — $r_{AP}(x) = -\frac{U''(x)}{U'(x)}$.

Comparative risk aversion (3)

Proposition (see Thm 3.4.1 and Ex 3.4.1)**:** The following four statements are equivalent.

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- $\phi(u) = U_2(U_1^{-1}(u))$ is a concave transformation of U_1.
- $\succsim_2$ has a higher degree of absolute risk aversion: for all outcomes α, $r_{AP,2}(\alpha) \geq r_{AP,1}(\alpha)$.

Mean-preserving spread

(On the board.)

Constant Absolute Risk Aversion

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$$U(x) = 1 - \exp(-rx)$$
$$x \in \mathbb{R}, r > 0.$$

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More generally (allowing for convex functions):

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(Sometimes rescaled as $U(x) = \frac{1-\exp(-rx)}{r}$.)

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(Sometimes rescaled as $U(x) = \frac{1-\exp(-rx)}{r}$.)

As suggested by the name, it has a constant (independent of x) Arrow-Pratt degree of risk aversion:
$r_{AP}(x) = -\frac{U''(x)}{U'(x)} = -\frac{-r^2 \exp(-rx)}{r \exp(-rx)} = r$

Constant Absolute Risk Aversion (2)

Proposition: Assume Structural Assumption 3.0.1 and that the utility function is differentiable. The following are equivalent:

- $\succsim$ is represented by CARA utility.
- The preference between two lotteries (x, y) is not affected if μ is added to both lotteries, for all $\mu \in \mathbb{R}$ and all (x, y).

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(Board.)

Decreasing Absolute Risk Aversion

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Preferences exhibit decreasing absolute risk aversion (DARA) if the risk premium $\pi(x)$ for any given lottery x weakly decreases if a sure payment $\mu \geq 0$ is added to the lottery, i.e. $\frac{\partial \pi(x+\mu)}{\partial \mu} \leq 0$.

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We get the following characterization.
Proposition: Under EU, preferences are DARA if and only if the Arrow-Pratt degree $r_{AP}(\alpha) = -\frac{U''(\alpha)}{U'(\alpha)}$ weakly decreases in α.

Constant Relative Risk Aversion

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(The *ln* curve is the unique function between the cases $r > 0$ and $r < 0$.)

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(The *ln* curve is the unique function between the cases $r > 0$ and $r < 0$.)

The function is often written as $U(x) = x^{1-\gamma}$ or $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

Lecture 8: Multiattribute utility

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(Multiattribute) outcome — $\alpha = (\alpha^1, ..., \alpha^m) \in X$

Health example

Example 3.7.1, $EU(Q, T)$, momontonicity in life duration, zero condition, SG invariance, Observation 3.7.2

Probabilistic multiattribute outcomes

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EU is defined analogously, too. See Figure 3.7.2.

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EU is defined analogously, too. See Figure 3.7.2.

Marginal prospect — $(p_1 : x_1^i, ..., p_n : x_n^i)$, the probability distribution over attribute set X^i generated by x.
See Figure 3.7.3 (and note the typo: the right panel should not depict a lottery between the marginals; it should depict just the marginals).

Multiattribute risk attitudes

Consider a sure attribute $\gamma^i \in X^i$ and let $\gamma^i\alpha$ denote outcome α but with its ith attribute replaced by γ^i.

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Consider attributes γ^i, δ^i and outcomes α, β with $\gamma^i\beta \succsim \gamma^i\alpha$, $\delta^i\beta \succsim \gamma^i\beta$ and $\delta^i\alpha \succsim \gamma^i\alpha$.

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Consider a choice between prospects:

$$\delta^i \alpha_{0.5} \gamma^i \beta \text{ and } \gamma^i \alpha_{0.5} \delta^i \beta$$

E.g. compare (3.7.2) and (3.7.3)

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Multiattribute risk aversion — $\delta^i\alpha_{0.5}\gamma^i\beta \succsim \gamma^i\alpha_{0.5}\delta^i\beta$ for all such $i, \alpha, \beta, \gamma^i, \delta^i$

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Additive decomposability

Multiattribute risk neutrality says that an improvement in one attribute i is evaluated independently of the other attributes.

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Proposition (see Thm 3.7.3)**:** The following three are equivalent.

(i) Multiattribute risk neutrality

(ii) $U(\alpha^1, ..., \alpha^m) = U(\alpha^1) + ... + U(\alpha^m)$

(iii) Marginal independence: Preference over prospects (x, y) depends only on the marginal prospects generated by x and y.

Identical dimensions

Anscombe and Aumann (1963) assume $X^1 = X^2 = ... = X^m = C$ (set of prizes).

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Note that under this interpretation, marginal independence is a consistency property, similar to SG consistency.

Lecture 9: Expected utility under uncertainty

Choice experiments

Figure 4.1.1 with $cand_1$ = Steinbrück, $cand_2$ = Merkel, in units of EUR 1,000.00

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Figure 4.1.4

Figure 4.1.5

Recall previous concepts

Structural Assumption 1.2.1 ("Decision under Uncertainty"): S is a finite or infinite state space and $\mathbb{R}$ is the outcome set. Prospects map states to outcomes, taking only finitely many values. $\succsim$ is a preference relation on the set of prospects, i.e. on all such maps. Nondegeneracy holds.

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Continuity — For every partition $\{E_i\}_{i=1}^{n}$ of S and for all prospects $y \in \mathbb{R}^n$, $y = (E_1 : y_1, ..., E_n : y_n)$, the better-than-set and worse-than-set, $\{x \in \mathbb{R}^n | x \succsim y\}$ and $\{x \in \mathbb{R}^n | y \succsim x\}$, are closed in $\mathbb{R}^n$.

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EV under Structural Assumption 1.2.1: Utility known, probabilities flexible

EU under Structural Assumption 2.5.2: Utility flexible, probabilities known

Definition of EU

Expected Utility — Under Structural Assumption 1.2.1, *expected utility (EU) holds* if there exist probabilities $P(E_i)$ for all events E_i in the state space and there exists a strictly increasing function $U : \mathbb{R} \to \mathbb{R}$ that depends only on outcomes, such that

$$E_1 x_1 \ldots E_n x_n \to \sum_{i=1}^{n} P(E_i) U(x_i) \equiv EU(x)$$

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(The assumption is often referred to as *Subjective Expected Utility*.)

Discussion of EU (— just like EV)

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Predicting choices

Exercise 4.2.1

Eliciting subjective parameters

Exercise 4.2.3

Getting used to EU

Exercise 4.2.5

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$\alpha_E x$ — A prospect that yields α if $s \in E$ and yields $x(s)$ otherwise.

Getting used to EU

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E is null — $\alpha_E x \sim \beta_E x$ for all prospects x and all outcomes α, β
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Getting used to EU

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Exercise 4.2.6

Exercise 4.2.7

Using your experimental choices

Excercise 4.3.1: Consider Figure 4.1.1, with $\alpha^0 = 10$. Show that the assumption of EU implies that $U(\alpha^k) - U(\alpha^{k-1})$ is constant in k.

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Figure 4.3.1, Figure 4.3.2

Note that we can measure U precisely with this method, hence also measure P, e.g. using standard gambles: for given E, select M, m, α such that

$$U(\alpha) = P(E)U(M) + (1 - P(E))U(m)$$
$$P(E) = U(\alpha)$$

Consistency under EU

Excercise 4.3.2

Consistency under EU

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Do not assume EU but only weak ordering and strong monotonicity ($x \succ y$ if $x \geq y$ and $\exists s$ with $x(s) > y(s)$).

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Exercise 4.3.5

Choices in Figure 4.1.1 — in slow motion

Consider Figure 4.1.1 (a) and (d)

$$\alpha_E^1 1 \sim \alpha_E^0 8$$

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Conditional an some event (here, E^c), $8 \ominus 1$ reflects the preference value of receiving the right prospect.

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We write this as

$$\alpha^1 \ominus \alpha^0 \sim_t \alpha^4 \ominus \alpha^3$$

Definition of t-indifference

Consider general prospects x, y, events E and outcomes $\alpha, \beta, \gamma, \delta$, and indifferences:

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Figure 4.5.1, Example 4.5.2

Lecture 10: Expected utility under uncertainty (2)

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t-indifference and EU

Exercise 4.5.3: Show that under EU,

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$$P(E)U(\alpha) + \sum_{s_j \notin E} P(s_j)U(x_j) = P(E)U(\beta) + \sum_{s_j \notin E} P(s_j)U(y_j)$$
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t-indifference and EU (2)

Suppose that in addition to $\alpha \ominus \beta \sim^t \gamma \ominus \delta$ we observe that $\alpha' \ominus \beta \sim^t \gamma \ominus \delta$ with $\alpha > \alpha'$ (see Example 4.6.1).

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Tradeoff consistency — Strictly improving an outcome in any t-indifference breaks that indifference.

EU representation theorem ($\sim$ Savage)

Theorem 4.6.4 — Under Structural Assumption 1.2.1, the following two statements are equivalent.

1. EU holds with continuous and strictly increasing $U(\cdot)$.

2. $\succsim$ satisfies weak ordering, monotonicity, continuity, and tradeoff consistency.

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Observation 4.6.4': Moreover: in (2), the probabilities P are uniquely determined and utility U is unique up to positive affine transformations.

Discussion of Theorem 4.6.4

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- The case of decision under risk is included in Structural Assumption 1.2.1. The "tradeoff method" of stating consistency works also for this domain of preferences. See Figure 4.7.1.
- Note again the degrees of freedom—e.g. allowing for arbitrary beliefs.

Proof of the theorem

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– For all $(x_1, x_2) \succ (y_1, y_2)$, there exists a large enough y_1' such that $(x_1, x_2) \sim (y_1', y_2)$, and analogously for y_2.
– Strong monotonicity: If $(x_1, x_2) \geq (y_1, y_2)$ and $(x_1, x_2) \neq (y_1, y_2)$, then $(x_1, x_2) \succ (y_1, y_2)$.

Proof of the theorem (2)

Assume that property 2. holds, and construct the EU function as follows.
Fix a small outcome $\alpha^0 = \beta^0$ and a larger outcome α^1. Define outcome β^1 by requiring

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$$(\alpha^{i+1}, \beta^0) \sim (\alpha^i, \beta^1).$$

Likewise, fix α^0, α^1 and define $\{\beta^{j+1}\}_{j=1}^{\infty}$

$$(\alpha^1, \beta^j) \sim (\alpha^0, \beta^{j+1}).$$

[See Figure 4.15.1]

Proof of the theorem (3)

We constructed sequences such that

$$\alpha^{i+1} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \qquad (1.)$$

and

$$\beta^{j+1} \ominus \beta^{j} \sim^{t} \beta^{1} \ominus \beta^{0} \qquad (1'.)$$

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for all i, j. For arbitrary $i, j > 0$, consider the indifference

$$(\alpha^{1}, \beta^{j}) \sim (\alpha^{0}, \beta^{j+1}) \qquad (2.)$$

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But for some large enough α^* we have

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(2.) and (3.) together imply the t-indifference:

$$\alpha^{*} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \qquad (4.)$$

Proof of the theorem (4)

$$\alpha^{i+1} \ominus \alpha^{i} \sim^{t} \alpha^{1} \ominus \alpha^{0} \quad (1.)$$

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(1.) and (4.) imply, by tradeoff consistency, that $\alpha^{*} = \alpha^{i+1}$.

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(1.) and (4.) imply, by tradeoff consistency, that $\alpha^* = \alpha^{i+1}$. Using (3.) we therefore know

$$(\alpha^{i+1}, \beta^j) \sim (\alpha^i, \beta^{j+1}).$$

Decreasing one superscript by 1 and increasing the other by 1 does not change the preference value of a prospect (α^i, β^j). Repeated application shows that decreasing one superscript by any $k \in \mathbb{N}$ and increasing the other by k does not change the preference value.

Proof of the theorem (5)

Therefore, defining $V_1(\alpha^i) = i$ and $V_2(\beta^j) = j$, the function

$$V_1(\alpha^i) + V_2(\beta^j)$$

represents $\succsim$ over all constructed prospects (α^i, β^j).

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But this representation does not yet have the right form. To arrive at EU representation, we need to find subjective probabilities $P(E_1)$ and $P(E_2)$ and a function $U : \mathbb{R} \to \mathbb{R}$ such that

$$V_1(\alpha^i) = P(E_1)U(\alpha^i)$$

and

$$V_2(\beta^j) = P(E_2)U(\beta^j).$$

Proof of the theorem (6)

The size ratio of the α's and β's determines the probabilities. Assume (arbitrarily) that $\alpha^1 - \alpha^0 = \beta^3 - \beta^0$.

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This suggests that E_2 is three times as likely as E_1.

Tradeoff consistency ensure that this reasoning is true (i.e. leads to the uniquely possible probabilities)—see next slides.

Proof of the theorem (7)

From $(\alpha^0, \beta^6) \sim (\alpha^3, \beta^3)$ and $(\alpha^0, \beta^3) \sim (\alpha^3, \beta^0)$ we obtain

$$\beta^6 \ominus \beta^3 \sim^t \beta^3 \ominus \beta^0.$$

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Substituting $\alpha^1 = \beta^3$ and $\alpha^0 = \beta^0$,

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Applying the same argument recursively gives $\beta^{3i} = \alpha^i$ for all i.

Proof of the theorem (8)

Once again, observe that because $V_1 + V_2 = i + j$ represents $\succsim$, a step of any size $(\alpha^i - \alpha^0)$ in β-direction increases utility by three times as much as a step of the same size in α-direction. That is,

$$3V_1(\alpha^i) = V_2(\alpha^i) \text{ or, equivalently, } V_1(\alpha^i) = \frac{1}{3}V_2(\alpha^i).$$

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Compare this to the EU function:

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Both are weighted sums. But we need more, namely that EU represents the same preferences, i.e. for all $(\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$,

$$P(E_1)U(\alpha_1^i) + P(E_2)U(\beta_1^j) \geq P(E_1)U(\alpha_2^i) + P(E_2)U(\beta_2^j)$$

$$\Leftrightarrow$$

$$\frac{1}{3}V_2(\alpha_1^i) + V_2(\beta_1^j) \geq \frac{1}{3}V_2(\alpha_2^i) + V_2(\beta_2^j)$$

Proof of the theorem (9)

There exists exactly one possibility to achieve this, namely the combination of (i) and (ii) as follows. (i) The weights have to be identical

$$P(E_1) = \frac{1}{4} \text{ and } P(E_2) = \frac{3}{4},$$

(otherwise one can find two pairs $(\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$ that are differently ranked by the two functions),

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$$U(\cdot) = \frac{4}{3} V_2(\cdot)$$

or positive affine transformations thereof. (Again because otherwise $\exists(\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$ that are differently ranked by the two functions.)

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(otherwise one can find two pairs $(\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$ that are differently ranked by the two functions), and (ii) U provides the same ordering of $\mathbb{R}$ as V_2, i.e.

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or positive affine transformations thereof. (Again because otherwise $\exists(\alpha_1^i, \beta_1^j), (\alpha_2^i, \beta_2^j)$ that are differently ranked by the two functions.) Overall we have shown that the function

$$EU = \frac{1}{4}\frac{4}{3} V_2(x_1) + \frac{3}{4}\frac{4}{3} V_2(x_2)$$

represents $\succsim$ over prospects on (E_1, E_2) and that only positive affine transformations $U(\cdot) = \frac{4}{3} V_2(\cdot)$ preserve the EU form. ■

Hybrid case I

In many choice contexts, we have objective probabilities for some events R but not for general events E.

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Structural Assumption 4.9.1 ("Uncertainty plus EU-for-risk"): Structural Assumption 1.2.1 (decision under uncertainty) holds. In addition, for some of the events, notated as *probabilized events* R, a probability $P(R)$ is given. If, for an event-contingent prospect $R_1 : x_1, ..., R_n : x_n$, all outcome events are probabilized with $P(R_j) = p_j$, then this prospect generates a probability distribution $p_1 : x_1, ..., p_n : x_n$ (a probability-contingent prospect) over the outcomes. All event-contingent prospects that generate the same probability-contingent prospect are preference equivalent. Preferences over probability-contingent prospects satisfy EU.

Hybrid case I (2)

To make the probabilized events comparable to the others, look for a suitable $P(R)$:

Matching probability of E — q is a probability such that $1_E 0 \sim 1_q 0$.

Matching probabilities may or may not exist under Structural Assumption 4.9.1.

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(Note the different property name "addivity" on p. 120.) See Figure 4.9.2.

Hybrid case I (3)

Another consistency, relating to complex prospects:

Probabilistic matching — For each partition $E_1, ..., E_n$, the indifference

$$E_1 : x_1, ..., E_n : x_n \sim q_1 : x_1, ..., q_n : x_n$$

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See Figure 4.9.3.

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See Figure 4.9.3.

Theorem 4.9.4 — Under Structural Assumption 4.9.1, the following two statements are equivalent.

1. EU holds.

2. $\succsim$ satisfies weak ordering, existence and additivity of matching probabilities, and probabilistic matching.

Lecture 11: Probability weighting under risk

Motivation

Back to Structural Assumption 2.5.2 (risk).

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- Decision-makers often pay extra attention to small probabilities.

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Back to Structural Assumption 2.5.2 (risk).

Is EU's linearity in probabilitites is a reasonable way to organize choices?

- Why should attitudes towards lotteries be determined solely through attitudes towards sure outcomes?
- Decision-makers often pay extra attention to small probabilities.
- For small gambles, a smooth U is close to linear, contradicting risk aversion vis-a-vis small gambles.

Motivation (2): Example

Non-linearity of U was a modelling choice that we made. Consider preferences in Figure 5.1.1.

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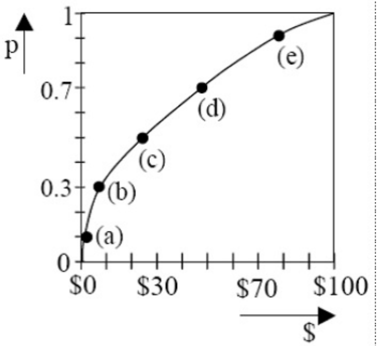

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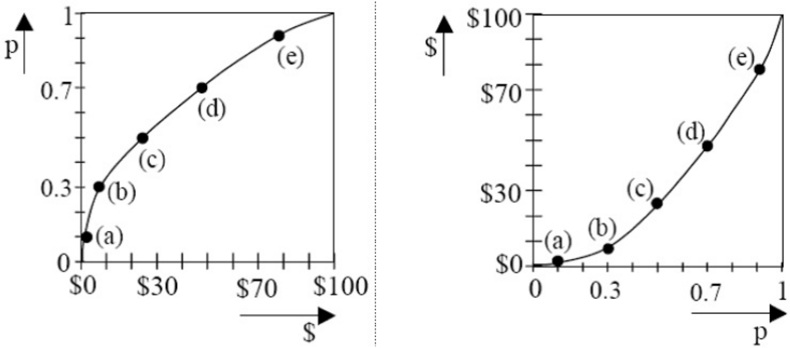

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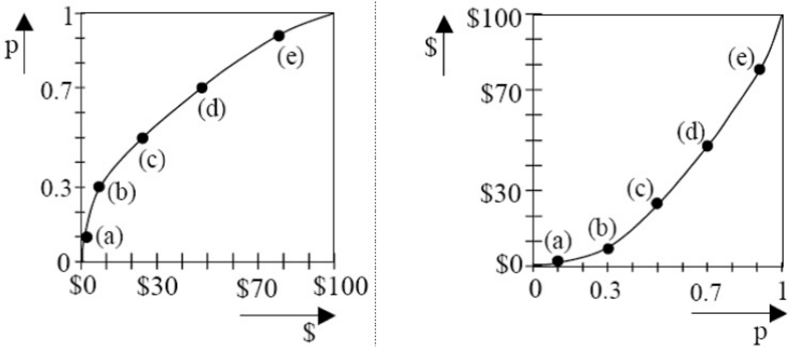

The convex shape is akin to arguing that the decisionmaker dislikes lotteries: each probability p of receiving the high outcome lies below the p-weighted average of receiving the sure outcomes.

Transforming the probability axis

For lotteries with $n > 2$ outcomes, we have to be careful how to transform probabilities.

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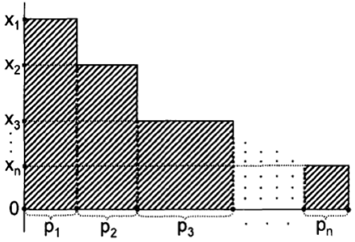
Recall the transformation from EV to EU:

Figure 5.2.1. Expected value



The area shaded by ▨ is the expected value $p_1x_1 + p_2x_2 + \cdots + p_nx_n$.

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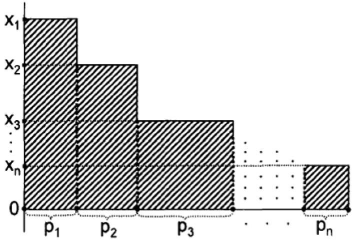

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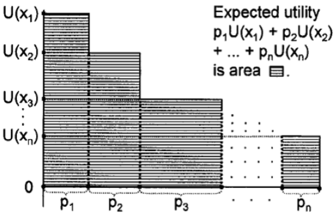
Transforming the outcome axis, the height of each column in the integral was changed according to $U : x \to U(x)$—see next slide.

Transforming the probability axis (2)

Figure 5.2.3. Expected utility

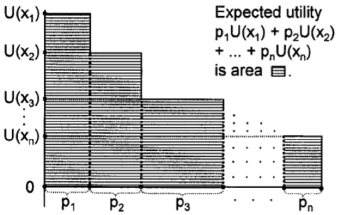


To calculate expected utility, the distance from x_j ("all the way") down to the x-axis has been transformed into the distance $U(x_j)$ for all j.

$$EU(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} p_j\right)\left(U(x_i) - U(x_{i+1})\right)$$

Transforming the probability axis (2)

Figure 5.2.3. Expected utility



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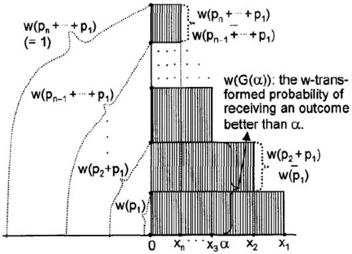
$$EU(x) = \sum_{i=1}^{n} (\sum_{j=1}^{i} p_j)(U(x_i) - U(x_{i+1}))$$

Now, instead transform the probability axis: change the length of each "row" in the integral, and swap axes.

Transforming the probability axis (3)

Figure 5.4.1. Rank-dependent utility with linear utility



Distances of endpoints of layers ("all the way") down to the x-axis are transformed, similar to Figure 5.2.3. The endpoint of the last layer now remains at a distance 1 from the x-axis, reflecting normalization of the bounded probability scale.

Transforming the probability axis (3)

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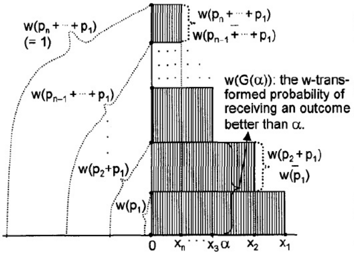

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Figure 5.5.2 in the book (not 5.4.1).

Transforming the probability axis (3)

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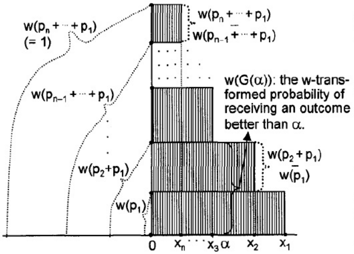

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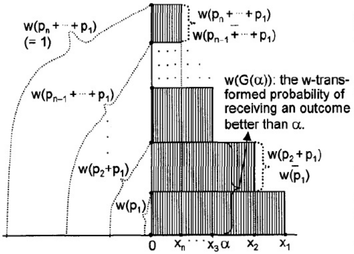

Distances of endpoints of layers ("all the way") down to the x-axis are transformed, similar to Figure 5.2.3. The endpoint of the last layer now remains at a distance 1 from the x-axis, reflecting normalization of the bounded probability scale.

Figure 5.5.2 in the book (not 5.4.1). The transformation assigned non-constant weights to cumulative probabilities.

Rank of outcome x_i — The probability of receiving strictly more than x_i: $p_{i-1} + \ldots + p_1 = \sum_{j=1}^{i-1} p_j$, for $x_1 \geq x_2 \geq \ldots \geq x_n$.

A formulaeic analogue to EU

Consider again the transformation from EV to EU.

$$EV(x) = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} (\sum_{j=1}^{i} p_j)(x_i - x_{i+1}).$$

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Notice that p_i and x_i have 'different roles' in this change of indices:absolute (x_i measures the distance from 0) versus marginal (p_i).

A formulaeic analogue to EU (2)

Now consider the equivalent expression $\sum_{i=1}^{n}(\sum_{j=1}^{i} p_j)(x_i - x_{i+1})$.
Here, $(x_i - x_{i+1})$ is the marginal increase in outcome, and $(\sum_{j=1}^{i} p_j)$ is the (absolute) rank of outcome $i+1$.

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EU(x) &= \sum_{i=1}^{n} p_i U(x_i) \\
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To transform the probability axis, we do the same but in reverse roles.

A formulaeic analogue to EU (3)

$w : [0, 1] \to [0, 1]$ is a probability weighting function — w is strictly increasing and satisfies $w(0) = 0$ and $w(1) = 1$.

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If $w(p) = p$, then $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j) = p_i$.

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If $w(p) = p$, then $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j) = p_i$.

Now construct an expression where the w axis has the marginal role. The marginal w contribution of outcome i is $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)$. (See Figure 5.5.2.)

A formulaeic analogue to EU (4)

Rank-dependent preferences with linear utility — Preferences are represented by

$$RDLU(x) = \sum_{i=1}^{n}[w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)]x_i$$

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Another reason that we do not simply transform p, but rather transform ranks, is that the model with transformed p violates first-order stochastic dominance.

A formulaeic analogue to EU (5)

Final step: both transformations at once, of outcomes and ranks.

A formulaeic analogue to EU (5)

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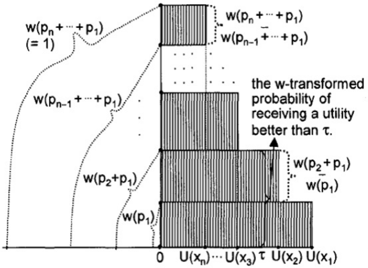
Figure 5.4.2. Rank-dependent utility with general utility



For points on the y-axis ("endpoints of layers"), their distances down to the x-axis are transformed using w. For points on the x-axis ("endpoints of columns"), their distances leftwards to the y-axis are transformed using U.

A formulaeic analogue to EU (6)

Rank-dependent utility — Under Structural Assumption 2.5.2, *rank-dependent utility* (RDU) holds if there exist a strictly increasing utility function $U : \mathbb{R} \to \mathbb{R}$ and a probability weighting function w such that preferences over lotteries $(p_1 : x_1, ...p_n : x_n)$ with rank-ordered outcomes $x_1 \geq ... \geq x_n$ are represented by

$$RDU(x) = \sum_{i=1}^{n}[w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)]U(x_i).$$

Remarks on RDU

- *RDU* is sometimes written as

$$\begin{aligned} RDU(x) &= \sum_{i=1}^{n} \pi_i U(x_i) \text{ where} \\ \pi_i &= w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j). \end{aligned}$$

  Importantly, note that the "decision weight" π_i is a function of all $p_j, j = 1...i$.
- For the best outcome x_1, the formula requires that we find the expression $\sum_{j=1}^{1-1} p_j$. We use the notational convention that $\sum_{j=1}^{0} p_j = 0$.

Remarks on RDU (2)

- For the worst outcome x_n, we use the weighting function's boundary restriction $w(1) = 1$: $w(\sum_{j=1}^{n} p_j) = w(1) = 1$
- If outcomes are not rank-ordered $(x_1 \geq ... \geq x_n)$ we simply re-label them to ensure rank-ordering. Under the assumption that preferences respond only to the distribution over money (see Assumption 2.1.2) this is wlog.

Example

See Section 5.6

Lecture 12: Probability weighting under risk (2)

Recall

Ranked probability p^r — A pair (p, r) where p is the probability of an outcome and r is its rank, in a given prospect.

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In RDU, the decision weight depends on both p and r:

$$RDU(x) = \sum_{i=1}^{n} \pi_i U(x_i)$$

$$= \sum_{i=1}^{n} \pi(p_i^{(p_{i-1}+...+p_1)}) U(x_i)$$

$$= \sum_{i=1}^{n} (w(p_i + ...p_1) - w(p_{i-1} + ... + p_1)) U(x_i)$$

Optimism and Pessimism

Figures 6.3.1 and 6.3.2

Optimism and Pessimism

Figures 6.3.1 and 6.3.2

Pessimism — Worsening the rank increases the decision weight, i.e. $\pi(p^{r'}) \geq \pi(p^r)$ whenever $r' \geq r$.

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Observation: Under RDU, pessimism holds iff w is convex.

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Observation: Under RDU, pessimism holds iff w is convex.

Proof: Plug the definition of π into the definition of optimism and optimism. ■

Typical w

Figure 6.1.1

Behavioral foundation of RDU

Consider Figure 4.1.1 again and make the assumption that Steinbrück wins with probability 0.5.

Behavioral foundation of RDU

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We can assume that Structural Assumption 2.5.2 holds for this example and investigate RDU's prediction.

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Analogously,

$$\pi(0.5^0)(U(\alpha^2) - U(\alpha^1)) = \pi(0.5^{0.5})(U(8) - U(1))$$
$$\pi(0.5^0)(U(\alpha^3) - U(\alpha^2)) = \pi(0.5^{0.5})(U(8) - U(1))$$
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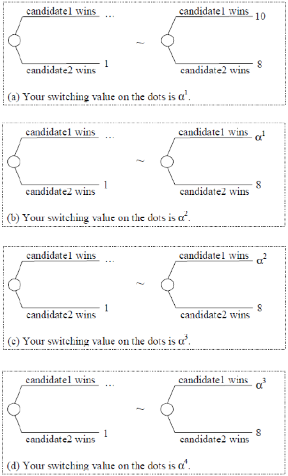
$\rightarrow$ U can be measured under RDU.

Behavioral foundation of RDU (2)

FIGURE 4.1.1. Eliciting $\alpha^1 \dots \alpha^4$ for unknown probabilities



Indicate in each Fig. which outcome on the dots ⋯ makes the two prospects indifferent (the switching value).

Behavioral foundation of RDU (3)

$8 \ominus 1$ — "Receiving 8 instead of 1"

Conditional an some probabilized event (here, candidate 2 wins), $8 \ominus 1$ reflects the preference value of receiving the right prospect.

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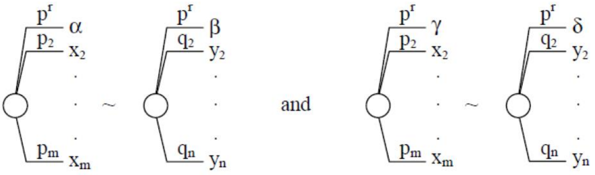
Conditional an some probabilized event (here, candidate 2 wins), $8 \ominus 1$ reflects the preference value of receiving the right prospect. This value depends on the utility difference between 8 and 1 and on the decision weight of the event.

$\alpha \ominus \beta \sim_c^t \gamma \ominus \delta$ — The indifferences in Figure 6.5.1 hold for some outcome probability p and some rank r and some prospects x, y.



The superscript r indicates the rank of p, which is the same for all prospects.

Behavioral foundation of RDU (4)

Observation 6.5.3: Under RDU,

$\alpha \ominus \beta \sim_c^t \gamma \ominus \delta \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$

Behavioral foundation of RDU (4)

Observation 6.5.3: Under RDU,
$\alpha \ominus \beta \sim_c^t \gamma \ominus \delta \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta)$

Proof:

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Proof: (We consider only the case where $\alpha \neq x_i$ for all i, and similarly for β, γ, δ. See the book for the more general case.)

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Proof: (We consider only the case where $\alpha \neq x_i$ for all i, and similarly for β, γ, δ. See the book for the more general case.)
The two indifferences are, under RDU,

$$\pi(p^r)U(\alpha) + \sum_{i=2}^{m} \pi_i U(x_i) = \pi(p^r)U(\beta) + \sum_{j=2}^{n} \pi_j U(y_j)$$

$$\pi(p^r)U(\gamma) + \sum_{i=2}^{m} \pi_i U(x_i) = \pi(p^r)U(\delta) + \sum_{j=2}^{n} \pi_j U(y_j)$$

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$w' > 0 \Rightarrow \pi(p^r) > 0 \Rightarrow U(\alpha) - U(\beta) = U(\gamma) - U(\delta).$ ■

Behavioral foundation of RDU (5)

It could be that

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and

$$\alpha' \ominus \beta \sim_c^t \gamma \ominus \delta$$

for $\alpha' \neq \alpha$.

Behavioral foundation of RDU (5)

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Not under RDU, by observation 6.5.3.

Behavioral foundation of RDU (5)

It could be that

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Rank-tradeoff consistency —Improving an outcome in any $\sim_c^t$ relationship breaks the relationship.

Behavioral foundation of RDU (5)

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for $\alpha' \neq \alpha$.

Not under RDU, by observation 6.5.3.

Rank-tradeoff consistency —Improving an outcome in any $\sim_c^t$ relationship breaks the relationship.

A variant of monotonicity is also implied by RDU:

Strict stochastic dominance — Shifting positive probability mass from an outcome to a strictly preferred outcome leads to a strictly preferred outcome.

Behavioral foundation of RDU (6)

Theorem 6.5.6 — Under Structural Assumption 2.5.2, the following two statements are equivalent.

1. RDU holds with continuous and strictly increasing $U(\cdot)$.

2. $\succsim$ satisfies weak ordering, strict stochastic dominance, continuity, and rank-tradeoff consistency.

Behavioral foundation of RDU (6)

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1. RDU holds with continuous and strictly increasing $U(\cdot)$.

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(No proof.)

Measuring w

(See also Sections 6.4, 7.1 and 7.2)

Measuring w

(See also Sections 6.4, 7.1 and 7.2)

Exercise 6.5.6

(First redo Figure 4.1.1 with 50/50 probabilites, then Figure 4.1.5.)

Likelihood insensitivity

Likelihood insensitivity assings high decision weight to the tails of the outcome distribution. It may be due to cognitive rather than motivational factors.

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RDU can combine it with pessimism. See Figures 7.1.2a and 7.1.2b. Also, see Figure 7.2.4 for a simple version.

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RDU can combine it with pessimism. See Figures 7.1.2a and 7.1.2b. Also, see Figure 7.2.4 for a simple version.

w is likelihood insensitive with insensitivity region $[b_{rb}, w_{rb}]$ — The boundaries b_{rb} (best-rank boundary) and w_{rb} (worst-rank boundary) delimit an intermediate region of ranks where the decision weights are smaller than for best-ranked probabilities and worst-ranked probabilities:

$$w(p) - w(0) \geq w(p + r) - w(r) \text{ if } r + p \leq w_{rb}$$

and

$$w(1) - w(1 - p) \geq w(r + p) - w(r) \text{ if } r \geq b_{rb}$$

See Figure 7.7.1'

Loss ranks

Recall Rank of outcome x_i — The probability of receiving strictly more than x_i: $p_{i-1} + ... + p_1 = \sum_{j=1}^{i-1} p_j$, for $x_1 \geq x_2 \geq ... \geq x_n$.

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Consider a weighting function z for loss ranks and decision weights

$$\pi(p_l) = z(p + l) - z(l),$$

the marginal contribution of the outcome to the loss-rank.

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Consider a weighting function z for loss ranks and decision weights

$$\pi(p_l) = z(p + l) - z(l),$$

the marginal contribution of the outcome to the loss-rank. RDU can be re-written as

$$\sum_{i=1}^{n} (z(p_i + ... + p_n) - z(p_{i+1} + ... + p_n))U(x_i),$$

Loss ranks (2)

Should/can we set $z = w$?

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$$w(p) = 1 - w(1 - p)$$

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Only if we are willing to assume that w is symmetric:

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We may not be willing to do so. (And why should we, considering that w was defined for (gain-)ranks not loss-ranks?)

But we can use the above natural alternative notation if defining z as the dual weighting function of w:

$$z(p) = 1 - w(1 - p)$$

Lecture 13: Prospect theory under risk

Gains and losses

Figure 8.1.1a

Gains and losses

Figure 8.1.1a

Figure 8.1.1b

Gains and losses

Figure 8.1.1a

Figure 8.1.1b

Notice that RDU or EU need to change their components if choice differs between a and b.

Asset integration versus narrow bracketing

Figure 8.1.1c

Asset integration versus narrow bracketing

Figure 8.1.1c

Isolation / mental accounting / narrow bracketing implies that choice in b and c are identical.

Asset integration versus narrow bracketing

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Isolation / mental accounting / narrow bracketing implies that choice in b and c are identical.
Note that this is in accordance with additivity, which in turn is equivalent to freedom from arbitrage (de Finetti).

Asset integration versus narrow bracketing

Figure 8.1.1c

Isolation / mental accounting / narrow bracketing implies that choice in b and c are identical.
Note that this is in accordance with additivity, which in turn is equivalent to freedom from arbitrage (de Finetti).

But problems a and c are the same if things are added – asset integration. The same consumption possibilities exist iff choice is identical between a and c.

Asset integration versus narrow bracketing

Figure 8.1.1c

Isolation / mental accounting / narrow bracketing implies that choice in b and c are identical.
Note that this is in accordance with additivity, which in turn is equivalent to freedom from arbitrage (de Finetti).

But problems a and c are the same if things are added – asset integration. The same consumption possibilities exist iff choice is identical between a and c.

Most discussions argue for asset integration as the only rational (or normatively sound) principle. Wakker (Ch. 8.2): the problem are the risk attitudes.

Loss aversion

The reference point may be viewed as the point where risk attitudes change discontinuously.

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A separate role: Utilities from sure outcomes are also evaluated differently: losses loom larger than gains—loss version.

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E.g. with a reference point of 0, a given function u satisfying $u(0) = 0$, and $\lambda \in \mathbb{R}_+$:

$$U(\alpha) = u(\alpha) \text{ for } \alpha \geq 0$$
$$U(\alpha) = \lambda u(\alpha) \text{ for } \alpha < 0$$

Loss aversion — Preferences are represented by RDU with the above utility function and $\lambda > 1$.

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Candidates for reference point:
(i) Status quo / initial wealth, and choice is framed as choice between changes in wealth

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Loss aversion — Preferences are represented by RDU with the above utility function and $\lambda > 1$.

Candidates for reference point:
(i) Status quo / initial wealth, and choice is framed as choice between changes in wealth
(ii) Expectation (See e.g. Koszegi/Rabin 2005, 2006)

Reference dependence is more than a fixed initial wealth

Figure 8.1.1 questions that the reference point is known and fixed.

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Rabin (2000). Choose between 0 and $11_{0.5}(-10)$, for different wealth levels.

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Figure 8.1.1 questions that the reference point is known and fixed.

Most of decision theory views the reference point as fixed, for the purpose for the present analysis.

Rabin (2000). Choose between 0 and $11_{0.5}(-10)$, for different wealth levels.
Consistently rejecting the lottery implies that that U is concave to an absurd extent.

Prospect theory — overview

Prospect theory (Tversky/Kahneman 1992) combines three elements that we studied: utility curvature (diminishing outcome sensitivity), probabilistic sensitivity and loss aversion.

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For a fixed reference point (which is a gross simplification that may or may not be misleading) PT is almost the same as RDU, with the exception that it uses two weighting functions: one for gains, one for losses.

Prospect theory — overview

Prospect theory (Tversky/Kahneman 1992) combines three elements that we studied: utility curvature (diminishing outcome sensitivity), probabilistic sensitivity and loss aversion.

For a fixed reference point (which is a gross simplification that may or may not be misleading) PT is almost the same as RDU, with the exception that it uses two weighting functions: one for gains, one for losses.

PT involves symmetry/reflection around the reference point: diminishing outcome sensitivity in gains and losses, and decision weights that depend on the reference point.

Prospect theory — formal

For a given prospect $p_1 x_1 ... p_n x_n$, assign labels $1 ... n$ and identify k to satisfy the complete sign-ranking:

$$x_1 \geq ... \geq x_k \geq 0 \geq x_{k+1} \geq ... \geq x_n$$

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Consider a weighting function w^+ that is applied only to outcomes $x_{k+1}, ..., x_n$ by weighting their gain-ranks, and another weighting function w^- that is applied to outcomes $x_1, ..., x_k$ by weighting their loss ranks.

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Consider a weighting function w^+ that is applied only to outcomes $x_{k+1}, ..., x_n$ by weighting their gain-ranks, and another weighting function w^- that is applied to outcomes $x_1, ..., x_k$ by weighting their loss ranks. Decision weights are:

$$\pi_i = \pi(p_i^{p_{i-1}+...+p_1}) = w^+(p_i + ... + p_1) - w^+(p_{i-1} + ... + p_1)$$

for $i \leq k$, and

$$\pi_j = \pi(p_{j_{p_{j+1}+...+p_n}}) = w^-((p_j + ... + p_n) - w^-(p_{j+1} + ... + p_n)$$

for $j > k$.

Prospect theory — formal (2)

Prospect theory — Under Structural Assumption 2.5.2, *prospect theory* (PT) holds if there exist a strictly increasing utility function $U : \mathbb{R} \to \mathbb{R}$ with $U(0) = 0$ and two probability weighting functions w^+ and w^- such that preferences over lotteries $(p_1 : x_1, ...p_n : x_n)$ with completely sign-ranked outcomes $x_1 \geq ... \geq x_k \geq 0 \geq x_{k+1} \geq ... \geq x_n$ for some $k \in \{1, ..., n\}$ are represented by

$$PT(x) = \sum_{i=1}^{k} \pi(p_i^{p_{i-1}+\ldots+p_1})U(x_i) + \sum_{j=k+1}^{n} \pi(p_{j_{p_{j+1}+\ldots+p_n}})U(x_j),$$

where $\pi(p_i^{p_{i-1}+\ldots+p_1})U(x_i)$ and $\pi(p_{j_{p_{j+1}+\ldots+p_n}})$ are given on the previous slide.

Calculating the prospect theory value

See pages 255-256.

Typical U, w^+ and w^-

Figures 8.4.1, 7.1.2b.

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PT can acount for the typical pattern of (experimental) findings:

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For losses, preferences are often risk seeking but closer to risk neutrality than for gains.

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- risk seeking for medium- and high-probability losses
- risk seeking for small-probability gains
- risk averison for small-probability losses
- $\lambda > 1$

Remarks on prospect theory

- Exercise 9.3.2: For a given prospect x define x^+ as the prospect that replaces all of x's negative outcomes by 0, and x^- as the prospect that replaces all of x's positive outcomes by 0. Show that $PT(x) = PT(x^+) + PT(x^-)$.

Remarks on prospect theory

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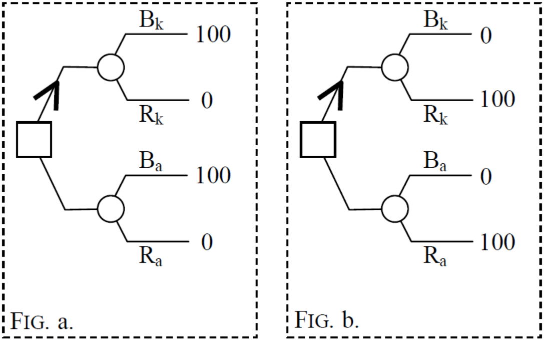
- For $x = x^-$, PT coincides with RDU with $w = 1 - w^-(1-p)$.
- Exercise 9.3.3: The decision weights need not sum to 1.

Lecture 14: Ambiguity preferences

The Ellsberg Paradox

The Ellsberg Paradox

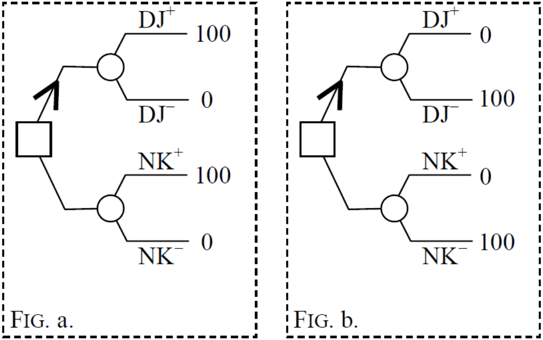
FIGURE 10.1.1. Ellsberg paradox



Arrows indicate majority preferences.

The Ellsberg Paradox (2)

FIGURE 10.1.2. Home Bias



Arrows indicate majority preferences in the United States.

The Ellsberg Paradox (3)

Source — A set of events.

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Source preference — For all events A from source $\mathbb{A}$ and all events B from source $\mathbb{B}$, it may be that

$$1_A 0 \succsim 1_B 0 \text{ and } 1_{A^c} 0 \succsim 1_{B^c} 0$$

but it cannot be that

$$1_B 0 \succsim 1_A 0 \text{ and } 1_{B^c} 0 \succsim 1_{A^c} 0.$$

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Probabilistic sophistication — There exists a probability measure P on S such that each event-contingent prospect is evaluated according to its corresponding probability-contingent prospect.

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Probabilistic sophistication — There exists a probability measure P on S such that each event-contingent prospect is evaluated according to its corresponding probability-contingent prospect.

The Ellsberg example shows a source preference and violates probabilistic sophistication.

Overview of RDU under uncertainty

Decision weights for EU under uncertainty: events are assigned (additive) probabilites $P(E)$

Decision weights for RDU under risk: ranked probabilities are assigned $w-$transformed weights

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Decision weights RDU under uncertainty: ranked events are assigned (non-additive) $W-$transformed weights.

Overview of RDU under uncertainty

Decision weights for EU under uncertainty: events are assigned (additive) probabilites $P(E)$
Decision weights for RDU under risk: ranked probabilities are assigned $w-$transformed weights

Decision weights RDU under uncertainty: ranked events are assigned (non-additive) $W-$transformed weights.

See p. 279 on piecing together and surprising lack of surprise.

Event weights

Under Structural Assumption 1.2.1, consider a prospect $x = (E_1 : x_1, ... E_n : x_n)$, where outcomes are rank-ordered, $x_1 \geq ... \geq x_n$.

Rank of outcome x_j — The event of receiving an outcome strictly better than x_j, denoted by $R = E_{j-1} \cup ... \cup E_1$.

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Ranked event — E^R, a pair (E, R) where R is event E's rank.

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Decision weight $\pi(E^R)$ — The W-contribution of event E to the rank: $\pi(E^R) = W(E \cup R) - W(R)$.

RDU under uncertainty – formal

RDU under uncertainty (Choquet expected utility) — Under Structural Assumption 1.2.1, rank-dependent utility (RDU) holds if there exist a strictly increasing continuous utility function $U : \mathbb{R} \to \mathbb{R}$ and a weighting function W such that preferences over prospects $x = (E_1 : x_1, ...E_n : x_n)$ (with $x_1 \geq ... \geq x_n$) are represented by

$$RDU(x) = \sum_i^n (W(E_i \cup ... \cup E_1) - W(E_{i-1} \cup ... \cup E_1))U(x_i)$$
$$= \sum_i^n \pi(E_i^{E_{i-1} \cup ... \cup E_1})U(x_i)$$

RDU can accommodate the Ellsberg paradox

Example 10.3.1

RDU can accommodate the Ellsberg paradox

Example 10.3.1

Note: W has many degrees of freedom – hard to use in empirical applications

Estimation of RDU

Estimation of RDU

The measurements in Figure 4.1.1 and 4.1.2 are still valid: $U(\alpha^k) - U(\alpha^{k-1})$ is constant in k. See Exercise 10.5.3.

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With U measured, we can find the weights:

$$\text{If } \alpha \sim 1_E 0, \text{ then } W(E) = U(\alpha)/U(1).$$

Pessimism and optimism

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Pessimism — Worsening the rank increases the decision weight, i.e. $\pi(E^{R'}) \geq \pi(E^{R})$ whenever $R' \supset R$.

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Exercise 10.4.2: Pessimism is equivalent to

$$W(A \cup B) \geq W(A) + W(B) - W(A \cap B)$$

Likelihood insensitivity

See Section 10.4.2 for a formulation of a likelihood insensitivity $[B_{rb}, W_{rb}]$, involving a behavioral definition of "revealed more likely than".

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$$RDU(x) = \alpha U(x_1) + (1 - \alpha)U(x_n)$$

("α-Hurwicz criterion")

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Neo-additive weighting function — There exist $(a, b) > 0$ with $a + b < 1$ and a probability measure P such that $W(\emptyset) = 0, W(S) = 1$ and $W(E) = b + aP(E)$ for all other E.

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With neo-additive weighting, we have:

$$RDU(x) = b\ sup_{s \in S} U(x(s)) + aEU(x) + (1 - a - b)inf_{s \in S} U(x(s))$$

Sets of probabilities

RDU with probability intervals — There exists α and for each event E there exists an interval I_E of probabilities such that:

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More popular, and related – but not a special case of RDU: Multiple priors (Gilboa/Schmeidler (1989)).

Maxmin expected utility — There exists a convex set C of probability measures (priors) on S, and preferences are represented by:

$$MEU(x) = inf_{P \in C} EU_p(x)$$

Behavioral foundation of RDU under uncertainty

Theorem 10.5.6 — Under Structural Assumption 1.2.1, the following two statements are equivalent.

1. RDU holds with continuous and strictly increasing $U(\cdot)$.

2. $\succsim$ satisfies weak ordering, monotonicity, continuity, and rank-tradeoff consistency.

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(Essentially the same as Theorem 6.5.6 for RDU under risk, except that rank-tradeoff consistency is now defined for ranked events, not ranked probabilities.)