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The correct formula of 1979 prospect theory for multiple outcomes

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Abstract
Whereas original prospect theory was introduced over 40 years ago, its formula for multi-outcome prospects has never yet been published, resulting in many misunderstandings. This note provides that formula.

Keywords Prospect theory · Separable prospect theory · Stochastic dominance · Multiple outcomes

1 Introduction

When Kahneman and Tversky (1979) introduced (original) prospect theory, they formally restricted their theory to lotteries with at most two nonzero outcomes. The extension to more outcomes is straightforward, as they wrote, but they only stated it in words and did not provide the actual formulas. Up to today, over 40 years after, and 20 years after the shared prize in memory of Nobel, the formulas of original prospect theory have never yet been written in public. While understood by specialists, the formulas nevertheless continue to cause numerous misunderstandings.[1] This note provides the formulas, hoping to put an end to the confusions.

[1] The twenty-first century includes Bernheim and Sprenger (2020 p. 1369 ll. 3–4), Berns et al., (2007 Eq. 1), Blake et al., (2021 Eq. 2), Blondel (2002 Eq. 8), De Giorgi, Hens and Mayer (2007 footnote 2), Grishina, Lucas, and Date (2017 Eq. 13), Harrison, List, and Towe (2007 footnote 23 and p. 451 below Eq. 7), Harrison and Rutström (2009 p. 140), Hey, Lotito and Maffioletti (2010 p. 108), Nagarajan and Shechter (2014 Eq. 1), Nilsson, Rieskamp, and Wagenmakers (2011 Eq. 2), Rieger (2014 Eq. 2), Smith, Levere, and Kurtzman (2009 p. 1548), Wibbenmeyer et al., (2012 Eq. 2), and Wu et al., (2021 Eq. 3). Nilsson, Rieskamp and Wagenmakers (2020) corrected their preceding paper.

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2 The formulas

I will refer to the 1979 version of prospect theory as original prospect theory (OPT). The current version (Tversky & Kahneman, 1992), called prospect theory here,[2] will not be discussed in this note.

By $(p_1{:}x_1, \ldots, p_n{:}x_n)$ we denote a *prospect* (lottery) assigning probability $p_j > 0$ to *outcome* $x_j \in \mathbb{R}$, j = 1, …, n. All outcomes are different and the variable n can be any natural number. Deviating from Kahneman and Tversky (1979), we always write the outcome 0, so that always $p_1 + \cdots + p_n = 1$. By $v{:}\ \mathbb{R} \to \mathbb{R}$ we denote the *value function*. It is nondecreasing with $v(0) = 0$. $\pi{:}\ [0,1] \to [0,1]$ is the *weighting function*. It is nondecreasing with $\pi(0) = 0$ and $\pi(1) = 1$.

Outcome 0 is the *reference point*, positive outcomes $\alpha > 0$ are *gains*, and negative outcomes $\alpha < 0$ are *losses*. We first consider *gain-prospects* with $x_1 > \cdots > x_n \geq 0$. Thus, x_n is the smallest outcome with positive probability, which may be 0. The *OPT value* of the gain-prospect $(p_1{:}x_1, \ldots, p_n{:}x_n)$ now is

$$v(x_n) + \pi(p_{n-1})(v(x_{n-1}) - v(x_n)) + \cdots + \pi(p_1)(v(x_1) - v(x_n)). \tag{1}$$

The minimum outcome x_n, called *riskless* by Kahneman and Tversky, plays a special role. It is received with certainty—maybe even more will be received. Its value ("utility") is, therefore, not weighted. For the other outcomes, their extra value, which is not certain but risky, is weighted.

For *loss-prospects* $(0 \geq x_1 > \cdots > x_n)$, the formula is reflected. Here x_1 is the "best" loss with positive probability, and x_1 may be 0. The *OPT value* of the loss-prospect $(p_1{:}x_1, \ldots, p_n{:}x_n)$ is

$$v(x_1) + \pi(p_2)(v(x_2) - v(x_1)) + \cdots + \pi(p_n)(v(x_n) - v(x_1)). \tag{2}$$

The minimum loss, x_1 (possibly 0), is what one surely loses—maybe more. As with gains, the outcome closest to the reference point is perceived as quasi-certain. For the other outcomes, their extra value loss, which is not certain, is weighted.

For *mixed prospects*, where both a gain and a loss occur with positive probability, there is no perception of a certain threshold and the *OPT value* of $(p_1{:}x_1, \ldots, p_n{:}x_n)$ then is

$$\pi(p_1)v(x_1) + \cdots + \pi(p_n)v(x_n). \tag{3}$$

It is important to notice that this equation is a natural mathematical combination of Eqs. 1, 2—and it is the only one so. To see this point, split a mixed prospect up into a gain-part (all losses replaced by 0) and a loss-part (all gains replaced by 0). Then Eq. 3 results as the sum of the OPT values of the gain- and loss-part. Note that both the gain- and loss-part then involve outcome 0 as quasi-certainty—whose weight is immaterial because $v(0) = 0$. Thus, Eq. 3 is indeed the only possible combination of Eqs. 1, 2. The three formulas on the three different subparts of the domain naturally

[2] Following Tversky's (personal communication) preference, I avoid the common term cumulative prospect theory, which is long and technical, making it unsuited for nontechnical audiences.

fit together as one well-defined functional on the domain of all simple (finite-valued) prospects.

Another point of note is that OPT of 1979 and the current prospect theory (Tversky & Kahneman, 1992) agree for prospects with no more than two outcomes ($n \leq 2$).[3] The two theories only diverge for three or more outcomes. Thus, Eqs. 1, 2 already contained part of the idea of rank dependence, central to current prospect theory.

There is no clear extension of OPT to infinite-valued prospects. Rieger and Wang (2008) considered plausible candidates but revealed several difficulties. For instance, extensions may depend on π only through $\pi'(0)$ and may depend much on the particular discrete approximations chosen.

A theory different than OPT, called *separable prospect theory* results if Eq. 3 is also used for nonmixed prospects, rather than Eqs. 1, 2. It was used in the very first experimental study of risk attitudes (Preston & Baratta, 1948) and was widely used in the psychological literature until the 1980s (Edwards, 1962). It lost popularity when it was discovered that it violates stochastic dominance (Fishburn, 1978). The violations are not only normatively undesirable, but also descriptively. They do not fit with the few violations that have been found empirically and that always crucially depend on misleading framings. Further, they become extreme, and the formulas become absurd, if there are many outcomes. The following lottery illustrates this point:

$$\left(0.01 : 1 + 1 \times 10^{-5}, \ldots, 0.01 : 1 + 99 \times 10^{-5}, 0.01 : 0\right).$$

Its certainty equivalent, with the parametric estimates of Tversky and Kahneman (1992), is 6.9, which exceeds the maximal outcome of the lottery more than sixfold. This does not make any sense, neither theoretically nor empirically. Yet, separable prospect theory has continued to be used sometimes, for instance by Camerer and Ho (1994) who do carefully distinguish it from OPT in their endnote 16. Unfortunately, other papers often confused separable prospect theory with OPT as referenced in §1, also for lotteries with only two outcomes that are both nonzero, even though Kahneman and Tversky (1979 Eq. 2) explicitly wrote a deviating formula for these lotteries. Finally, OPT and separable prospect theory do agree if outcome 0 has a positive probability.

3 Background

Kahneman and Tversky (1979) had much impact. Merigó et al. (2016) listed it as the most cited paper in economics. Kahneman and Tversky (1979) only gave a verbal statement of the general Eqs. 1, 2. We repeat their text here, where the additions between square brackets serve to extend to plurality of non-minimum outcomes.

[3] Strictly speaking, OPT is a special case because the new theory, unlike OPT, allows for a different weighting function for gains than for losses. This assumption is in fact at will for both theories.

> … prospects are segregated into two components: (i) the riskless component, i.e., the minimum gain or loss … which is certain to be obtained or paid; (ii) the risky component, i.e., the additional gain[s] or loss[es] … which is[are] actually at stake…. That is, the value of a strictly positive or strictly negative prospect equals the value of the riskless component plus the value-difference between the outcomes, multiplied by the weight associated with the more extreme outcome[s]. The essential feature … is that a decision weight is applied to the value difference … which represents the risky component of the prospect, but not to … the riskless component. (Kahneman & Tversky, 1979 p. 276).

That Eqs. 1, 2 are correct also appears from Kahneman and Tversky (1975 p. 18), a first working paper version of their 1979 paper. They did explicitly write the formula of PT for multiple outcomes there.[4] They also emphasized (their p. 12) that the riskless outcome should not be weighted, as in our Eqs. 1, 2 and as opposed to separable prospect theory where it is weighted. In other papers, whenever relevant, they pointed this out (Tversky & Kahneman, 1981 Footnote 5; Tversky & Kahneman, 1986 Footnote 2).

4 Conclusion

Kahneman and Tversky (1979) only described the formula of their original prospect theory (OPT) for multiple outcomes in words. This has led to many misunderstandings, and dozens of papers confused separable prospect theory with the—different—OPT. Amazingly, I am not aware of even one paper that wrote the correct formula of OPT for multiple outcomes. This paper has provided it.

References

Bernheim, B. D., & Sprenger, C. (2020). On the empirical validity of cumulative prospect theory: experimental evidence of rank-independent probability weighting. *Econometrica, 88*, 1363–1409.

Berns, G. S., Monica Capra, C., Moore, S., & Noussair, C. (2007). A shocking experiment: new evidence on probability weighting and common ratio violations. *Judgm Decis Making, 2*, 234–242.

[4] They treated the value function somewhat differently than in their 1979 paper, taking values of differences (their p. 12) rather than differences of values. The latter is preferable because it can be applied to nonquantitative outcomes. But they treated probability weighting exactly as in our equations.

Blake, D., Cannon, E., & Wright, D. (2021). Quantifying loss aversion: evidence from a UK population. *Journal of Risk and Uncertainty, 63*, 27–57.
Blondel, S. (2002). Testing theories of choice under risk: estimation of individual functionals. *Journal of Risk and Uncertainty, 24*, 251–265.
Camerer, C. F., & Ho, T.-H. (1994). Violations of the betweenness axiom and nonlinearity in probability. *Journal of Risk and Uncertainty, 8*, 167–196.
De Giorgi, E., Hens, T., & Mayer, J. (2007). Computational aspects of prospect theory with asset pricing applications. *Computational Econ, 29*, 267–281.
Edwards, W. (1962). Subjective probabilities Inferred from decisions. *Psychological Rev, 69*, 109–135.
Fishburn, P. C. (1978). On Handa's "new theory of cardinal utility" and the maximization of expected return. *J Political Econ, 86*, 321–324.
Grishina, N., Lucas, C. A., & Date, P. (2017). Prospect theory-based portfolio optimization: an empirical study and analysis using intelligent algorithms. *Quant Finance, 17*, 353–367.
Harrison, G. W., List, J. A., & Towe, C. (2007). Naturally occurring preferences and exogenous laboratory experiments: a case study of risk aversion. *Econometrica, 75*, 433–458.
Harrison, G. W., & Rutström, E. E. (2009). Expected utility theory and prospect theory: one wedding and a decent funeral. *Experimental Economics, 12*, 133–158.
Hey, J. D., Lotito, G., & Maffioletti, A. (2010). The descriptive and predictive adequacy of theories of decision making under uncertainty/ambiguity. *Journal of Risk and Uncertainty, 41*, 81–111.
Kahneman, D., & Tversky, A. (1975) "Value theory: an analysis of choices under risk," paper presented at the ISRACON conference on Public Economics, Jerusalem, 1975; http://personal.eur.nl/wakker/refs//pdf/ktpt75.pdf. Accessed 19 Mar 2022.
Kahneman, D., & Tversky, A. (1979). Prospect theory: an analysis of decision under risk. *Econometrica, 47*, 263–291.
Merigó, J. M., Rocafort, A., & Aznar-Alarcón, J. P. (2016). Bibliometric overview of business & economics research. *Journal of Business Economics and Management*, *17*, 397–413.
Nagarajan, M., & Shechter, S. (2014). Prospect theory and the newsvendor problem. *Management Science, 60*, 1057–1062.
Nilsson, H., Rieskamp, J., & Wagenmakers, E.-J. (2011). Hierarchical Bayesian parameter estimation for cumulative prospect theory. *Journal of Mathematical Psychology, 55*, 84–93.
Nilsson, H., Rieskamp, J., & Wagenmakers, E.-J. (2020). Commentary: hierarchical bayesian parameter estimation for cumulative prospect theory. *Journal of Mathematical Psychology, 98*, 102429.
Preston, M. G., & Baratta, P. (1948). An experimental study of the auction value of an uncertain outcome. *American Journal of Psychology, 61*, 183–193.
Rieger, M. O. (2014). Evolutionary stability of prospect theory preferences. *Journal of Mathematical Economics, 50*, 1–11.
Rieger, M. O., & Wang, M. (2008). Prospect theory for continuous distributions. *Journal of Risk and Uncertainty, 36*, 83–102.
Smith, G., Levere, M., & Kurtzman, R. (2009). Poker player behavior after big wins and big losses. *Management Science, 55*, 1547–1555.
Tversky, A., & Kahneman, D. (1981). The framing of decisions and the psychology of choice. *Science, 211*, 453–458.
Tversky, A., & Kahneman, D. (1986). Rational choice and the framing of decisions. *Journal of Business, 59*, S251–S278.
Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty, 5*, 297–323.
Wibbenmeyer, M. J., Hand, M. S., Calkin, D. E., Venn, T. J., & Thompson, M. P. (2012). Risk preferences in strategic wildfire decision making: a choice experiment with us wildfire managers. *Risk Analy, 32*, 1021–1037.
Wu, S., Huang, H.-W., Li, Y.-L., Chen, H., & Pan, Y. (2021). A Novel probability weighting function model with empirical studies. *Int J Computational Intell Syst, 14*, 208–227.