Source Theory: A Tractable and Positive Ambiguity Theory

Aurélien Baillon, Emlyon Business School, Lyon, France

Han Bleichrodt, Department of Economics (FAE), University of Alicante, Spain

Chen Li, Erasmus School of Economics, Erasmus University Rotterdam, the Netherlands

Peter P. Wakker, wakker@ese.eur.nl, Erasmus School of Economics, Erasmus University Rotterdam, the Netherlands

October 2023

Abstract. This paper introduces source theory, a new theory for decision under ambiguity. It shows how Savage's subjective probabilities, with source-dependent nonlinear weighting functions applied to them, can be used to model Ellsberg's ambiguity (unknown probabilities). It can do so in Savage's framework of state-contingent assets, and does not need complex two-stage gambles, multistage optimization principles, expected utility for risk (descriptively problematic), or any linear algebra. Still the mathematical analysis is simple, with intuitive preference axioms, tractable calculations and prescriptive implementability, empirically realistic fittings and predictions, and convenient graphical representations of ambiguity attitudes. We provide new ways to compare weighting functions, not between persons as is common, but within one person and between sources. So-called p-matchers turn out to capture uncertainty attitudes well, giving Arrow-Pratt-like transformations, however, "within" rather than "outside" functions. Within-person-between-sources comparisons are the main novelty of ambiguity over risk, first demonstrated by Ellsberg's paradox.

Keywords: subjective beliefs; ambiguity aversion; Ellsberg paradox; source of uncertainty

1. Introduction

Decision analysis has traditionally been based on expected utility (EU). Whereas EU had been used for measuring uncertainty attitudes long before 1954 (reviewed by Arrow 1951), and its formal framework of state-contingent assets had already been introduced by Debreu (1953) and others, Savage (1954) provided the main contribution to its foundation. He did so by providing a preference foundation, revealing the critical empirical tests and, more importantly, clarifying its normative status. Only then EU became widely used in prescriptive decision analysis. However, criticisms were raised soon.

Ellsberg (1961) showed that Savage's use of subjective probabilities is not only descriptively problematic (Trautmann & van de Kuilen 2015; l'Haridon et al. 2018), but, according to many, also normatively, due to ambiguity aversion (Gilboa & Marinacci 2016). Given the ubiquity of ambiguity in applications and everyday life, it may be surprising that ambiguity models, that had been available long before (Wald 1950) were not widely used in the first decades after 1961. Only when Gilboa & Schmeidler (1989) and Schmeidler (1989) provided preference foundations for these models, did the field of ambiguity take off, catching up with many urgent questions. Many non-Bayesian decision analysts use ambiguity models prescriptively nowadays, for instance, for climate change policies (Berger, Emmerling, & Tavoni 2017; Chandy et al. 2019).

Many ambiguity theories have been proposed (Gilboa & Marinacci 2016). However, they have as yet been based almost exclusively on Anscombe-Aumann's framework, rather than Savage's (1954). Savage's framework has generally been considered too difficult to use for ambiguity. Karni, Maccheroni, & Marinacci (2022, p. 229) wrote: "Savage's most brilliant measure-theoretic approach was not so easily extended beyond its original domain and this was a main reason why so little happened in the field for decades after his 1954 masterpiece." Anscombe-Aumann's framework simplifies the mathematical analysis but has to make restrictive assumptions. Further, many ambiguity models today are very general, making it hard to derive implications and predictions. Further discussion of related literature is in §9.

For fitting data on ambiguity, empirical studies often used the source method. Here, subjective probabilities are assumed and then source-dependent probability weighting functions are used.[1] For instance, a different weighting function for the known than for the unknown Ellsberg urn can be used to accommodate the Ellsberg paradox. And a different

[1] See, for instance, Abdellaoui et al. (2011), Abdellaoui et al. 2021; Baillon et al. (2018), Bleichrodt, Grant, & Yang (2023), Grevenbrock et al. (2021), Kemel & Gutierrez (2023), and Spiliopoulos & Hertwig (2023).

weighting function for foreign stocks than for domestic stocks accommodates the home bias in finance. This method has been suggested since the origins of the ambiguity literature (Fellner 1961 p. 672; Kahneman & Tversky 1979 p. 289). It has as yet lacked any theoretical modeling or foundation, though. Therefore, it was never used in theoretical analyses, prescriptive applications, or critical empirical tests. Surveys of ambiguity *theories* did not mention it (Gilboa & Marinacci 2016).

It may be amazing that for a useful empirical method that has been around for decades, no theoretical model has ever been provided yet. The reason is the aforementioned (suggested) difficulty of the framework most suited for this method: Savage's (1954).[2] We will show that ambiguity can be analyzed in Savage's framework after all. And, importantly, that it can be done in an intuitive and tractable manner, making it suited not only for empirical applications but also for theoretical analyses and for non-Bayesian prescriptive applications.

This paper introduces source theory (ST). We, first, provide a formal framework for the source method. Second, we provide a preference axiomatization for ST and for the main attitudinal components of ambiguity. In general, besides showing theoretical soundness, empirical testability, and the normative status of a model, preference foundations also show what the relevant concepts are to be measured and determined, such as Savage's (1954) subjective probabilities or Gilboa & Schmeidler's (1989) sets of priors. Our foundation shows that pmatchers, defined later, are the relevant concepts for uncertainty attitudes.

We will not only analyze aversion to ambiguity, but also a component called insensitivity. It captures ambiguity perception (Baillon et al. 2021) while adding dependence on cognitive ability, a subjective element. This dependence, while irrelevant for normative applications, is useful for descriptive applications, Insensitivity accommodates the prevailing four-fold pattern of ambiguity (Trautmann & van de Kuilen 2015).

Unlike with aversion, there are no standard tools from convex analysis or other mathematical disciplines available for analyzing insensitivity. This paper is even the first to define insensitivity formally.[3] The analysis of insensitivity is, admittedly, complex, but is

---

[2] It provides a rich domain of events convenient for analyzing weighting functions. We will use Gilboa's (1987) extension of Savage (1954) to ambiguity or, yet more precisely, a biseparable generalization of that. Hence, our results are valid under many ambiguity and risk theories (§2).

[3] Compare Tversky & Wakker (1995 Footnote 7) and Lewandowski (2017 Definition 7), which do not specify the logical status of their boundary events. Section 9 discusses a cavexity condition that has often been used in the literature. Insensitivity shows that two seemingly separate concepts in the literature, ambiguity perception and inverse-S probability weighting for risk, are two sides of the same coin. For the latter property, so widely

indispensable for empirical applications. Our Theorems 9 and 14 show that it is also indispensable to fully capture ambiguity theoretically. The problems of insensitivity should be resolved rather than ignored.

ST's use of weighting functions allows for convenient graphical representations of ambiguity attitudes. For instance, Figure 1 in §8 will completely capture the risk, uncertainty, and ambiguity attitudes of a given subject for the CAC40 stock index. This feature makes ST convenient for the practice of decision analysis. Decision makers can easily express their desired ambiguity attitude graphically. Extra aversion is captured by vertically moving the curve downward, and extra perceived uncertainty about probabilities by horizontally moving weight ("probability mass") from the expected middle to the unexpected extremes, as-if increasing variance.

The outline of this paper is as follows. Standard concepts are in §2. Section 3 formalizes the theoretical framework underlying ST, including sources. We then define the well-known source preference and the new source insensitivity, both comparatively (§4) and absolutely (§5), as yet for uncertainty in general. Section 6 uses these concepts to obtain our first main result, Theorem 9, an axiomatization of ST. It captures the meaning of probabilistic sophistication directly in terms of ambiguity: ambiguity should be uniform across the source. Section 7 then assumes ST and shows that so-called pmatchers provide a proper tool to analyze uncertainty attitudes. They map subjective probabilities, formally called a-neutral, of one source into gambling-equivalent a-neutral probabilities of another source. Dimmock, Kouwenberg, & Wakker (2016 Theorem 3.1) showed that matching probabilities are well-suited to analyze ambiguity attitudes. Our pmatchers generalize those matching probabilities to general uncertainty. Our second and third main results, Theorems 11 and 12 in §7, show that, under the specific ST, source preference is equivalent to a disliked (roughly, lower) pmatcher and insensitivity of sources (ambiguity perception) is equivalent to an insensitive (inverse-S shaped) pmatcher. This result further illustrates the tractability of ST because pmatchers can directly be derived from the a-neutral probabilities. Remarkably, based on intuitive arguments, Kemel & Gutierrez (2023) used pmatchers empirically and demonstrated their usefulness. We axiomatically justify their choice. The rest of §7 and §8 further illustrate the tractability of ST. Section 9 discusses related frameworks and Section 10 concludes. Proofs are in Appendix B.

---

documented in empirical studies, it is extra remarkable that no fully formalized definition had been provided in the literature yet.

2. Basic Definitions

S denotes a *state space*. Its subsets are *events*.[4] Later assumptions will imply that S is infinite. A *weighting function* W is defined on all events and satisfies $W(\emptyset) = 0$, $W(S) = 1$, and $A \supset B \Rightarrow W(A) \geq W(B)$. *Probability measures* P are weighting functions that satisfy additivity. Following Savage (1954), we only impose finite additivity.

Γ denotes a set of consequences, or *outcomes*, and can be finite or infinite. We assume that all outcomes are gains, leaving reference-dependence to future studies. An *act* maps S to Γ and is finite-valued. We denote acts by $x = (E_1 : x_1, \ldots, E_n : x_n)$, mapping every $s \in E_j$ to x_j. It is implicit in this notation that the E_js partition S and that the x_js are outcomes. $\alpha_E \beta$ denotes $(E : \alpha, E^c : \beta)$. A *preference relation* $\succcurlyeq$ of an agent is given over acts, with $\preccurlyeq$, $\succ$, $\prec$, and $\sim$ as usual. We assume that $\succcurlyeq$ is a *weak order* (transitive and complete). As usual, we identify constant acts with outcomes, so that $\succcurlyeq$ also denotes a preference relation over outcomes.

We assume *biseparable utility*: there exist a *utility function* $U : \Gamma \to \mathbb{R}$ and a weighting function W such that preferences over binary acts $\gamma_E \beta$, with $\gamma \succcurlyeq \beta$, maximize

$$W(E)U(\gamma) + (1 - W(E))U(\beta). \tag{1}$$

An event is *null* if its outcomes never affect preference. We assume that $\succcurlyeq$ satisfies *strong monotonicity:* strictly improving an outcome of an act on a nonnull event strictly improves the act.

In Eq. 1, $W(E)$ is called the *decision weight* of event E when ranked best, i.e., when yielding the best outcome(s). We interpret it as the share of the agent's attention given to event E if it is ranked best. We sometimes denote it as $\pi^b(E)$, or π^b for short if E is understood. The complementary share of attention, $1 - W(E)$, is the *decision weight* of event E^c when *ranked worst* (yielding the worst outcome(s)), also denoted $\pi^w(E^c)$, or π^w for short if E^c is understood. For acts $(E_1 : x_1, \ldots, E_n : x_n)$ with more than two outcomes $x_1 \succ \cdots \succ x_n$ there are "middle" events with neither best nor worst outcomes. The remaining share of attention is divided among them. Empirical studies usually find that the attention paid to a fixed event E_i when ranked middle is more or less constant (even though theoretically it could depend on several aspects of the act). In informal interpretations we will, therefore, sometimes use the term "middle" attention/weight $\pi^m(E_i)$ without further specification, or π^m. The preference conditions defined later will specify all comparisons between π^b, π^m,

[4] Our results do not change if we endow S with an algebra or a σ-algebra of events and consider only measurable acts.

and π^w, and later theorems will confirm that these comparisons capture the relevant aspects for uncertainty and ambiguity attitudes.

Rank-dependent utility (*RDU*) (or Choquet expected utility), a special case of biseparable utility, holds if $(E_1{:}x_1, \dots, E_n{:}x_n)$, with $x_1 \succcurlyeq \cdots \succcurlyeq x_n$, is evaluated by $\sum_{j=1}^{n} \pi_j U(x_j)$ with $\pi_j = W(E_j \cup \cdots \cup E_1) - W(E_{j-1} \cup \cdots \cup E_1)$; here, $\pi_1 = W(E_1) = \pi^b(E_1)$. *Expected utility* (*EU*) holds if, further, W is a probability measure. EU holds if and only if the decision weight of a fixed event is the same for all acts, in particular, when ranked best or worst. This weight then always is the probability of that event. Deviations from EU are characterized by the way in which the decision weight of a fixed event varies over different acts. We will analyze uncertainty attitudes from this perspective.

For events A, B, we define $A \succcurlyeq B$ (A is *preferred* to B) if $\gamma_A\beta \succcurlyeq \gamma_B\beta$ for some $\gamma \succ \beta$. Under biseparable utility, $\succcurlyeq$ is represented by W. The *event interval* $[E, G]$ contains all events F with $W(E) \leq W(F) \leq W(G)$, i.e, $E \preccurlyeq F \preccurlyeq G$.

The function $\pi^w(E) = 1 - W(E^c)$ is called the *dual* of W. Although duality is not needed in our formal analysis, it facilitates conceptual understanding. Insensitivity conditions defined later always involve two conditions that are in fact one condition but imposed both on the weighting function and its dual.

We assume Gilboa's (1987) Savage-type richness: (i) there are at least three nonequivalent outcomes; (ii) *convex-rangedness* holds: for all events $A \subset C$ and $W(A) \leq \mu \leq W(C)$, there exists $A \subset B \subset C$ with $W(B) = \mu$. The theorems in this paper can be turned into complete preference foundations by adding the necessary and sufficient preference conditions that Gilboa gave for RDU. For brevity, we do not repeat them. RDU, i.e., prospect theory for gains, is the primary model of interest for our analysis. Intuitive discussions that need specific models will be targeted to this model.

Our formal results are valid under all special cases of biseparable utility that, besides rank-dependent utility/Choquet expected utility for uncertainty (Gilboa 1987; Schmeidler 1989; Tversky & Kahneman 1992 for gains), include various multiple prior models (maxmin EU; α-maxmin: Ghirardato, Maccheroni, & Marinacci 2004). For risk, a special case of uncertainty and part of our model, our results are valid under many models (Wakker 2010 Observation 7.11.1). ST generalizes Chateauneuf, Eichberger, & Grant's (2007) neo-additive utility.

3. Source Theory

We first formalize the concept of a source. Formally, *sources* are algebras of events. Uncertainty attitudes may differ for different sources. For instance, Tversky & Fox (1995) showed that basketball fans are ambiguity averse for Ellsberg urns but ambiguity seeking for basketball games. This illustrates that ambiguity theories have to reckon with source dependence. Our formal results will always assume that weighting functions satisfy convex-rangedness when restricted to sources. That is, we only consider "rich", infinite, sources. We sometimes use finite sources in illustrations. An act x is *from a source* if it is measurable w.r.t. that source, i.e., $x^{-1}(\alpha)$ is in the source for each outcome α.

Our results can in principle be applied to any algebra of events, taking it as a source. In applications, people will usually specify sources that are of special interest to them, and then sources are exogenous. In Ellsberg's paradox they are also exogenous, determined by information about urns irrespective of preference. Therefore, sources will mostly be exogenous, similarly as commodities in consumer theory are, and this is our primary interpretation. Other authors have preferred endogenous interpretations of sources. Most of our results concern uniform sources (defined later), and they can be taken endogenous. Observation 17 will list the related results, valid both if sources are endogenous and if they are exogenous. Grant, Rich, & Stecher (2022) similarly adopted a flexible interpretation of sources.

Risk, denoted $\mathcal{R}$, is a special source of uncertainty for which the probabilities of its events R are known, denoted $K(R)$. *Risky acts* are acts from that source. We throughout assume *stochastic dominance*, i.e., for $P = K$ and all risky acts x, y:

$$\left[\text{For all } \alpha \in \Gamma\colon P(x \succcurlyeq \alpha) \geq P(y \succcurlyeq \alpha)\right] \quad \Rightarrow \quad x \succcurlyeq y. \tag{2}$$

It implies that risky acts that induce the same probability distribution over outcomes are indifferent (apply Eq. 2 both ways.) In other words, preferences over risky acts depend only on the probability distribution they induce over outcomes. We usually identify risky acts with their induced probability distributions. w is a *probability weighting function* if $w\colon [0,1] \to [0,1]$, w is strictly increasing, $w(0) = 0$, and $w(1) = 1$. Our assumptions imply:

OBSERVATION 1. There exists a continuous probability weighting function w such that, for all risky events R,

$$W(R) = w(K(R))\ . \tag{3}$$

Because $R_1 \succcurlyeq R_2 \Leftrightarrow K(R_1) \geq K(R_2)$ for risky events R_1 and R_2, we can rewrite Eq. 2 for risky acts x, y as:

$$\left[\text{For all } \alpha \in \Gamma\colon \{s \in S\colon x(s) \succcurlyeq \alpha\} \succcurlyeq \{s \in S\colon y(s) \succcurlyeq \alpha\}\right] \quad \Rightarrow \quad x \succcurlyeq y. \tag{4}$$

Cumulative dominance holds if Eq. 4 also holds for all acts x, y that are not risky. We assume cumulative dominance throughout the paper.

DEFINITION 2. *Source theory* (*ST*) holds for a source $\mathcal{S}$ if, besides biseparable utility, *local probabilistic sophistication* holds for $\mathcal{S}$. That is, there exists a probability measure $P_\mathcal{S}$ on $\mathcal{S}$ such that Eq. 2 holds with $P = P_\mathcal{S}$ for all acts x, y from that source. Such sources are called *uniform*.

In the definition, $P_\mathcal{S}$ may concern subjective probabilities, or it may merely be a mathematical device without any particular interpretation. Risk is a special case of local probabilistic sophistication. As with risk, preferences over acts from a uniform source $\mathcal{S}$ are entirely determined by the probability distributions over outcomes induced by $P_\mathcal{S}$ under ST (but they depend on the source $\mathcal{S}$). Consequently, as for risk, we can define a continuous source-dependent probability weighting function $w_\mathcal{S}$ under ST such that:

$$\text{For a uniform } \mathcal{S}\text{: } W = w_\mathcal{S} \circ P_\mathcal{S}. \tag{5}$$

The proof is identical to Observation 1 and is omitted. We call $w_\mathcal{S}$ the *source function* of $\mathcal{S}$. $w = w_\mathcal{R}$ is the source function for risk. The source $\mathcal{S}$ is *ambiguity neutral* if it satisfies local probabilistic sophistication, the risky source $\mathcal{R}$ is present in the domain of events, and $w_\mathcal{S} = w$. That is, the probabilities $P_\mathcal{S}$ are treated as if objective. An agent is *ambiguity neutral* if the algebra of all events is ambiguity neutral. In general, for any uniform source $\mathcal{S}$, we call $P_\mathcal{S}$ the *a-neutral probability measure* because it would serve as a regular probability measure had the agent been ambiguity neutral (with unchanged risk attitude). It can be interpreted as the beliefs of a “Bayesian twin” of the agent.

Chew & Sagi (2008) demonstrated the importance of local probabilistic sophistication for ambiguity and, thus, “revived” the use of probabilities to analyze ambiguity. ST combines this insight with the classical idea of probability weighting for ambiguity. It, further, specifies the relevant phenomena and concepts with formalizations and axiomatizations.

Expected utility holds *in a source* $\mathcal{S}$ if EU represents preferences over all acts from $\mathcal{S}$. Then the source is uniform and $w_\mathcal{S}$ is the identity function. EU for risk thus means that w is the identity. We summarize the assumptions made so far. They are assumed explicitly in theorems, and implicitly elsewhere.

ASSUMPTION 3 [Structural Assumption]. S is a state space, and Γ an outcome set. Acts $x = (E_1: x_1, \ldots, E_n: x_n)$ are finite-valued maps from S to Γ, endowed with a weak order $\succcurlyeq$, the preference relation. Γ contains at least three nonindifferent outcomes. Sources are sub-algebras of events. Biseparable utility holds, with a utility function $U: \Gamma \to \mathbb{R}$, a weighting function W on S, and a representation $W(E)U(\gamma) + (1 - W(E))U(\beta)$ over binary acts $\gamma_E\beta$ ($\gamma \succcurlyeq \beta$). W is convex-ranged for every source. The relation $\succcurlyeq$ is extended to outcomes via constant acts, maximizing U, and to events via bets on them, maximizing W. Strong monotonicity and cumulative dominance hold.

4. Comparative Uncertainty and Ambiguity Attitudes

This section analyzes the main novelty of uncertainty relative to risk: within-person-between-sources comparisons. This was a big but not always sufficiently appreciated novelty in Ellsberg (1961).

4.1. Introduction

We take risk as a single source of uncertainty, partly for tractability reasons. The domain of ambiguity, however, is too rich to be taken as one single source, similarly as the domain of non-monetary commodities is too rich to be taken as one. We will, therefore, distinguish between different sources of ambiguity. This leads to within-person between-sources comparisons.

We consider comparisons between two sources $\mathcal{A}$ and $\mathcal{C}$. Although our analysis is symmetric between the sources, an asymmetric presentation is more convenient. In elucidations, we take $\mathcal{C}$ as an established source used for calibration, and $\mathcal{A}$ as a new source to be compared with $\mathcal{C}$. Ambiguity concerns the special case of $\mathcal{C} = \mathcal{R}$. Generic elements of $\mathcal{A}$ are A, A_i, A_j, and those of $\mathcal{C}$ are C, C_i, C_j.

The preference conditions introduced next cover all comparisons between decisions weights: source dispreference will imply that π^b loses more weight than π^w (§4.2), and insensitivity that π^m loses more than either π^b or π^w (§4.3).

4.2. Source Preference

We first consider changes of decision weights for two-fold partitions (A_1, A_2) and (C_1, C_2) with $\pi^b(A_1) = \pi^b(C_1)$.[5] It means $\pi^w(A_2) = \pi^w(C_2)$ so that the two partitions

[5] Following mathematical conventions, we denote partitions using brackets rather than braces here, to indicate that the ordering of the events may sometimes be relevant.

involve the same decision weights. We call such two-fold partitions *matching*. If $\mathcal{C} = \mathcal{R}$, then $K(C_1)$ is the *matching probability* of A_1. It is the gambling equivalent objective probability of A_1.

DEFINITION 4. (*Source) preference* for $\mathcal{C}$ over $\mathcal{A}$ holds, or $\mathcal{C}$ is *preferred* to $\mathcal{A}$, if for all partitions (C_1, C_2) from $\mathcal{C}$ and (A_1, A_2) from $\mathcal{A}$ we have

$$W(A_1) = W(C_1) \Rightarrow W(A_2) \le W(C_2). \quad (6)$$

Thus if, in the two matching partitions, we change the ranks of A_2 and C_2 from worst to best then $\pi^b(A_2) < \pi^b(C_2)$ may occur (whereas we had $\pi^w(A_2) = \pi^w(C_2)$), meaning that A_2 loses more weight than C_2 (or gains less weight). There is less attention for favorable events, and more for unfavorable events, for source $\mathcal{A}$ than for source $\mathcal{C}$, leading to more dislike of $\mathcal{A}$. Formally, source preference (or preference for short) for $\mathcal{C}$ over $\mathcal{A}$ allows for such inequalities but precludes any reversed inequalities.

Source indifference means source preference both ways. We then also say that the two sources are *equally preferred*. *Ambiguity aversion* for source $\mathcal{A}$ means source preference for $\mathcal{R}$ over $\mathcal{A}$. Within-person between-sources comparative results for ambiguity and uncertainty coincide: *more ambiguity aversion* for source $\mathcal{A}$ than for source $\mathcal{C}$ means source preference for $\mathcal{C}$ over $\mathcal{A}$. *Ambiguity indifference* for source $\mathcal{A}$ means that there is both ambiguity aversion and ambiguity seeking for $\mathcal{A}$. Then $\mathcal{A}$ is equally preferred as $\mathcal{R}$.

Given the richness and monotonicity that we assume, source preference for $\mathcal{C}$ over $\mathcal{A}$ is equivalent to the following condition (Appendix A):

$$W(A_1) \ge W(C_1) \Rightarrow W(A_2) \le W(C_2). \quad (7)$$

Similar conditions have been used in many models in the literature.

4.3. Insensitivity

Uncertainty attitudes include, besides source preference (including ambiguity aversion) which is a motivational component, also a cognitive component, insensitivity, which reflects (lack of) discriminatory power. The behavioral implication of insensitivity is extremity-orientedness, focusing on extreme events rather than middle events.[6] Such focusing moves the linear perception of likelihood in expected utility in the direction of a flat default of "just don't know," where middle events become one blur.

Many ambiguity models in the literature contain a component of ambiguity perception, determined by perceived vagueness of information (Marinacci 2015). For instance, in the α-maxmin model, the size of the set of priors can be interpreted this way. Increasing the set of priors increases insensitivity, with more weight for extreme events. Baillon et al. (2021) theoretically analyzed indexes of insensitivity. They showed that those indexes agree with popular indexes of perception in many ambiguity models (that, if taken normative, involve no cognitive limitations), e.g., sizes of sets of priors in several multiple priors models.

Our insensitivity component generalizes ambiguity perception by allowing dependence on the cognitive ability of the agent, a subjective element. Many studies have shown that, empirically, insensitivity does indeed depend on cognitive ability.[7] Hence, for empirical work, this subjective generalization of ambiguity perception is desirable. For risk, where vagueness of probabilities does not play any role, insensitivity leads to inverse-S shaped probability weighting, and it is the prevailing empirical pattern (Fehr-Duda & Epper 2012). Gonzalez & Wu (1999) provided an excellent explanation of insensitivity for risk. Insensitivity is yet stronger under ambiguity (Trautmann & van de Kuilen 2015),[8] being reinforced by the perceived uncertainty about the probabilities. Thus, our insensitivity component brings together ambiguity perception and inverse-S probability weighting. Henkel (2023) found strong empirical relations between these concepts.

For insensitivity, we compare middle events with extreme events. To consider middle events, we need threefold partitions (A_1, A_2, A_3) and (C_1, C_2, C_3). We consider partitions where $\pi^b(A_1) = \pi^b(C_1)$ and $\pi^w(A_3) = \pi^w(C_3)$. Then the remaining share of attention, informally denoted $\pi^m = 1 - \pi^b - \pi^w$, will also be the same for the two partitions, and the partitions involve the same decision weights. We, therefore, also call such threefold partitions *matching*. The following condition entails that changing middle weights π^m to extreme weights π^b or π^w involves more gain of weight as there is more insensitivity. The

---

[6] Following Savage (1954), events in themselves carry no value. Their favorability and extremity is determined by the outcomes they yield, which depends on the act considered. This is implicitly understood throughout this paper.

[7] These studies include Choi et al. (2022), Dimmock et al. (2021), and Grevenbrock et al. (2021).

[8] In general, phenomena for risk also occur for ambiguity, but to a more pronounced degree. See Fellner (1961 p. 684) and many other references in Wakker (2010 p. 292). More recent references include Kemel & Paraschiv (2013) and Maafi (2011).

insensitivity region below, discussed further after the definition, serves to ensure, through the inequalities in the premises, that middle events are really middle and not best or worst. For instance, when comparing π^m with π^b, we ensure that π^m is not π^w, and is neither close to it. This way we avoid comparisons between the two extremes.

DEFINITION 5. There is *more insensitivity* for source $\mathcal{A}$ than for source $\mathcal{C}$ (or $\mathcal{A}$ is *more insensitive than* $\mathcal{C}$) with *insensitivity region* $[B, D]$, if for all partitions (C_1, C_2, C_3) from $\mathcal{C}$ and (A_1, A_2, A_3) from $\mathcal{A}$:

$$W(A_1) = W(C_1) \ \& \ W(A_3^c) = W(C_3^c) \leq W(D) \implies W(A_2) \geq W(C_2) \tag{8}$$

and

$$W(B) \leq W(A_1) = W(C_1) \ \& \ W(A_3^c) = W(C_3^c) \implies W(A_2^c) \leq W(C_2^c)\,. \tag{9}$$

Thus, for $\mathcal{A}$ there is more focus on extreme events, i.e., more insensitivity. Eq. 9 is Eq. 8 but applied to the dual of W. A verbal statement of Definition 5: Assume two threefold matching partitions. If a middle event changes rank with an extreme event, where they are safely bounded away from the other extreme, then the more weight is gained as there is more insensitivity.

The inequality $W(C_3^c) \leq W(D)$ in Eq. 8 precludes cases such as $C_3 = A_3 = \emptyset$, in which case C_2 and A_2 would actually be ranked worst in the matching partitions and not be genuinly middle. We would then in fact have twofold partitions and would be observing source preference.

In the insensitivity region $[B, D]$, there will be less discriminatory power with W shallower for $\mathcal{A}$ events than for $\mathcal{C}$ events. The above definition is the more restrictive and informative the larger $[B, D]$ is. Empirically, we can usually take the events B and D with matching probabilities 0.05 and 0.95, which is strong enough for most applications. We can often even take B empty.

Again, definitions of absolute and relative ambiguity attitudes readily follow. Thus, *ambiguity-generated insensitivity*, or *a-insensitivity* for short, holds for a source if it is more insensitive than $\mathcal{R}$. Comparative results for ambiguity and uncertainty again coincide: more a-insensitivity for $\mathcal{A}$ than for $\mathcal{C}$ is the same as more insensitivity.

5. Absolute Uncertainty Attitudes

Absolute conditions follow from comparative conditions by choosing a neutrality point. The following proposition, which readily follows from substitution, suggests that EU is a natural neutrality point for uncertainty.

PROPOSITION 6. If EU holds for two sources, then they are equally preferred, and equally insensitive with the maximal insensitivity region $[\emptyset, S]$.

We thus obtain the following definitions. W is *liked* if always $W(E) \geq 1 - W(E^c)$. If there is a source available on which EU is maximized, then W is liked if and only there is source preference for W's entire domain over the EU source, as follows from substitution. This illustrates how the (not directly observable) EU model can serve as neutrality point with no need to actually specify or measure it. *Disliked* results from the reversed inequality. W is *insensitive with insensitivity region* $[B, D]$ if for all partitions (E_1, E_2, E_3):

$$W(E_2) \geq W(E_1 \cup E_2) - W(E_1) \text{ whenever } W(E_1 \cup E_2) \leq W(D) \quad (10)$$

and

$$1 - W(E_2^c) \geq W(E_1 \cup E_2) - W(E_1) \text{ whenever } W(E_1) \geq W(B)\ . \quad (11)$$

The inequalities compare the decision weight of E_2 when ranked middle and when ranked extreme, safely bounded away from the other extreme. The conditions ensure that W is shallow and "insensitive" for events between B and D, i.e., on the insensitivity region $[B, D]$. Again, if there is a source available on which EU is maximized, then W is insensitive if and only if its entire domain is more insensitive than the EU source. And, again, our definitions need not specify the EU model. Eq. 10 (and similarly 12 below) without the boundary restriction is sometimes called subadditivity. Insensitivity amounts to imposing subadditivity and its dual, and imposing boundary conditions to avoid that the two conditions "bite" each other.

Similarly, for risk, w is *liked* if $w(p) \geq 1 - w(1-p)$ for all p and *disliked* if the reversed inequality holds. Further, w is *insensitive with insensitivity region* $[b, d]$ $(0 \leq b < d \leq 1)$ if, for all probabilities p_1, p_2, p_3 summing to 1:

$$w(p_2) \geq w(p_1 + p_2) - w(p_1) \text{ whenever } p_1 + p_2 \leq d \quad (12)$$

and

$$1 - w(p_1 + p_3) \geq w(p_1 + p_2) - w(p_1) \text{ whenever } p_1 \geq b\ . \quad (13)$$

For w, the insensitivity region $[b, d]$ is a subinterval of the reals. If risk is available as a source with objective probability measure K, then $E[b, d]$ denotes the corresponding event interval $[B, D]$, i.e., $K(B) = b$ and $K(D) = d$. In other words, it contains the events with matching probabilities between b and d.

6. Preference Conditions to Axiomatize Source Theory and Its Main Attitudinal Comparisons

Preference conditions to capture the aforementioned comparative properties readily follow because all conditions were in terms of inequalities and equalities for W that immediately translate into preferences and indifferences between events. We, therefore, use the same terms. *Source preference holds for $\mathcal{C}$ over $\mathcal{A}$* if, for all partitions (A_1, A_2) from $\mathcal{A}$ and (C_1, C_2) from $\mathcal{C}$:

$$A_1 \sim C_1 \Rightarrow A_2 \preccurlyeq C_2 \; . \tag{14}$$

There is *more insensitivity* for source $\mathcal{A}$ than for source $\mathcal{C}$ (or $\mathcal{A}$ is *more insensitive than* $\mathcal{C}$) with *insensitivity region* $[B, D]$, if for all partitions $\{C_1, C_2, C_3\}$ from $\mathcal{C}$ and $\{A_1, A_2, A_3\}$ from $\mathcal{A}$:

$$A_1 \sim C_1 \;\&\; A_3^c \sim C_3^c \preccurlyeq D \;\; \Longrightarrow \;\; A_2 \succcurlyeq C_2 \tag{15}$$

and

$$B \preccurlyeq A_1 \sim C_1 \;\&\; A_3^c \sim C_3^c \;\; \Longrightarrow \;\; A_2^c \preccurlyeq C_2^c \; . \tag{16}$$

The following result follows immediately from substitution.

OBSERVATION 7. W shows more source preference (or insensitivity) for one source over another if and only if preferences do.

We further have:

OBSERVATION 8. The source preference and source insensitivity relations are transitive. For insensitivity, the new insensitivity region is the intersection of the other two.

Two sources are *equally preferred*, or *equally insensitive*, if the comparative relations hold in both directions. In several results presented later, insensitivity regions should be large enough to avoid triviality. An insensitivity region $[B, D]$ is *regular* for source $\mathcal{S}$ if for every threefold partition (C_1, C_2, C_3) of S from $\mathcal{S}$ we have $C_j \succcurlyeq B$ and $C_j^c \preccurlyeq D$ for at least one j.

Intuitively, the region should capture at least the middle 1/3 of the event domain. If we take $\mathcal{S}$ uniform and B and D from $\mathcal{S}$, then $P_{\mathcal{S}}(B) \leq \frac{1}{3} \leq \frac{2}{3} \leq P_{\mathcal{S}}(D)$ follows (take $P_{\mathcal{S}}(C_j) = \frac{1}{3}$ for all j). Empirically, insensitivity regions are commonly found to be larger.

Under the assumption of EU for risk (or another source), commonly made in the literature on ambiguity, the absolute conditions can readily be axiomatized by applying Observation 7 to comparisons with those EU preferences.

We need one technical condition. The *Archimedean axiom* holds for source $\mathcal{S}$ if there is no infinite sequence of disjoint nonnull event $E_1, E_2, \ldots$ in S with $E_i \sim E_j$ for all i, j. The axiom is, in the presence of the other axioms, necessary and sufficient to ensure that probabilities are real-valued.

THEOREM 9. Under Structural Assumption 3, [source theory (Def. 2) holds for a source $\mathcal{S}$] if and only if [$\mathcal{S}$ is equally preferred and insensitive to itself w.r.t. a regular insensitivity region and the Archimedean axiom holds].

There have been several axiomatizations of general probabilistic sophistication (Chew & Sagi 2008; Grant, Rich, & Stecher 2022 Theorem 5). Buchak (2013) axiomatized probabilistic sophistication combined with rank-dependent expected utility, an important special case of our model. The main feature of our axiomatization is that it captures the meaning of probabilistic sophistication (within one source) directly in terms of ambiguity attitudes: the source must be equal to itself regarding ambiguity aversion and a-insensitivity. This observation suggests that our two components capture the essence of ambiguity/uncertainty attitudes.

7. Pmatchers to Capture Uncertainty Attitudes under Source Theory

This section provides comparative uncertainty/ambiguity results for ST.

ASSUMPTION 10 [for this section]. ST holds for sources $\mathcal{C}$ and $\mathcal{A}$ with generic events $C, C_1, C_2, \ldots$ and $A, A_1, A_2, \ldots$, a-neutral probability measures $P_{\mathcal{C}}$ and $P_{\mathcal{A}}$, and source functions $w_{\mathcal{C}}$ and $w_{\mathcal{A}}$.

We next present the second and third main result of this paper, illustrated in Figure 1 in §8.

THEOREM 11. Under Structural Assumption 3 and Assumption 10, [$\mathcal{C}$ is preferred to $\mathcal{A}$] if and only if [there exists a disliked transformation φ such that $w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi$].

THEOREM 12. Under Structural Assumption 3 and Assumption 10, [$\mathcal{A}$ is more insensitive than $\mathcal{C}$] if and only if [$w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi$ for an insensitive transformation φ]. Furthermore, if the insensitivity region for the preference condition is $[B, D]$ where the boundary events are from source $\mathcal{A}$, then the insensitivity region for φ is $[P_{\mathcal{A}}(B), P_{\mathcal{A}}(D)]$.

In the above theorem, for a general insensitivity region $[B, D]$, we can always take B' and D' from $\mathcal{A}$ (take $B' \sim B$ and $D' \sim D$) and then the insensitivity region for φ is $[P_{\mathcal{A}}(B'), P_{\mathcal{A}}(D')]$.

The theorems identify as central tool for analyzing uncertainty the transformation φ, which satisfies

$$\varphi = w_{\mathcal{C}}^{-1} \circ w_{\mathcal{A}} \quad \text{and} \quad w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi \; . \tag{17}$$

By solvability and strong monotonicity, φ is well-defined, continuous, and strictly increasing. It calibrates, for every a-neutral probability p in source $\mathcal{A}$, the gambling-equivalent a-neutral probability $\varphi(p)$ in source $\mathcal{C}$. That is, we take $C \sim A$ and then, for $P_{\mathcal{A}}(A) = p$, we have $P_{\mathcal{C}}(C) = \varphi(p)$. We, therefore, call φ the *pmatcher from $\mathcal{A}$ to $\mathcal{C}$*. We henceforth use this notation φ. φ can readily be obtained empirically if the a-neutral probabilities of the sources are available. Dimmock, Kouwenberg, & Wakker (2016 Theorem 3.1) showed that matching probabilities conveniently capture ambiguity attitudes. This concerns the special case where source $\mathcal{C}$ is risk. Thus, our Theorems 11 and 12 have generalized their result to general uncertainty, showing that pmatchers are suited to analyze uncertainty attitudes in general. For easy reference, we display a result essentially just demonstrated.

OBSERVATION 13. Under Structural Assumption 3 and Assumption 10, $\varphi(P_{\mathcal{A}}(A)) = P_{\mathcal{C}}(C)$ if and only if $A \sim C$. Consequently, $\varphi(P_{\mathcal{A}}(A)) > P_{\mathcal{C}}(C)$ if and only if $A \succ C$ and $\varphi(P_{\mathcal{A}}(A)) < P_{\mathcal{C}}(C)$ if and only if $A \prec C$.

The following theorem shows that we fully capture ambiguity and uncertainty attitudes through the two components of preference and insensitivity. We do not impose Assumption 10 in the following theorem because it is implied by the other conditions.

THEOREM 14. Under Structural Assumption 3, sources $\mathcal{A}$ and $\mathcal{C}$ are equally preferred and insensitive with a regular insensitivity region and satisfy the Archimedean axiom if and only if both are uniform and $w_{\mathcal{A}} = w_{\mathcal{C}}$. $\mathcal{A}$ and $\mathcal{C}$ then are equally insensitive w.r.t. the maximal insensitivity region $[\emptyset, S]$.

Ambiguity concerns comparisons of uncertainty with risk, i.e., $\mathcal{C} = \mathcal{R}$. Hence, our results for general uncertainty immediately imply the following results for ambiguity.

COROLLARY 15. Assume $\mathcal{C} = \mathcal{R}$. Under Structural Assumption 3 and Assumption 10, the pmatcher φ is the matching probability function. Ambiguity aversion holds if and only if matching probabilities are disliked. A-insensitivity holds if and only if matching probabilities are insensitive. Ambiguity neutrality holds for source $\mathcal{A}$ if and only if it is equally preferred[9] and insensitive as risk (with a regular insensitivity region). They are then equally insensitive with the maximal insensitivity region $[\emptyset, S]$. Ambiguity neutrality holds for all events if and only if the source of all events is equally preferred and insensitive as risk.

The following example shows that aversion/source preference alone is not enough to capture ambiguity and uncertainty attitudes.

EXAMPLE 16. Suppose an agent behaves according to EU for a known urn (risk), but has an insensitive, symmetric[10] but nonlinear, source function $w_{\mathcal{A}}$ for an unknown urn. Then $\gamma_A \beta \sim \gamma_p \beta \Rightarrow \gamma_{A^c} \beta \sim \gamma_{1-p} \beta$ so that ambiguity indifference holds. This has often been defined as ambiguity neutrality in the literature. However, the agent is less sensitive to the unknown urn and ambiguity greatly impacts the agent.

The example underscores a new insight we obtain from analyzing insensitivity: we have to distringuish between ambiguity indifference and the, strictly more restrictive, ambiguity neutrality. This distinction is usually not made in the literature.

As announced before, sources can be taken endogenous in our analysis. An *endogenous uniform source* is any algebra of events that is uniform, with convex-rangedness of the restriction of W. All results of this section can then be applied.

---

[9] Being equally preferred as risk is also called ambiguity indifference.

[10] That is, $w_{\mathcal{A}}(p) + w_{\mathcal{A}}(1-p) = 1$ for all p, for instance if $w_{\mathcal{A}}(p) = \frac{p^2}{p^2+(1-p)^2}$.

OBSERVATION 17 [Endogenous results]. Observation 13, Theorems 11, 12, 14, and Corollary 15 remain valid if $\mathcal{A}$ and $\mathcal{C}$ are endogenous uniform sources.

8. Tractability of Source Theory

Besides being empirically tractable, ST also provides tractable calculations. The formula

$$\int_0^\infty w_{\mathcal{S}}\left(G_{f,U}(\alpha)\right)d\alpha \tag{18}$$

captures the rank-dependent utility value of an act f from a uniform source $\mathcal{S}$, assuming nonnegative outcomes and utilities. Here, with $P_{\mathcal{S}}$ the a-neutral probability measure on $\mathcal{S}$, $G_{f,U}$ denotes the dual of the distribution function that f induces over outcome utilities by $P_{\mathcal{S}}$, and $w_{\mathcal{S}}$ denotes the source function. The mere addition of the transformation $w_{\mathcal{S}}$ to Bayesian expected utility is straightforward and, we think, easier than, for instance, adding higher-order distributions and calculating double integrals as in the smooth model, or solving extra maximization and minimization problems as in multiple prior models. Alternative formulas for rank-dependent utility require a ranking of outcomes and authors have complained about that (Spiliopoulos & Hertwig 2023 Footnote 12). However, again, ranking outcomes is easier than carrying out integrations or solving optimization problems.

Because of their mathematical tractability the functionals that we use, nonlinear transformations of subjective additive probabilities, provide the most popular risk measures nowadays (Artzner et al. 1999), called law-invariant or distorted risk measures. There is extensive literature on these measures, including Liu, Schied, & Wang (2021) and Wang, Wei, & Willmot (2020). Hence, this paper also serves as a preference foundation of those risk measures, and provides new tools for analyzing them.

The analysis of the preceding section shows that we can easily manipulate uncertainty attitudes analytically and numerically, by inserting the bold transformation $\varphi_{\mathcal{A}}$ in Eq. 19 to capture how $\mathcal{A}$ deviates from the calibration source $\mathcal{C}$. In particular, it has to be inserted to the right of $w_{\mathcal{C}}$, and not to the left as has been commonly done with Pratt-Arrow type utility transformations:

$$\int_0^\infty w_{\mathcal{C}} \circ \boldsymbol{\varphi}_{\boldsymbol{\mathcal{A}}}\left(G_{f,U}(\alpha)\right)d\alpha \tag{19}$$

Adding more source dislike or insensitivity can readily be done by adding $\varphi_{\mathcal{A}}$ accordingly. Ambiguity is handled by taking $\mathcal{C} = \mathcal{R}$. Then ambiguity aversion is added by adding a disliked $\varphi_{\mathcal{A}}$, as we proved axiomatically; and so on.

FIGURE 1. Attitudes of subject 2 of Abdellaoui et al. (2011)

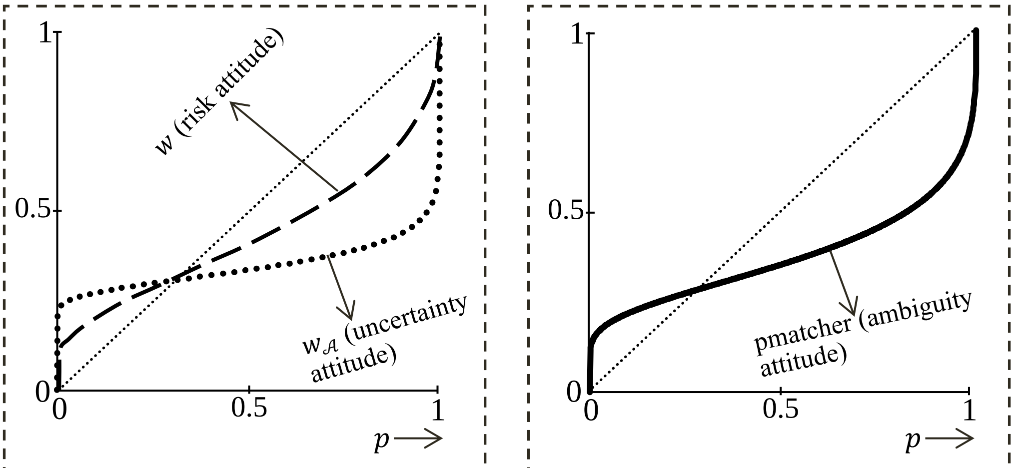

Fig. 1a. Source functions for CAC40 and for risk

Fig. 1b. Matching probabilities for CAC40

An attractive feature of ST is that we can apply it graphically. Figure 1 shows the ease with which uncertainty attitudes can be completely captured and compared visually under ST. It shows the data of subject 2 of Abdellaoui et al. (2011, Figure 10).[11] Source $\mathcal{A}$ is the CAC40 stock index, which passed a test of uniformity. Source $\mathcal{C}$ is risk. Fig. 1a shows the source functions $w_{\mathcal{A}}$ and w. Risk is disliked ($w(p) + w(1 - p) \leq 1$) and also exhibits insensitivity. Dislike and insensitivity are reinforced by the extra uncertainty about probabilities due to the ambiguity of CAC40. Fig. 1b shows that the pmatcher ($w^{-1} \circ w_{\mathcal{A}}$), i.e., the matching probability function, indeed has the corresponding properties, so that the comparative conditions of Theorems 11 and 12 hold.

A non-Bayesian decision analyst who wants to use ambiguity theory, does not have to consider any formula but can directly work with graphs as in Figure 1. Extra uncertainty about the a-neutral probabilities due to perceived ambiguity is captured by graphically shifting weight to the tails, thus reckoning more with deviations from standard expectations.[12]

---

[11] Abdellaoui et al. have $w_{\mathcal{A}} = exp(-1.14(-ln(\mathrm{p})^{0.15})$ and $w = exp(-1.06(-ln(\mathrm{p})^{0.47})$. By Eq. 17, the pmatcher is $exp(-1.17(-ln(\mathrm{p})^{0.32})$. The maximal insensitivity regions are [0, 0.9993], [0, 0.965], and [0, 0.975], respectively.

[12] It reinforces both the ambiguity seeking and aversion that are present. In the literature, authors often focus on universal ambiguity aversion, and then insensitivity/perception only reinforces that aversion.

The distance between the graph and the diagonal can be taken as a (utility independent) measure of insensitivity (Baucells & Borgonovo 2014).

9. Discussion

Risk as one source. Following Tversky & Fox (1995 p. 271), we assumed one fixed w for all objective probabilities. We let parsimony prevail over fit here for tractability reasons. The assumption holds approximately for emotion-neutral risky events and outcomes and we focus on those.[13] Objective probabilities served as the neutrality benchmark for ambiguity attitudes.

*Insensitivity versus cavexity.*Inverse-S shapes are usually described informally as cavexity; i.e., concave up to an inflection point and convex after. This definition requires prior specification of the inflection point similarly as our definition of insensitivity requires prior specification of the insensitivity region. We next discuss pros and cons.

We take insensitivity as cognitive/informational, moving perfect sensitivity (w linear) in the direction of perfect insensitivity with a flat w in the middle suggesting a simple three-valued logic. Insensitivity concernes a global phenomenon of steepness at extremes versus shallowness in the middle, and not a local development of curvature as with cavexity. Further, under cavexity the exact location of the inflection point is theoretically critical (a change leads to opposite requirements in-between) whereas empirically it is noncritical and volatile, weighting functions being approximately linear in the interior (Baucells & Villasís 2015). Different insensitivity regions do not impose opposite requirements but only differ on the region where they impose the (same!) requirements. For applications, insensitivity regions only have to be “big enough” and their exact size can be taken flexibly.

Another problem for cavexity concerns the location of the inflection point relative to the diagonal (Lewandowski 2017 pp. 305-307).[14] Empirically, it will not be exactly on the diagonal. If it is too far above or below the diagonal then cavexity does not capture insensitivity. The main drawback of cavexity is that it is not easily extended to uncertainty,

---

[13] We assume a fixed outcome set. Violations have been found for events and outcomes inducing particular emotions, e.g., if referring to complex arithmetics (Armantier & Treich 2016) or particular familiarities (Chew et al. 2008).

[14] The logical status of the inflection point and the intersection with the diagonal was never formalized.

especially for nonuniform sources. We are not aware of a link of cavexity with ambiguity perception.

Related literature. Many studies on ambiguity used the Anscombe-Aumann (AA) framework. Here, acts do not assign outcomes to states but probability distributions over prizes. They are, thus, two-stage. EU is assumed for risk[15], and a backward induction evaluation is applied to the two-stage acts. This framework makes it possible to use linear algebra to analyze ambiguity, which greatly facilitates the mathematical analysis and, thus, has propelled the ambiguity field. Multistage optimization should be studied for applications anyhow, and is nontrivial for ambiguity. Yet, there is also interest in studying ambiguity in a single-stage framework such as Savage's (1954). Multistage stimuli are complex for tests and applications. Regarding backward induction and, in general, multistage optimization, as unproblematic and self-evident as they are under classical EU, so problematic and controversial they are under ambiguity and nonEU.[16] Many studies have, thus, criticized backward induction in the AA framework.[17] Further, especially for empirical work, it is desirable to allow for violations of EU. ST introduces its ambiguity concepts while avoiding dynamic complications and while allowing violations of EU for risk. As a price to pay, our results had to be derived without resorting to linear algebra as a tool to simplify the mathematics. As we have shown, our tools still remain tractable, while their validity is immune to violations of backward induction or EU for risk.

ST provides a specification of Choquet expected utility (Gilboa 1987; Schmeidler 1989) and prospect theory (Tversky & Kahneman 1992). Those theories use nonadditive measures. However, it has often been argued that nonadditive measures are too general to be tractable beyond the simplest state spaces.[18] Basu & Echenique (2020) showed that this holds even more for multiple prior models. Second-order distributions, as used in the smooth model (Klibanoff, Marinacci, & Mukerji 2005), are of yet higher cardinality and have this problem even more. This problem is yet bigger for various generalizations of these theories provided in the literature (Cerreia-Vioglio et al. 2011). Spiliopoulos & Hertwig (2023) discussed this

---

[15] Hill (2019) and Wang (2022) allowed violations of EU for risk, sacrificing some tractability.

[16] Normative criticisms include Epstein & Le Breton (1993 p. 4), and Machina (1989). Empirical criticisms are referenced next.

[17] See König-Kersting, Kops, & Trautmann (2023 pp. 2-3) and Schneider & Schonger (2018).

[18] See Basu & Echenique (2020) and Tversky & Kahneman (1992 p. 311).

problem for their extensive empirical study and, hence, used the source method because of its tractability.[19] Chew, Bin, & Zhong (2017) also pointed out this problem.

The aforementioned ambiguity models, using high-dimensional parameters, have been used in empirical studies, but then strong parametric assumptions had to be added, especially if the underlying models were very general. Then those extra assumptions drove the results more than the underlying model (Abdellaoui, Bleichrodt, & l'Haridon 2008 p. 246). ST uses nonadditive weighting functions but adds uniformity restrictions and, thus, achieves better parsimony. Abdellaoui et al. (2011) and Dimmock, Kouwenberg, & Wakker (2016) showed that the source method is tractable enough to even allow for nonparametric measurements, i.e., without any parametric assumption added. The source method outperformed other ambiguity theories in prediction tests (Georgalos 2019; Kothiyal, Spinu, & Wakker 2014), underscoring its good parsimony. It provided a better fit than the smooth model in Abdellaoui et al. (2021).

Gul & Pesendorfer (2015) did not need the Anscombe-Aumann framework. However, their requirement that all ideal events (interpreted as unambiguous) should be elicited, needed to determine their inner and outer measures is intractable. Further, their assumption that diffuse events exist, which involve extreme unrealistic decision attitudes violating monotonicity (Grant, Rich, & Stecher 2022 p. 10), is unrealistic.

In many ambiguity theories, ambiguity attitudes depend mainly on the set of outcomes and not on the events. Examples include Chew et al. (2008), Grant, Rich, & Stecher (2022), and Kontek & Lewandowski (2018). The latter proposed to use subjective (a-neutral) probabilities as in ST (their p. 2818). The most well-known theory of this kind is Klibanoff, Marinacci, & Mukerji's (2005) smooth model. Such theories cannot accommodate the fourfold pattern of ambiguity, or insensitivity (König-Kersting, Kops, & Trautmann 2023).[20] Like Machina (2009 p. 390), we think that ambiguity attitudes are mainly event-driven rather than outcome-driven. Chew, Bin, & Zhong (2017) found that event-driven models fit data better than the smooth model. Chateauneuf, Eichberger, & Grant's (2007) neo-additive model is a popular and efficient special case of ST that focuses only on overweighting infimum and

---

[19] They used the term two-stage model (Fox & Tversky 1998), but this model uses a decomposition $w(P)$ where w is the risk-probability weighting function and ambiguity is captured solely through a, nonadditive, P, usually based on introspective measurements. Spiliopoulos & Hertwig (2023) instead used w to capture ambiguity attitudes. That is, they used the source method.

[20] Kontek & Lewandowski (2018)'s model can accommodate extremity orientedness though.

supremum values, as does α-maxmin EU. This ignores attitudes towards intermediate values, relevant for instance in values at risk and their generalizations in finance.

This paper extends concepts, source preference and insensitivity, of Tversky & Wakker (1995), the theoretical counterpart to Tversky & Fox (1995). Tversky & Wakker used traditional between-subjects comparisons (except their §7, discussed further below). Nascimento & Ng (2021) extended their results to conditions on weighting functions and derived advanced comparative results. We, to the contrary, focussed on the main novelty of uncertainty: within-subject between-sources comparisons. Observation 7, our, trivial, starting result, was given by Tversky & Wakker (1995 §7). Other than that, our results are new. In particular, we compare the same function in different subdomains. All preference-axiomatizations of Pratt-Arrow-type transformations in the literature, including Lewandowski (2017 Result 11), Nascimento & Ng (2021), Tversky & Wakker (1995), and Wang (2022) compared different functions on the same domain and applied transformations to images of functions ("outside"). We instead apply transformations to arguments of functions ("inside"), formalizing and justifying Kemel & Gutierrez's (2023) empirical implementation. Wakker (2004) axiomatized a simple version of an "inside" transformation.

10. Conclusion

For modeling uncertainty and ambiguity attitudes through different probability weighting functions, an idea that has been alluded to for decades because of its plausibility, and that has been informally used in many empirical studies, we have provided the first formal framework and axiomatization. In particular, we have shown what the right formulas are (e.g., Eq. 19). No formal theory or foundation had been provided before because the proper framework for it, Savage's (1954), was considered too difficult to handle. We showed that it can be made tractable through source theory. A pro of Savage's framework, relative to the popular one of Anscombe-Aumann, is that we need no multistage stimuli and we can allow for violations of expected utility under risk. Source theory is specific enough to allow for measurements and predictions, even without parametric assumptions. Now ambiguity and uncertainty can be analyzed tractably, both analytically and graphically, and empirically realistically.

Our axiomatic analysis brought many new insights:

- Matching partitions are a useful tool to compare general uncertainty attitudes. They generalize Dimmock, Kouwenberg, & Wakker's (2016) matching probabilities (§4).
- Pmatchers capture uncertainty attitudes quantitatively (Theorems 11 and 12 and Eq. 19). They, again, generalize Dimmock et al.'s matching probabilities.

- Eq. 19 shows the proper formula for using pmatchers.
- Probabilistic sophistication and uniform sources can be characterized directly in terms of ambiguity attitudes (Theorem 9).
- Insensitivity is admittedly complex to analyze, but empirical reality imposes it upon us (Trautmann & van de Kuilen 2015). Insensitivity is needed for completely capturing uncertainty and ambiguity (explained at end of §4.1, and confirmed by Theorems 9 and 14).
- Theorem 12 directly connects ambiguity perception and inverse-S probability weighting of the pmatcher, showing their equivalence. They are two sides of the same insensitivity coin.
- Distinguishing between ambiguity indifference and ambiguity neutrality is important (Example 16).

All our concepts coherently fit together in source theory, supporting the properness of our new definitions (including insensitivity). Uncertainty and ambiguity can be tractably analyzed in the Savage-Gilboa framework.

Appendix A. Reformulations Using Weak Preferences

We give reformulations of some conditions using weak preferences instead of indifferences in the premises. They would give less powerful axiomatizations, but are better suited for empirical tests when indifferences are not easy to obtain.

LEMMA 18. Eq. 6 is equivalent to Eq. 7.

PROOF. Eq. 7 immediately implies Eq. 6. Next assume Eq. 6. Assume $W(A_1) \geq W(C_1)$. By convex-rangedness, we can move part of A_1 to A_2 so that the premise of Eq. 6 follows. The resulting conclusion in that equation and set-monotonicity of W imply Eq. 7. □

We next give the corresponding reformulations of insensitivity.

LEMMA 19. Eq. 8 is equivalent to:

$$W(C_1) \geq W(A_1) \;\&\; W(C_3^c) \leq W(A_3^c) \leq W(D) \Longrightarrow W(A_2) \geq W(C_2)\,. \tag{20}$$

Eq. 9 is equivalent to:

$$W(C_3^c) \leq W(A_3^c) \;\&\; W(C_1) \geq W(A_1) \geq W(B) \Longrightarrow W(A_2^c) \leq W(C_2^c)\,. \tag{21}$$

Eq. 14 is equivalent to:

For all partitions (A_1, A_2) from $\mathcal{A}$ and (C_1, C_2) from $\mathcal{C}$:

$$A_1 \succcurlyeq C_1 \Rightarrow A_2 \preccurlyeq C_2\,. \tag{22}$$

Eq. 15 is equivalent to:

For all partitions (C_1, C_2, C_3) from $\mathcal{C}$ and (A_1, A_2, A_3) from $\mathcal{A}$:

$$C_1 \succcurlyeq A_1 \;\&\; C_3^c \preccurlyeq A_3^c \preccurlyeq D \Longrightarrow A_2 \succcurlyeq C_2\,. \tag{23}$$

Eq. 16 is equivalent to:

For all partitions (C_1, C_2, C_3) from $\mathcal{C}$ and (A_1, A_2, A_3) from $\mathcal{A}$:

$$B \preccurlyeq A_1 \preccurlyeq C_1 \;\&\; A_3^c \succcurlyeq C_3^c \Longrightarrow A_2^c \preccurlyeq C_2^c\,. \tag{24}$$

PROOF. Eq. 20 immediately implies Eq. 8. Next assume Eq. 8. For Eq. 20, assume $W(C_1) \geq W(A_1)$ and $W(C_3^c) \leq W(A_3^c)$. By convex-rangedness, we can move part of C_1 to C_2 to get $W(C_1) = W(A_1)$. Similarly, $W(C_3^c) = W(A_3^c)$ by moving part of C_3 to C_2. (The move of part of C_1 to C_2 did not affect C_3^c.) Eq. 8 and set-monotonicity of W imply Eq. 20. The equivalence of Eqs. 9 and 21 follows similarly. It in fact is dual to Eqs. 8 and 20. The remaining results in terms of preferences follow from the corresponding results on W. □

Appendix B. Proofs

PROOF OF OBSERVATION 1. We define $w(r) = W(R)$ for event R with $K(R) = r$. It is well-defined because $K(R) = K(R') = r$ implies, by stochastic dominance, $W(R) = W(R') = w(r)$. Convex-rangedness readily implies that w's domain is the entire [0,1]. w is strictly increasing (Online Appendix). So, Eq. 3 for all risky events R. We have $w(0) = 0,\ w(1) = 1$. Surjectivity, implied by convex-rangedness, implies that w is continuous. □

PROOF OF OBSERVATION 8. If π^b (π^m) loses more weight, relative to π^w (π^b and π^w) under $\mathcal{A}$ than under $\mathcal{B}$, and more under $\mathcal{B}$ than under $\mathcal{C}$, then it loses more under $\mathcal{A}$ than under $\mathcal{C}$. □

PROOF OF THEOREM 9. The proof will be stated in terms of preference conditions. Throughout, all events assumed to be from the source $\mathcal{S}$. By convex-rangedness, boundary events can be assumed to be from $\mathcal{S}$. We write P for $P_{\mathcal{S}}$ throughout. We first show that the axioms are necessary. For source preference, assume $A_1 \sim B_1$. Then $P(A_1) = P(B_1)$, so, with subscript 2 indicating complements: $P(A_2) = P(B_2)$; $W(A_2) = W(B_2)$; $A_2 \sim B_2$ so $A_2 \succcurlyeq B_2$, as required by source preference. For insensitivity, consider threefold partitions as before. $A_1 \sim B_1$ and $A_3^c \sim B_3^c$ imply $P(A_1) = P(B_1)$ and $P(A_3) = P(B_3)$, so, $P(A_2) = P(B_2)$ and hence, both $A_2 \sim B_2$ and $A_2^c \sim B_2^c$. We have $A_2 \succcurlyeq B_2$ and $A_2^c \preccurlyeq B_2^c$, as required by more insensitivity. The boundary conditions were not needed here. This shows that the insensitivity region can be taken maximal: $[\emptyset, S]$. The Archimedean axiom follows directly. It can also be directly seen that cumulative dominance is implied by biseparable utility and probabilistic sophistication.

We next consider the reversed implication. We assume the Archimedean axiom, and that $\mathcal{S}$ is equally preferred and insensitive to itself w.r.t. a regular insensitivity region. We use cumulative dominance. We throughout use the reformulated conditions of Appendix A. By source preference: $A \succcurlyeq B \Rightarrow A^c \preccurlyeq B^c$. The reversed implication, $A^c \preccurlyeq B^c \Rightarrow A \succcurlyeq B$ follows by taking complementary events. Hence:

$$A \succcurlyeq B \Leftrightarrow A^c \preccurlyeq B^c \tag{25}$$

within the a-uniform source $\mathcal{S}$. Assume a regular insensitivity region $[B, D]$.

LEMMA 20. Assume $E \cap G = F \cap G = \emptyset$. Then $E \succcurlyeq F \Leftrightarrow E \cup G \succcurlyeq F \cup G$.

PROOF OF LEMMA 20.

CASE 1. First assume $E \succcurlyeq F$. We derive $E \cup G \succcurlyeq F \cup G$. Define $A_1 = F,\ A_2 = (F \cup G)^c, A_3 = G$, and $C_1 = E,\ C_2 = (E \cup G)^c, C_3 = G$.

CASE 1.1. Assume $A_3^c \preccurlyeq D$. By Eq. 23, $A_2 \succcurlyeq C_2$, by Eq. 25 implying $A_2^c \preccurlyeq C_2^c$, i.e., $E \cup G \succcurlyeq F \cup G$.

CASE 1.2. Assume $A_1 \succcurlyeq B$. By Eq. 24, $A_2^c \preccurlyeq C_2^c$, i.e., $E \cup G \succcurlyeq F \cup G$.

CASE 1.3. Assume $A_1 \prec B$ and $A_3 \prec D^c$ (so that $B \succ \emptyset$ and $D \prec S$). Then, by regularity, G exceeds all these events. By convex-rangedness, there exists $G_1 \subset G$ such that $A_1 \cup G_1 \sim B$. Define $G_2 = G - G_1$ and $A_3' = A_3 \cup G_2$. Now ${A_3'}^c = A_1 \cup G_1 \sim B \preccurlyeq D$. Therefore, by Case 1.1 applied to $\{A_1, G_1, A_3'\}$ and $\{C_1, G_1, C_3'\}$ with $C_3' = C_3 \cup G_2$, we have that $A_1 \succcurlyeq C_1$ implies $A_1 \cup G_1 \succcurlyeq C_1 \cup G_1$.

Now define $A_1' = A_1 \cup G_1$, $C_1' = C_1 \cup G_1$. By Case 1.2 applied to $\{A_1', G_2, A_3\}$ and $\{C_1', G_2, C_3\}$ we have that $A_1 \cup G_1 \succcurlyeq C_1 \cup G_1$ implies $A_1 \cup G_1 \cup G_2 \succcurlyeq C_1 \cup G_1 \cup G_2$, i.e., $E \cup G \succcurlyeq F \cup G$. Case 1 is done.

CASE 2. Next assume $E \succ F$. We derive $E \cup G \succ F \cup C$. By convex-rangedness, there exists $A_1 \subset E$ with $A_1 \sim F$. By Case 1, $A_1 \cup G \succcurlyeq F \cup G$. $A_2 \coloneqq E - A_1$ is nonnull. By monotonicity, $E \cup G \succ A_1 \cup G$. Transitivity gives $E \cup G \succ F \cup G$. QED

By Lemma 20, weak ordering of $\succcurlyeq$, convex-rangedness, and Krantz et al. (1971 Theorem 5.2.2), there exists a probability measure P on source $\mathcal{S}$ that represents the preference relation $\succcurlyeq$ on events. So does W and, hence, $W = w_{\mathcal{S}} \circ P$ for a strictly increasing $w_{\mathcal{S}}$. Cumulative dominance implies Eq. 2 w.r.t. $P = P_{\mathcal{S}}$ for all x, y from $\mathcal{S}$, i.e., local probabilistic sophistication for $\mathcal{S}$. □

PROOF OF THEOREM 11. Assume matching partitions (C_1, C_2), (A_1, A_2) with a-neutral probabilities p_1, p_2 and q_1, q_2, respectively. We have $\varphi(q_1) = p_1$. The implication $A_2 \preccurlyeq C_2$ is equivalent to $\varphi(q_2) \le p_2 = 1 - p_1 = 1 - \varphi(q_1) = 1 - \varphi(1 - q_2)$. Preference for $\mathcal{C}$ over $\mathcal{A}$ is equivalent to a disliked pmatcher from $\mathcal{A}$ to $\mathcal{C}$. For ambiguity, take $\mathcal{C} = \mathcal{R}$. □

PROOF OF THEOREM 12. Assume matching partitions (C_1, C_2, C_3), (A_1, A_2, A_3) with a-neutral probabilities p_1, p_2, p_3 and q_1, q_2, q_3, respectively. Consider Eqs. 8 and 12. (C_1, C_2, C_3) is matching with (A_1, A_2, A_3) iff $\phi(q_1) = p_1$ and $\varphi(q_1 + q_2) = p_1 + p_2$. Then $A_2 \succcurlyeq C_2$ if and only if $\varphi(q_2) \ge p_2 = (p_1 + p_2) - p_1 = \varphi(q_1 + q_2) - \varphi(q_1)$. The boundary condition $A_3^c \preccurlyeq D$ means $q_1 + q_2 \le P_{\mathcal{A}}(D)$. The q probabilities are the arguments of φ. Hence, the worst-rank bound for φ is $P_{\mathcal{A}}(D)$. In general, if B, D are from a source $\mathcal{S}$, then the worst-rank bound for φ is $m(D)$ where m is the p-matcher from $\mathcal{S}$ to $\mathcal{A}$.

Eqs. 9 and 13 are similar (above case for dual of W). For ambiguity, take $\mathcal{C} = \mathcal{R}$. □

PROOF OF OBSERVATION 13. The definition of φ gives the first iff. For the second iff, $A \succ C$ implies the existence of $A' \subset A$ with $A' \sim C$ and $\varphi(P_{\mathcal{A}}(A')) = P_{\mathcal{C}}(C)$. The second iff now mainly follows from monotonicity and transitivity. The third is similar. □

PROOF OF THEOREM 14. We derive uniformity and $w_{\mathcal{A}} = w_{\mathcal{C}}$ from the other conditions. (The reversed implication is direct.) R abbreviates source preference or source insensitivity. $\mathcal{C}R\mathcal{A}$ and $\mathcal{A}R\mathcal{C}$ and transitivity (Observation 8) imply $\mathcal{C}R\mathcal{C}$, similarly with $\mathcal{A}$. Hence, by Theorem 9, both sources are uniform. With φ the pmatcher from $\mathcal{A}$ to $\mathcal{C}$, by twofold source preference, $C \sim A \Rightarrow C^c \sim A^c$, i.e., $\varphi\big(P_{\mathcal{A}}(A)\big) = P_{\mathcal{C}}(C) \Rightarrow \varphi\big(1 - P_{\mathcal{A}}(A)\big) = 1 - P_{\mathcal{C}}(C)$; that is, $\varphi(p) + \varphi(1-p) = 1$. Hence, $\varphi\left(\frac{1}{2}\right) = \frac{1}{2}$ and we need to prove our result only on $[0, \frac{1}{2}]$.

Because $\mathcal{A}$ is more insensitive than $\mathcal{C}$, $\varphi(\varepsilon) \geq \varphi(p+\varepsilon) - \varphi(p)$ for every $p > 0$ and ("small") $\varepsilon > 0$ if $p + \varepsilon$ is below the upper bound of the insensitivity region which, by regularity, exceeds $\frac{1}{2}$. Hence, it holds on the entire $[0, \frac{1}{2}]$. Because $\mathcal{C}$ is more insensitive than $\mathcal{A}$, $\varphi^{-1}(\varepsilon) \geq \varphi^{-1}(p+\varepsilon) - \varphi^{-1}(p)$ for every $p > 0$ and ("small") $\varepsilon > 0$ as long as $p + \varepsilon$ is below the upper bound of the insensitivity region which, by regularity, exceeds $\frac{1}{2}$. Hence, it holds on the entire $[0, \frac{1}{2}]$. The two inequalities can only hold if φ is linear on $[0, \frac{1}{2}]$. Because $\varphi\left(\frac{1}{2}\right) = \frac{1}{2}$, φ must be the identity on $[0, \frac{1}{2}]$ and, then, on $[0,1]$. By substitution, the insensitivity region can be taken maximal. □

ACKNOWLEDGMENT. Ruodu Wang and Hendrik Rommeswinkel made helpful comments.

References

Abdellaoui, Mohammed, Aurélien Baillon, Laetitia Placido, & Peter P. Wakker (2011) "The Rich Domain of Uncertainty: Source Functions and Their Experimental Implementation," *American Economic Review* 101, 695–723.

Abdellaoui, Mohammed, Han Bleichrodt, Emmanuel Kemel, & Olivier L'Haridon (2021) "Measuring Beliefs under Ambiguity," *Operations Research* 69, 599–612.

Abdellaoui, Mohammed, Han Bleichrodt, & Olivier L'Haridon (2008) "A Tractable Method to Measure Utility and Loss Aversion under Prospect Theory," *Journal of Risk and Uncertainty* 36, 245–266.

Armantier, Olivier & Nicolas Treich (2016) "The Rich Domain of Risk," *Management Science* 62, 1954–1969.

Arrow, Kenneth J. (1951) "Alternative Approaches to the Theory of Choice in Risk-Taking Situations," *Econometrica* 19, 404–437.

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, & David Heath (1999) "Coherent Measures of Risk," *Mathematical Finance* 9, 203–228.

Baillon, Aurélien, Han Bleichrodt, Chen Li, & Peter P. Wakker (2021) "Belief Hedges: Measuring Ambiguity for All Events and All Models," *Journal of Economic Theory* 198, 105353.

Baillon, Aurélien, Zhenxing Huang, Asli Selim, & Peter P. Wakker (2018) "Measuring Ambiguity Attitudes for All (Natural) Events," *Econometrica* 86, 1839–1858.

Basu, Pathikrit & Federico Echenique (2020) "On the Falsifiability and Learnability of Decision Theories," *Theoretical Economics* 15, 1279–1305.

Baucells, Manel & Emanuele Borgonovo (2014) "Invariant Probabilistic Sensitivity Analysis," *Management Science* 59, 2536–2549.

Baucells, Manel & Antonio Villasís (2015) "Equal Tails: A Simple Method to Elicit Utility under Violations of Expected Utility," *Decision Analysis* 12, 190–204.

Berger, Loïc, Johannes Emmerling, & Massimo Tavoni (2017) "Managing Catastrophic Climate Risks under Model Uncertainty Aversion," *Management Science* 63, 749–765.

Bleichrodt, Han, Simon Grant, & Jingni Yang (2023) "Testing Hurwicz Expected Utility," *Econometrica* 2023, 1393–1416.

Borgonovo, Emanuele & Massimo Marinacci (2015) "Decision Analysis under Ambiguity," *European Journal of Operational Research* 244, 823–836.

Buchak, Lara (2013) "*Risk and Rationality*." Oxford University Press, New York.

Cerreia-Vioglio, Simone, Fabio Maccheroni, Massimo Marinacci, & Luigi Montrucchio (2011) "Uncertainty Averse Preferences," *Journal of Economic Theory* 146, 1275–1330.

Chandy, Rajesh, Glen Dowell, Colin Mayer, Erica Plambeck, George Serafeim, Michael Toffel, Beril Toktay, & Elke Weber (2019) "Management Science—Special Issue on Business and Climate Change; Call for Papers," *Management Science* 65, 3447 –3448.

Chateauneuf, Alain, Jürgen Eichberger, & Simon Grant (2007) "Choice under Uncertainty with the Best and Worst in Mind: NEO-Additive Capacities," *Journal of Economic Theory* 137, 538–567.

Chew, Soo Hong, Miao Bin, & Songfa Zhong (2017) "Partial Ambiguity," *Econometrica*, 85, 1239–1260.

Chew, Soo Hong, King King Li, Robin Chark, & Songfa Zhong (2008) "Source Preference and Ambiguity Aversion: Models and Evidence from Behavioral and Neuroimaging Experiments." *In* Daniel Houser & Kevin McCabe (eds.) *Neuroeconomics*, 179–201, JAI Press, Emerald Group Publishing Limited, Bingley, UK.

Chew, Soo Hong & Jacob S. Sagi (2008) "Small Worlds: Modeling Attitudes toward Sources of Uncertainty," *Journal of Economic Theory* 139, 1–24.

Choi, Syngjoo, Jeongbin Kim, Eungik Lee, & Jungmin Lee (2022) "Probability Weighting and Cognitive Ability," *Management Science* 68, 5201–5215.

Debreu, Gérard (1953) "Une Economie de l'Incertain," Electricité de France. Translated as "Economics of Uncertainty" in Gérard Debreu (1983) *Mathematical Economics: Twenty Papers of Gérard Debreu*, 115–119, Cambridge University Press, Cambridge.

Dimmock, Stephen G., Roy Kouwenberg, Olivia S. Mitchell, & Kim Peijnenburg (2021), "Household Portfolio Underdiversification and Probability Weighting: Evidence from the Field," *Review of Financial Studies* 34, 4524–4563.

Dimmock, Stephen G., Roy Kouwenberg, & Peter P. Wakker (2016) "Ambiguity Attitudes in a Large Representative Sample," *Management Science* 62, 1363–1380.

Ellsberg, Daniel (1961) "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics* 75, 643–669.

Epstein, Larry G. & Michel Le Breton (1993) "Dynamically Consistent Beliefs Must Be Bayesian," *Journal of Economic Theory* 61, 1–22.

Fehr-Duda, Helga & Thomas Epper (2012) "Probability and Risk: Foundations and Economic Implications of Probability-Dependent Risk Preferences," *Annual Review of Economics* 4, 567–593.

Fellner, William (1961) "Distortion of Subjective Probabilities as a Reaction to Uncertainty," *Quarterly Journal of Economics* 75, 670–689.

Fox, Craig R. & Amos Tversky (1998) "A Belief-Based Account of Decision under Uncertainty," *Management Science* 44, 879–895.

Georgalos, Konstantinos (2019) "An Experimental Test of the Predictive Power of Dynamic Ambiguity Models," *Journal of Risk and Uncertainty* 59, 51–83.

Ghirardato, Paolo, Fabio Maccheroni, & Massimo Marinacci (2004) "Differentiating Ambiguity and Ambiguity Attitude," *Journal of Economic Theory* 118, 133–173.

Gilboa, Itzhak (1987) "Expected Utility with Purely Subjective Non-Additive Probabilities," *Journal of Mathematical Economics* 16, 65–88.

Gilboa, Itzhak & Massimo Marinacci (2016) "Ambiguity and the Bayesian Paradigm." *In* Horacio Arló-Costa, Vincent F. Hendricks, & Johan F.A.K. van Benthem (eds.), *Readings in Formal Epistemology,* 385–439. Springer, Berlin.

Gilboa, Itzhak & David Schmeidler (1989) "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics* 18, 141–153.

Gonzalez, Richard & George Wu (1999) "On the Shape of the Probability Weighting Function," *Cognitive Psychology* 38, 129–166.

Grant, Simon, Patricia Rich, & Jack Stecher (2022) "Bayes and Hurwicz without Bernoulli," *Journal of EconomicTheory* 199 105027.

Grevenbrock, Nils, Max Groneck, Alexander Ludwig, & Alexander Zimper (2021) "Cognition, Optimism, and the Formation of Age-Dependent Survival Beliefs," *International Economic Review* 62, 887–918.

Gul, Faruk & Wolfgang Pesendorfer (2015) "Hurwicz Expected Utility and Subjective Sources," *Journal of Economic Theory* 159, 465–488.

Henkel, Luca (2023) "Experimental Evidence on the Relationship between Perceived Ambiguity and Likelihood Insensitivity," working paper.

Hill, Brian (2019) "A Non-Bayesian Theory of State-Dependent Utility," *Econometrica* 87, 1341–1366.

Kahneman, Daniel & Amos Tversky (1979) "Prospect Theory: An Analysis of Decision under Risk," *Econometrica* 47, 263–291.

Karni, Edi, Fabio Maccheroni, & Massimo Marinacci (2022) "David Schmeidler's Contributions to Decision Theory," *Theory and Decision* 93, 219–235.

Kemel, Emmanuel & Cedric Gutierrez (2023) "Measuring Source Preferences."

Kemel, Emmanuel & Corina Paraschiv (2013) "Prospect Theory for joint Time and Money Consequences in Risk and Ambiguity," *Transportation Research Part B: Methodological* 50, 81–95.

Klibanoff, Peter, Massimo Marinacci, & Sujoy Mukerji (2005) "A Smooth Model of Decision Making under Ambiguity," *Econometrica* 73, 1849–1892.

König-Kersting, Christian, Christopher Kops, & Stefan T. Trautmann (2023) "A Test of (Weak) Certainty Independence," *Journal of Economic Theory* 209, 105623.

Kontek, Krzysztof & Michal Lewandowski (2018) "Range-Dependent Utility," *Management Science* 64, 2812–2832.

Kothiyal, Amit, Vitalie Spinu, & Peter P. Wakker (2014) "An Experimental Test of Prospect Theory for Predicting Choice under Ambiguity," *Journal of Risk and Uncertainty* 48, 1–17.

Krantz, David H., R. Duncan Luce, Patrick Suppes, & Amos Tversky (1971), "*Foundations of Measurement, Vol. I (Additive and Polynomial Representations)."* Academic Press, New York. (2nd edn. 2007, Dover Publications, New York.)

Lewandowski, Michal (2017) "Prospect Theory Versus Expected Utility Theory: Assumptions, Predictions, Intuition and Modelling of Risk Attitudes," *Central European Journal of Economic Modelling and Econometrics* 9, 275–321.

l'Haridon, Olivier, Ferdinand Vieider, Diego Aycinena, Augustinus Bandur, Alexis Belianin, Lubomir Cingl, Amit Kothiyal, & Peter Martinsson (2018) "Off the Charts: Massive Unexplained Heterogeneity in a Global Study of Ambiguity Attitudes," *Review of Economics and Statistics* 100, 664–677.

Liu, Peng, Alexander Schied, & Ruodu Wang (2021) “Distributional Transforms, Probability Distortions, and Their Applications,” *Mathematics of Operations Research* 46, 1490–1512.

Maafi, H. (2011) “Preference Reversals under Ambiguity,” *Management Science* 57, 2054–2066.

Machina, Mark J. (1989) “Dynamic Consistency and Non-Expected Utility Models of Choice under Uncertainty,” *Journal of Economic Literature* 27, 1622–1688.

Machina, Mark J. (2009) “Risk, Ambiguity, and the Rank-Dependence Axioms,” *American Economic Review* 99, 385–392.

Marinacci, Massimo (2015) “Model Uncertainty,” *Journal of the European Economic Association* 13, 1022–1100.

Nascimento, Arnaldo & Che Tat Ng (2021) “Measuring Attractiveness and Discriminability.”

Savage, Leonard J. (1954) “*The Foundations of Statistics.”* Wiley, New York. (2nd edn. 1972, Dover Publications, New York.)

Schmeidler, David (1989) “Subjective Probability and Expected Utility without Additivity,” *Econometrica* 57, 571–587.

Schneider, Florian & Martin Schonger (2018) “An Experimental Test of the Anscombe-Aumann Monotonicity Axiom,” *Management Science* 65, 1667–1677.

Spiliopoulos, Leonidas & Ralph Hertwig (2023) “Variance, Skewness and Multiple Outcomes in Described and Experienced Prospects: Can One Descriptive Model Capture It All?,” *Journal of Experimental Psychology: General*, 1188–1222.

Trautmann, Stefan T. & Gijs van de Kuilen (2015) “Ambiguity Attitudes.” *In* Gideon Keren & George Wu (eds.), *The Wiley Blackwell Handbook of Judgment and Decision Making* (Ch. 3), 89–116, Blackwell, Oxford, UK.

Tversky, Amos & Craig R. Fox (1995) “Weighing Risk and Uncertainty,” *Psychological Review* 102, 269–283.

Tversky, Amos & Daniel Kahneman (1992) “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty* 5, 297–323.

Tversky, Amos & Peter P. Wakker (1995) “Risk Attitudes and Decision Weights,” *Econometrica* 63, 1255–1280.

Wakker, Peter P. (2004) “On the Composition of Risk Preference and Belief,” *Psychological Review* 111, 236–241.

Wakker, Peter P. (2010) “*Prospect Theory: For Risk and Ambiguity*.*”* Cambridge Univ. Press.

Wald, Abraham (1950) “*Statistical Decision Functions.*” Wiley, New York.

Wang, Fan (2022) “Rank-Dependent Utility under Multiple Priors,” *Management Science* 68, 8166–8183.

Wang, Ruodu, Yunran Wei, & Gordon E. Willmot (2020) “Characterization, Robustness, and Aggregation of Signed Choquet Integrals, *Mathematics of Operations Research* 45, 993–1015.

ONLINE APPENDIX TO
“SOURCE THEORY: A TRACTRABLE AND POSITIVE AMBIGUITY THEORY

Aurélien Baillon, Han Bleichrodt, Chen Li, & Peter P. Wakker

October 2023

ADDITION TO PROOF OF OBSERVATION 1

We assume w well-defined as in Observation 1. We first show that the domain of w is entire $[0,1]$. This proof would be easy if K were countably additive and defined on a sigma algebra. However, K is only finitely additive and defined only on an algebra. Assume, for contradiction, that the w-range RK of K is a strict subset of [0,1]. We can standardly define w well on RK as indicated above, and it is nondecreasing. By convex-rangedness of W, the image $w(RK)$ is the entire $[0,1]$. So, RK is uncountable. For each “small” $\epsilon > 0$ there exist probabilities $p < q$ in RK with $q - p \leq \varepsilon$. Take event A with $K(A) = q$. By convex-rangedness, there is a subset $B \subset A$ with $W(B) = w(p)$, i.e., $K(B) = p$. So, $K(A - B) < \varepsilon$. Using convex-rangedness of W, we can keep on extending a disjoint array A_1 , …,A_i with $K(A_i) = K(A - B)$ as long as $K(A_1 \cup \cdots \cup A_i)^c \geq\ K(A - B)$ so that also $W(A_1 \cup \cdots \cup A_i)^c \geq\ W(A - B)$. Such standard sequences for smaller and smaller ε readily show that RK is dense in $[01]$. Now, if a p is missing from RK, then we must have $0 < p < 1$, and $w([0, p))$ and $w(p, 1)$ provide a partition of [0,1] of two open nonempty sets, violating connectedness of [0,1]. This shows that w’s domain is the entire $[0,1]$.

We next show that w is strictly increasing. If w is constant on $[q, q + \varepsilon]$ for $\varepsilon > 0$, then an event with K value $p = \varepsilon$ is null, but then so is every event with K value $\frac{1}{n} < \varepsilon$, but then so are all their finite unions including the entire $[0,1]$. Then all outcomes are equivalent, and we have a contradiction. □