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A unified derivation of classical subjective expected utility models through cardinal utility

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Abstract

Classical foundations of expected utility were provided by Ramsey, de Finetti, von Neumann and Morgenstern, Anscombe and Aumann, and others. These foundations describe preference conditions to capture the empirical content of expected utility. The assumed preference conditions, however, vary among the models and a unifying idea is not readily transparent. Providing such a unifying idea is the purpose of this paper. The mentioned derivations have in common that a cardinal utility index for outcomes, independent of the states and probabilities, can be derived. Characterizing that feature provides the unifying idea of the mentioned models. © 1999 Elsevier Science S.A. All rights reserved.

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1. Introduction

A characteristic property of expected utility is the separation of probabilities, describing the uncertainty of a decision maker regarding a state space, and utilities, describing the value of outcomes. That separation is reflected in existing

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preference axiomatizations, that can be classified into two groups accordingly. First, there are the likelihood-oriented axiomatizations, in which the central property of subjective expected utility is *probabilistic sophistication* (i.e., uncertainty is expressed in terms of probabilities). A rich structure is imposed on the relevant uncertainties and preference axioms are based on that structure. This approach can be used in decision under risk, where probabilities are already given (von Neumann and Morgenstern, 1944). Also, the approach of Savage (1954), which does not assume probabilities given, is likelihood oriented. His postulate P4 allows the derivation of probabilities. Utilities are then derived similarly to von Neumann and Morgenstern (1944).

The second group of axiomatizations is utility-oriented and is the subject of this paper. Here a rich structure is imposed on the outcome set and preference axioms are based on that structure. The central property of subjective expected utility (SEU) now is *cardinality of utility*, i.e., a meaningful ordering of utility differences (Vickrey, 1945). The approaches of Ramsey (1931), de Finetti (1931, 1937), and Anscombe and Aumann (1963) can be classified in this group. We will show that all these models can be derived from one unifying principle, ensuring the existence of cardinal utility, invariant across different states of nature or context. Probability then results from the utility-exchange rate between different states.

In a formal manner, the unifying principle can also be introduced in the likelihood-oriented axiomatization of von Neumann and Morgenstern (1944). A corresponding derivation of their model will thus be provided. The remaining classical axiomatization (Savage, 1954) can be derived in a similar fashion if the results of qualitative probability theory (Fishburn, 1986) can be used; this idea is not elaborated here. Hence Savage's axiomatization is not covered in this paper. Derivations of SEU that were directly based on the unifying principle have been presented by Wakker (1984, 1989b, 1993) and Wakker and Tversky (1993). The principle was used in experimental measurements of utility by Bouzit and Gleyses (1996), Wakker and Deneffe (1996), Fennema and van Assen (1997), Abdellaoui (1998), Bleichrodt and Luis Pinto (1998). It can already be recognized in the 'standard sequence invariance' of Krantz et al. (1971).

Nonexpected utility models can be characterized by appropriate weakenings of the unifying principle. For example, Wakker (1989a); Wakker (1989b, Chapter VI) used a 'comonotonic' weakening to characterize Choquet expected utility (Schmeidler, 1989). Wakker (1994) used a similar comonotonic weakening for risk to characterize rank-dependent utility (Quiggin, 1981). In these models, there is no complete separation of decision weight and utility because the decision weight of an event depends on the rank-ordering of the associated outcome. The invariant ordering of utility-differences is therefore not generally applicable but only under special circumstances ('comonotonicity'). Nonexpected utility models will not be discussed in this paper.

Section 2 presents the basic method for deriving expected utility from utility tradeoffs, invariant across states of nature. In subsequent sections, it is demonstrated that the utility tradeoffs can be recognized in the classical axiomatizations of SEU, i.e., by de Finetti (1937) in Section 3, Anscombe and Aumann (1963) in Section 4, von Neumann and Morgenstern (1944) in Section 5, and Ramsey (1931) in Section 6. Appendix A briefly demonstrates that utility tradeoffs can also be recognized in recent SEU axiomatizations. Appendix B presents proofs not given in the main text.

For each SEU axiomatization discussed, it is first shown how the unifying principle can be recognized in the axioms used and then how the principle can serve to provide alternative derivations.

2. The SEU model

Throughout this paper the following notation is used. Γ denotes the set of *outcomes* and $S := \{1,\dots,n\}, n \in IN$ is a finite set of *states* (*of nature*) where exactly one state is true and the others are not true. A decision maker does not know for sure which state is the true state. Subsets of S are *events*. An act f is a mapping from S to Γ, assigning the outcome $f(j)$ (or f_j for short) to each state j. f_j is the outcome obtained by the decision maker if he chose f and the true state is j. For an event E, $f_E g$ denotes the act that assigns outcome f_i to each state $i \in E$ and outcome g_j to each state $j \notin E$. Similarly, for state i and outcome α, $\alpha_i f$ is (f with f_i replaced by α).

$\succcurlyeq$ denotes the preference relation of the decision maker on Γ^n, the set of acts. A function V represents $\succcurlyeq$ if $V: \Gamma^n \to IR$ and $f \succcurlyeq g \Leftrightarrow V(f) \geq V(g)$. If a representing function exists, then $\succcurlyeq$ is a *weak order*, i.e., it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all f,g) and transitive. As usual, strict preference $\succ$ denotes the asymmetric part of $\succcurlyeq$ and $\sim$ the symmetric part, and $\preccurlyeq$ and $\prec$ denote reversed preferences. We call $\succcurlyeq$ *weak preference*. Outcomes are often identified with constant acts. Thus, the preference relation $\succcurlyeq$ generates a preference relation, also denoted by $\succcurlyeq$, over the outcomes.

Subjective expected utility (*SEU*) holds if there exist *probabilities* $p_1,\dots,p_n$ (non-negative and summing to one) and a *utility function* $U: \Gamma \to IR$ such that $f \mapsto \sum_{j=1}^n p_j U(f_j)$ represents $\succcurlyeq$. The utility function U will be *cardinal* in all results in this paper, meaning that it can be replaced by another utility function U^* if and only if there exist real τ and positive σ such that $U^* = \tau + \sigma U$.

In all main results the following *monotonicity* will be assumed: $f \succcurlyeq g$ whenever $f_j \succcurlyeq g_j$ for all j, where the preference between f and g is strict if the preference between f_j and g_j is strict for at least one j. This condition rules out null states, i.e., impossible states that have probability 0. Ruling those out simplifies some subsequent definitions and notation, but is not a restriction because null events do not affect preference.

In this section, Γ is assumed to be a convex subset of IR^m and Γ and Γ^n are endowed with the Euclidean topology. We use the term *linear* as equivalent to the mathematical term affine, i.e., linearity of a function need not imply that it assigns 0 to the origin. $\succcurlyeq$ is *continuous* if the sets $\{f \in \Gamma^n | f \succ g\}$ and $\{f \in \Gamma^n | f \prec g\}$ are open for all acts g. Next the preference condition is defined that captures cardinal utility and its invariance across different states, and that will be used later to derive the other expected utility models. As a preparation, the following notation is defined for outcomes $\alpha, \beta, \gamma, \delta \in \Gamma$:

$$\alpha\beta \succcurlyeq^* \gamma\delta$$

if there exist acts $f, g \in \Gamma^n$ and a state i such that $[\alpha_i f \succcurlyeq \beta_i g$ and $\gamma_i f \preccurlyeq \delta_i g]$. Note that, at this stage, no properties of $\succcurlyeq^*$ can be claimed yet. Under SEU, some can be derived from Eq. (1) hereafter. We similarly write

$$\alpha\beta \succ^* \gamma\delta$$

if there exist acts $f, g \in \Gamma^n$ and a state j such that $[\alpha_j f \succcurlyeq \beta_j g$ and $\gamma_j f \prec \delta_j g]$. Substitution of SEU elementarily shows the following implications:

$$\alpha\beta \succcurlyeq^* \gamma\delta \Rightarrow U(\alpha) - U(\beta) \geq U(\gamma) - U(\delta); \alpha\beta \succ^* \gamma\delta \Rightarrow U(\alpha) - U(\beta) > U(\gamma) - U(\delta). \tag{1}$$

Thus, the * relations capture orderings of utility differences. We now express invariance of cardinal utility across states, the characteristic condition of SEU, in observable and testable terms, i.e., in terms of a preference condition. $\succcurlyeq$ satisfies *tradeoff consistency* if there do not exist outcomes $\alpha, \beta, \gamma, \delta$ such that $\alpha\beta \succ^* \gamma\delta$ and $\gamma\delta \succcurlyeq^* \alpha\beta$. By (1), it immediately follows that tradeoff consistency is a necessary condition for SEU. Wakker (1984, 1989b) proved that it is also sufficient in the utility-oriented approach to SEU.

Theorem 1. *Let Γ be a convex subset of IR^m and $n \geq 2$. For the binary relation $\succcurlyeq$ on Γ^n the following two statements are equivalent.*

(i) SEU holds, with continuous utility and positive probabilities.

(ii) $\succcurlyeq$ is a monotonic continuous weak order that satisfies tradeoff consistency.

Further, utility in (i) is cardinal and the probabilities are uniquely determined whenever U is not constant. □

In the notation $\alpha\beta \succcurlyeq^* \gamma\delta$, the state i and the acts f, g are suppressed. That reflects the characteristic property of SEU, that the revelations of the * relations, and the belonging utility difference orderings, have a meaning independent of states and context. Indeed, Eq. (1) shows that the belonging ordering of utility differences always follows, independently of the specific nature of i, f, g. In nonexpected utility models with cardinal utility, such as Choquet expected utility (Schmeidler, 1989), restrictions must be imposed on revelations $\alpha\beta \succcurlyeq^* \gamma\delta$. For instance, in Choquet expected utility, the revelation $\alpha\beta \succcurlyeq^* \gamma\delta$ is only valid if a 'comonotonicity' restriction is verified (Wakker, 1989b, Chapter VI; Wakker and Tversky, 1993) to preclude distortions due to 'rank-dependence.'

The theorem has been adapted to arbitrary (infinite) state spaces by Wakker (1993). The purpose of the present paper is to demonstrate as clearly as possible a common aspect of the SEU models, hence the technical complications of infinite models will not be discussed. The remainder of the paper shows how classical derivations of SEU can be derived from Theorem 1.

3. De Finetti's approach

An early result, with the real numbers as outcome set, has been provided by de Finetti (1931, 1937). In his model, the tradeoff consistency condition for the binary relation is replaced by additivity and monotonicity with respect to the natural ordering on IR, conditions that jointly imply tradeoff consistency as will be shown in Observation 3. The restrictive nature of the additivity condition appears from the implied linearity of utility. The work of Wakker (1984) (Theorem 1 in this paper) originated as an attempt to generalize de Finetti's result to nonlinear utility.

In this section, $\Gamma = IR$. *Strong monotonicity* means monotonicity with respect to the natural ordering on IR, i.e., $f \succ g$ whenever $f_j \geq g_j$ for all j and $f_j > g_j$ for some j. $\succcurlyeq$ is additive if $[f \succcurlyeq g] \Rightarrow [f + h \succcurlyeq g + h]$ for all acts $f, g, h \in IR^n$. Now a version of de Finetti's theorem is given.

Theorem 2. *The following two statements are equivalent for the binary relation* $\succcurlyeq$ *on* IR^n.

(i) There exist positive probabilities $p_1, \ldots, p_n$ *such that* $f \mapsto \sum_{j=1}^{n} p_j f_j$ *represents* $\succcurlyeq$.

(ii) The binary relation $\succcurlyeq$ *is a strongly monotonic, additive, continuous weak order on* IR^n. □

Although de Finetti presented his result in different terms, Theorem 2 captures the essence. Let us briefly compare our terms to de Finetti's. His term 'Dutch book' comes down to additivity and strong monotonicity, and his assumption that a fair price exists for each act comes down to completeness of preference and continuity. He did not assume the entire domain IR^n but his result can be seen to hold on any linear subset.

The following observation shows that tradeoff consistency follows from de Finetti's conditions and is the central step in this section.

Observation 3. Let $\succcurlyeq$ be a strongly monotonic and additive weak order on IR^n. Then it satisfies tradeoff consistency.

Proof. Assume strong monotonicity, additivity, and weak ordering. For contradiction, assume a violation of tradeoff consistency. That is, there exist outcomes $\alpha,\beta,\gamma,\delta \in IR$ such that $\alpha\beta \succcurlyeq^* \gamma\delta$ and $\gamma\delta \succ^* \alpha\beta$. By definition this means that for a state i and acts $f,g \in IR^n$ we have

$$\alpha_i f \succcurlyeq \beta_i g \text{ and } \gamma_i f \preccurlyeq \delta_i g, \tag{2}$$

and for a state j and acts $v,w \in IR^n$ we have

$$\gamma_j v \succcurlyeq \delta_j w \text{ and } \alpha_j v \prec \beta_j w. \tag{3}$$

The preferences in Eq. (2) and additivity imply

$$(\alpha-\beta)_i(f-g) \succcurlyeq 0 \text{ and } (\gamma-\delta)_i(f-g) \preccurlyeq 0,$$

which because of transitivity and monotonicity implies that $\alpha-\beta \geq \gamma-\delta$. The preferences in Eq. (3) and additivity imply

$$(\gamma-\delta)_j(v-w) \succcurlyeq 0 \text{ and } (\alpha-\beta)_j(v-w) \prec 0,$$

which because of transitivity and monotonicity implies that $\gamma-\delta > \alpha-\beta$. Contradictory inequalities between $\alpha-\beta$ and $\gamma-\delta$ have been derived. Thus, tradeoff consistency cannot be violated. □

By means of Observation 3, the theorem of de Finetti can be derived from Theorem 1. The implication (i) ⇒ (ii) is obvious. Next, (ii) is assumed and (i) is derived. The result is trivial for $n=1$, hence $n \geq 2$ is assumed. All conditions of Statement (ii) in Theorem 1 are satisfied, hence SEU holds for $\succcurlyeq$ with U continuous and probabilities positive. U is strictly increasing because of strong monotonicity. All that remains to be shown is linearity of U (so that U can be the identity). That follows from additivity and is elaborated in Appendix B.

4. Anscombe and Aumann's approach

Another classical result for decision under uncertainty has been provided by Anscombe and Aumann (1963). We adapt their results, formulated in the modern version in which there is no prior mixing of acts, to the notation of this paper, and discuss differences at the end of this section. Let us note here that, whereas Anscombe and Aumann base their proof on the von Neumann–Morgenstern expected utility derivation, our proof does not invoke that derivation.

Let $X = \{x_1,\ldots,x_m\}$ be a finite set of prizes. Γ denotes the set of *lotteries*, i.e., the set of all probability distributions $(q_1,x_1;\ldots;q_m,x_m)$ assigning probability q_i to prize x_i for all $i=1,\ldots,m$ ($\sum_{i=1}^m q_i = 1$, $q_i \geq 0$ for all $i=1,\ldots,m$). Γ is the $m-1$ dimensional probability simplex; it is a convex subset of IR^m. We shall deal with linear (i.e., affine) functions U from Γ to IR. A probability distribution $(q_1, x_1; \ldots; q_m, x_m)$ is a convex combination $\sum_{i=1}^{m} q_i(1, x_i)$ of the degenerate probability distributions $(1, x_i)$ and, for linear U, its U value is therefore $\sum_{i=1}^{m} q_i U(1, x_i) = \sum_{i=1}^{m} q_i u(x_i)$ with $u(x) = U(1, x)$ for all outcomes x. The latter sum can be interpreted as an *expected utility form*, hence linear utility can be identified with expected utility on Γ.

$S := \{1, \ldots, n\}$ is again a finite state space and Γ^n is the set of acts $f{:}S \to \Gamma$, $f{:}j \to f(j) = f_j$. On Γ^n a binary relation, denoted by $\succcurlyeq$, is assumed. Mixing on Γ^n is defined pointwise, i.e., $[\lambda f + (1-\lambda)g](s) = \lambda f(s) + (1-\lambda)g(s)$ for all $s \in S$ and $0 \le \lambda \le 1$.

The binary relation $\succcurlyeq$ satisfies *vNM-independence* if, for all $f, g, h \in \Gamma^n$ and all $0 < \lambda < 1$, we have $[f \succcurlyeq g] \Leftrightarrow [\lambda f + (1-\lambda)h \succcurlyeq \lambda g + (1-\lambda)h]$. Following Fishburn (1970, 1982), the continuity condition of Jensen (1967) is used: $\succcurlyeq$ is *J-continuous* if, for all acts $f \succ g$, and $h \in \Gamma^n$ there exist $\lambda, \mu \in (0,1)$ such that $\lambda h + (1-\lambda) f \succ g$ and $f \succ \mu h + (1-\mu) g$. J-continuity is weaker than continuity (i.e., Euclidean continuity as defined in Section 2). However, in the presence of vNM-independence and monotonicity, J-continuity implies continuity for a weak order on Γ^n, as the next lemma shows.

Lemma 4. *Let $\succcurlyeq$ be a monotonic, J-continuous, and vNM-independent weak order on Γ^n. Then $\succcurlyeq$ is continuous on Γ^n.* □

The proof of Lemma 4 is given in Appendix B.

The approach of this section is in fact a two-stage model, where in the first stage the uncertainty concerning the true state of nature is resolved and in the second stage the resulting lottery over prizes is played. A 'folding-back' maximization is adopted, with expected utility in the second stage. In a mathematical sense that comes down to linear utility over the outcomes. This linearity greatly simplifies mathematical derivations, hence the popularity of this approach. It has been used in many classical papers (Anscombe and Aumann, 1963; Pollak, 1967; Keeney and Raiffa, 1976; Fishburn, 1970, 1982 and the references therein). Schmeidler (1989) used this approach for his famous introduction of Choquet expected utility. Numerous other papers on nonexpected utility have used the approach likewise (Drèze, 1987; Gilboa and Schmeidler, 1989; Hazen, 1989; Klibanoff, 1995; etc.). These papers still assume folding back and expected utility (thus linear utility) in the second stage, but consider deviations from expected utility in the first. A method for further simplifying the mathematics and avoiding the complications of the two-stage approach was described by Sarin and Wakker (1997) for the SEU of Anscombe and Aumann (1963) and by Sarin and Wakker (1992) for Choquet expected utility of Schmeidler (1989). The following observation is central in this section. The observation is similar to Observation 3 because it also derives tradeoff consistency from a linearity assumption.

Observation 5. Let $\succcurlyeq$ be a vNM-independent monotonic weak order on Γ^n. Then tradeoff consistency holds.

Proof. Assume that $\succcurlyeq$ satisfies the conditions in the Observation but assume, for contradiction, that it violates tradeoff consistency. Thus, there exist outcomes $\alpha,\beta,\gamma,\delta \in \Gamma$ such that $\alpha\beta \succcurlyeq^* \gamma\delta$ and $\gamma\delta \succ^* \alpha\beta$. By definition of the relations $\succ^*$ and $\succcurlyeq^*$, this means that for a state i and acts $f,g \in \Gamma^n$ we have

$$\alpha_i f \succcurlyeq \beta_i g \tag{4}$$

and

$$\gamma_i f \preccurlyeq \delta_i g, \tag{5}$$

and for a state j and acts $\upsilon,w \in \Gamma^n$ we have

$$\gamma_j \upsilon \succcurlyeq \delta_j w \tag{6}$$

and

$$\alpha_j \upsilon \prec \beta_j w. \tag{7}$$

vNM-independence and Eqs. (4)–(7), respectively, imply

$$\frac{1}{2}(\alpha_i f) + \frac{1}{2}(\delta_i g) \succcurlyeq \frac{1}{2}(\beta_i g) + \frac{1}{2}(\delta_i g), \tag{8}$$

$$\frac{1}{2}(\beta_i g) + \frac{1}{2}(\gamma_i f) \preccurlyeq \frac{1}{2}(\beta_i g) + \frac{1}{2}(\delta_i g), \tag{9}$$

$$\frac{1}{2}(\gamma_j \upsilon) + \frac{1}{2}(\beta_j w) \succcurlyeq \frac{1}{2}(\delta_j w) + \frac{1}{2}(\beta_j w), \text{ and} \tag{10}$$

$$\frac{1}{2}(\delta_j w) + \frac{1}{2}(\alpha_j \upsilon) \prec \frac{1}{2}(\delta_j w) + \frac{1}{2}(\beta_j w). \tag{11}$$

Eqs. (8) and (9) and transitivity imply

$$\frac{1}{2}(\alpha_i f) + \frac{1}{2}(\delta_i g) \succcurlyeq \frac{1}{2}(\beta_i g) + \frac{1}{2}(\gamma_i f) \tag{12}$$

and Eqs. (10) and (11) and transitivity imply

$$\frac{1}{2}(\gamma_j \upsilon) + \frac{1}{2}(\beta_j w) \succ \frac{1}{2}(\delta_j w) + \frac{1}{2}(\alpha_j \upsilon). \tag{13}$$

Eq. (12) implies, because of monotonicity, that $(1/2)\alpha + (1/2)\delta \geq (1/2)\beta + (1/2)\gamma$, Eq. (13) implies that $(1/2)\alpha + (1/2)\delta < (1/2)\beta + (1/2)\gamma$, i.e., a contradiction results. Thus, $\succcurlyeq$ cannot violate tradeoff consistency. □

Next, the Anscombe and Aumann (1963) representation theorem is derived. Actually, Anscombe and Aumann considered general, possibly infinite, prize sets X and assumed that Γ was the set of *simple* (finitely supported) lotteries on X. We have assumed X finite for simplicity of the presentation. The extension of the following theorem to general sets X is easily obtained and is presented in Proposition 14 in Appendix B. Another deviation is that Anscombe and Aumann assume that also lotteries over acts are available. Their Assumption 2 guarantees that lotteries over acts can be captured by lotteries over prizes, i.e., that their setup can be reduced to ours.

Theorem 6. *Let Γ be the set of probability distributions over a finite set of prizes and let $n \geq 2$. The following two statements are equivalent for the binary relation $\succcurlyeq$ on Γ^n:*

(i) SEU holds, with utility U linear (i.e., U is an expected utility functional) and all probabilities positive.

(ii) The binary relation $\succcurlyeq$ is a weak order; it satisfies monotonicity, vNM-independence, and J-continuity.

Proof. Obviously, Statement (i) implies Statement (ii), therefore Statement (ii) is assumed and Statement (i) is derived. By Lemma 4 and Observation 5, Statement (ii) in Theorem 1 is implied by Statement (ii) in Theorem 6. Hence, there exists an SEU representation with continuous utility U. Linearity of U is derived in Appendix B. □

5. von Neumann and Morgenstern's approach

The third approach, analyzed here, is the representation theorem of von Neumann and Morgenstern (1944). In their SEU model, probabilities are given in advance and only utilities are derived from the preference relation. The von Neumann and Morgenstern representation theorem can be considered the version of Theorem 6 for $n = 1$. The proof of Theorem 6 cannot be invoked for $n = 1$, hence the case is treated separately. The case $n = 1$ can be derived as a corollary from the case $n = 2$ and that is our approach. Thus, whereas Anscombe and Aumann derived their result from the von Neumann–Morgenstern result, we reverse the order and derive the von Neumann–Morgenstern result from Anscombe and Aumann's.

Theorem 7. *Let $\succcurlyeq$ be a binary relation on the set Γ of simple probability distributions over a (possibly infinite) set X of prizes. The following two statements are equivalent:*

(i) $\succcurlyeq$ can be represented by expected utility (i.e., a linear U).

(ii) $\succcurlyeq$ is a weak order that satisfies vNM-independence and J-continuity.

Proof. The implication (i) $\Rightarrow$ (ii) is elementary, hence (ii) is assumed and (i) is derived. Theorem 6 and its extension to general prize sets X, i.e., Proposition 14 in Appendix B, are used to deduce Statement (i). Let $\bar{\Gamma}$ be defined as follows:

$$\bar{\Gamma} := \{(f_1, f_2) | f_1, f_2 \in \Gamma\}.$$

On $\bar{\Gamma}$ the binary relation $\overline{\succcurlyeq}$ is defined for $\bar{f}, \bar{g} \in \bar{\Gamma}$ as:

$$\overline{f \succcurlyeq g} \Leftrightarrow \frac{1}{2} f_1 + \frac{1}{2} f_2 \succcurlyeq \frac{1}{2} g_1 + \frac{1}{2} g_2,$$

where $\bar{f} := (f_1, f_2)$ and $\bar{g} := (g_1, g_2)$ for $f_1, f_2, g_1, g_2 \in \Gamma$ are interpreted as acts in a two-states Anscombe and Aumann model with $\bar{\Gamma} = \Gamma^2$. It follows from elementary substitution that $\overline{\succcurlyeq}$ inherits all properties from $\succcurlyeq$ described in Statement (ii) of Theorem 7. In addition, monotonicity of $\overline{\succcurlyeq}$ follows from vNM-independence. (The following four preferences are all equivalent: $(f,f) \overline{\succcurlyeq} (g,g)$; $f \succcurlyeq g$; $(1/2)f + (1/2)h \succcurlyeq (1/2)g + (1/2)h$; $(f,h) \overline{\succcurlyeq} (g,h)$; similarly, $(h,f) \overline{\succcurlyeq} (h,g)$ can be derived. The equivalences for weak preferences immediately imply the equivalences for strict preferences.)

Therefore, the assumptions in Statement (ii) of Theorem 6 are satisfied and SEU holds for outcome set $\bar{\Gamma}$, preference relation $\overline{\succcurlyeq}$, probabilities p_1, p_2 and linear utility $\bar{U}$ (invoke Proposition 14 if X is infinite). The expected utility model in Statement (i) follows from restriction to the constant acts in $\bar{\Gamma}$. It can, but need not, be seen that $p_1 = (1/2) = p_2$. □

6. Ramsey's approach: the equiprobable state case

An appealing axiomatization of subjective expected utility is possible if all states are equally likely. In that case, essentially, sure-thing principle alone of Savage's characterizes subjective expected utility. Because the result is an almost trivial corollary of additive representation theorems, it has not received much attention in the literature. It has been used as a tool in more complex results by Blackorby et al. (1977) and Chew and Epstein (1989). Let us now present the result.

$\succcurlyeq$ satisfies the *sure-thing principle* if $c_i f \succcurlyeq c_i g$ implies $c'_i f \succcurlyeq c'_i g$ for all states i, 'common' outcomes c, c', and acts f, g. That is, the preference between two acts is independent of a common outcome (e.g., c), hence remains invariant if that common outcome is replaced by another common outcome (such as c'). Then preference is obviously independent of any number of common outcomes, by repeated application of the principle. The sure-thing principle was used by Debreu (1960) and Krantz et al. (1971) to characterize additive representations. That is, as soon as there are three or more states, then under the usual conditions (weak ordering, continuity, monotonicity) the sure-thing principle is necessary and sufficient for additive representability through $\sum_{j=1}^{n} V_j(f_j)$. That representation is reduced to the equally-likely subjective expected utility representation $\sum_{j=1}^{n}(1/n)U(f_j)$ by the following *exchangeability* condition: $f \sim g$ whenever g is obtained from f by permuting the outcomes. Let us state the result formally.

Theorem 8. *Let $\succcurlyeq$ be a binary relation on Γ^n, with $n \geq 3$ and Γ a convex subset of IR^m. The following two statements are equivalent.*

(i) There exists a continuous utility $U: \Gamma \to IR$ such that $\succcurlyeq$ is represented by $\sum_{j=1}^{n}(1/n)U(f(s_j))$.

(ii) $\succcurlyeq$ is a monotonic continuous weak order that satisfies the sure-thing principle and exchangeability.

Utility in (i) is cardinal.

The derivation of Theorem 8 from existing additive representation theorems is elementary, hence no separate proof is needed. We nevertheless demonstrate informally how exchangeability and the sure-thing principle imply a version of tradeoff consistency. Lemma 9 is derived from preference conditions without invoking the additive representation. Following Lemma 9, a proof of Theorem 8 is suggested. The purpose of this analysis is to demonstrate once more the unity of expected utility models, comprising cardinal utility that is invariant across all states.

Lemma 9. *Let, for $n \geq 3$, $\succcurlyeq$ be a weak order on Γ^n that satisfies the sure-thing principle and exchangeability. Then the three preferences $(x,\alpha,c_3,\ldots,c_n) \preccurlyeq (y,\beta,c_3,\ldots,c_n)$, $(x,\gamma,c_3,\ldots,c_n) \succcurlyeq (y,\delta,c_3,\ldots,c_n)$, and $(\alpha,v,c_3,\ldots,c_n) \succcurlyeq (\beta,w,c_3,\ldots,c_n)$ imply the fourth preference $(\gamma,v,c_3,\ldots,c_n) \succcurlyeq (\delta,w,c_3,\ldots,c_n)$.*

Proof. The outcomes for states $s_4,\ldots,s_n$ are fixed and are suppressed. We use the sure-thing principle and exchangeability throughout the proof without explicit mentioning. The first antecedent preference in the lemma implies $(x,\alpha,\delta) \preccurlyeq (y,\beta,\delta)$, the second implies $(y,\beta,\delta) \preccurlyeq (x,\beta,\gamma)$. Because of transitivity, $(x,\alpha,\delta) \preccurlyeq (x,\beta,\gamma)$, hence $(\beta,v,\gamma) \succcurlyeq (\alpha,v,\delta)$. The third antecedent preference implies $(\alpha,v,\delta) \succcurlyeq (\beta,w,\delta)$. Transitivity now implies $(\beta,v,\gamma) \succcurlyeq (\beta,w,\delta)$. From this, finally, the consequent preference in the lemma follows. □

Let us sketch, informally, how Lemma 9 can be used to prove Theorem 8. First, the sure-thing principle in Theorem 8 implies an additive representation $\sum_{j=1}^{n} V_j(x_j)$.

Lemma 9 implies that V_1 and V_2 order utility differences the same way. Due to exchangeability, V_1 and V_j order utility differences the same way for all j. That implies an expected utility representation $\sum_{j=1}^{n} p_j U(x_j)$. Exchangeability implies that all probabilities p_j are equal $((\alpha,\beta,\ldots,\beta) \sim (\beta,\alpha,\beta,\ldots,\beta))$, etc.

We next turn to the case of $n=2$ with equally likely states s_1 and s_2. The preceding analysis essentially used $n \geq 3$, not only for invoking the additive representation but also in its derivation of tradeoff consistency in Lemma 9. The case $n=2$ is of historical interest because it underlies the expected utility derivation of Ramsey (1931), with s_1 and s_2 the ‘ethically neutral’ events. The way of deriving a strength of preference relation $\succcurlyeq^{**}$, used by Ramsey and many after him (e.g., d’Aspremont and Gevers, 1990, see Appendix A), is through

$$\alpha\beta \succcurlyeq^{**} \gamma\delta \text{ if } (\alpha,\delta) \succcurlyeq (\beta,\gamma). \tag{14}$$

This method requires equal likelihood of the two states. It implies $\alpha\beta \succcurlyeq^{*} \gamma\delta$ because of $(\alpha,\delta) \succcurlyeq (\beta,\gamma)$ and $(\gamma,\delta) \preccurlyeq (\delta,\gamma)$ (implied by equal likelihood and hence exchangeability). Weak ordering of $\succcurlyeq^{**}$, studied in the next lemma, was used by d’Aspremont and Gevers (1990).

Lemma 10. *Let $\succcurlyeq$ be an exchangeable weak order on Γ^2. If $\succcurlyeq^{**}$, defined in Eq. (14), is a weak order on Γ^2, then it agrees with $\succcurlyeq^{*}$ and tradeoff consistency holds.*

Proof. Because of exchangeability, $\succ^{**}$, defined by $\alpha\beta \succ^{**} \gamma\delta$ if $(\alpha,\delta) \succ (\beta,\gamma)$, is the asymmetric part of $\succcurlyeq^{**}$ (not $(\gamma,\beta) \succcurlyeq (\delta,\alpha)$ implies not $\gamma\delta \succcurlyeq^{**} \alpha\beta$).

Assume $\alpha\beta \succcurlyeq^{*} \gamma\delta$, say $(\mu,\alpha) \succcurlyeq (\nu,\beta)$ and $(\mu,\gamma) \preccurlyeq (\nu,\delta)$. Then, using $(\alpha,\mu) \succcurlyeq (\beta,\nu)$, we have $\alpha\beta \succcurlyeq^{**} \nu\mu \succcurlyeq^{**} \gamma\delta$. Transitivity of $\succcurlyeq^{**}$ implies $\alpha\beta \succcurlyeq^{**} \gamma\delta$. We already saw above the lemma that $\alpha\beta \succcurlyeq^{**} \gamma\delta$ implies $\alpha\beta \succcurlyeq^{*} \gamma\delta$. Hence $\succcurlyeq^{*}$ and $\succcurlyeq^{**}$ are identical.

Similarly, $\alpha\beta \succ^{*} \gamma\delta$ implies $\alpha\beta \succ^{**} \gamma\delta$. $\alpha\beta \succcurlyeq^{*} \gamma\delta$ and $\gamma\delta \succ^{*} \alpha\beta$ would now imply $\alpha\beta \succcurlyeq^{**} \gamma\delta$ and $\gamma\delta \succ^{**} \alpha\beta$, contradicting weak ordering of $\succcurlyeq^{**}$. Hence tradeoff consistency must hold. □

Corollary 11. *Let $\succcurlyeq$ be a binary relation on Γ^2, where Γ is a convex subset of IR^m. The following two statements are equivalent.*

(i) There exists a continuous utility function U such that $(\alpha,\beta) \mapsto (1/2)U(\alpha) + (1/2)U(\beta)$ represents preference.

(ii) $\succcurlyeq$ is an exchangeable continuous weak order and $\succcurlyeq^{**}$ is also a weak order.

Proof. (i) $\Rightarrow$ (ii) is elementary. Now assume Statement (ii). If both states are null then U is constant and the corollary follows. If one state is nonnull then so is, by exchangeability, the other, and that is what we assume henceforth. According to Lemma 10, (ii) implies tradeoff consistency, hence subjective expected utility (monotonicity in Theorem 1 is only needed to imply positive probabilities and it is not needed for the subjective expected utility representation). Because of exchangeability, $(\alpha,\beta) \sim (\beta,\alpha)$, implying $p_1 = p_2$. □

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Appendix A. Modern SEU derivations

This appendix briefly discusses some recent derivations of SEU. First, a 'bisymmetry axiom' is discussed for two states; that is, $n = 2$ is assumed for now. Acts are sometimes denoted as pairs (x,α), etc. We assume $\Gamma = IR$ and continuity and strong monotonicity. Then for each act f there exists a unique *certainty equivalent* CE(f), i.e., an outcome equivalent to f. *Bisymmetry* holds if

$$\mathrm{CE}(\mathrm{CE}(x,\alpha),\mathrm{CE}(\beta,y)) = \mathrm{CE}(\mathrm{CE}(x,\beta),\mathrm{CE}(\alpha,y))$$

whenever all CEs exist. Substitution of SEU shows that the condition is necessary, because both the left- and right-hand sides have utility

$$p_1^2 U(x) + p_1 p_2 U(\alpha) + p_1 p_2 U(\beta) + p_2^2 U(y).$$

Pfanzagl (1968) and Krantz et al. (1971, Theorem 6.10) show that bisymmetry, together with some other axioms, implies SEU. The following lemma shows how the bisymmetry axiom implies invariant cardinal utility across different states.

Lemma 12. *Let $\Gamma = IR$ and $n = 2$. Assume that $\succcurlyeq$ is a continuous strongly monotonic weak order. Assume $(x,\alpha) \preccurlyeq (y,\beta)$, $(x,\gamma) \succcurlyeq (y,\delta)$, and $(\alpha,\upsilon) \succcurlyeq (\beta,w)$. Then bisymmetry implies that $(\gamma,\upsilon) \succcurlyeq (\delta,w)$.*

Proof. $\mathrm{CE}(x,\gamma) \geq \mathrm{CE}(y,\delta)$ and $\mathrm{CE}(\alpha,\upsilon) \geq \mathrm{CE}(\beta,w)$ imply, because of monotonicity, that $\mathrm{CE}(\mathrm{CE}(x,\gamma),\mathrm{CE}(\alpha,\upsilon)) \geq \mathrm{CE}(\mathrm{CE}(y,\delta),\mathrm{CE}(\beta,w))$. Applying bisymmetry to both sides of the inequality yields $\mathrm{CE}(\mathrm{CE}(x,\alpha),\mathrm{CE}(\gamma,\upsilon)) \geq \mathrm{CE}(\mathrm{CE}(y,\beta),\mathrm{CE}(\delta,w))$. Because $\mathrm{CE}(x,\alpha) \leq \mathrm{CE}(y,\beta)$, monotonicity implies $\mathrm{CE}(\gamma,\upsilon) \geq CE(\delta,w)$, i.e., $(\gamma,\upsilon) \succcurlyeq (\delta,w)$. □

The lemma shows an alternative way for deriving SEU from bisymmetry. First, an additive representation is derived (see, e.g., Krantz et al., 1971, Section 6.10.1).

Next, Lemma 12 shows that utility differences are ordered the same way across different states. That implies SEU.

Many variations of the bisymmetry condition, and extensions to more than two states, have been presented in the literature. In such cases Lemma 12 can be applied to each two-dimensional subspace to show that additive value functions in an additive representation are linearly related. An example is Münnich et al. (1997). Often, ‘comonotonic’ restrictions of such axioms have been used to characterize rank-dependent generalizations of SEU (Quiggin, 1981, Axiom 4 and Quiggin and Wakker, 1994, Axiom 4; Chew, 1989, ‘weak commutativity’; Nakamura, 1990, 1992, 1995, ‘weak multisymmetry’). Its most appealing interpretation refers to multistage resolutions of uncertainty (Luce, 1988, Eqs. (22) and (23); Segal, 1993, ‘order indifference’; Luce, 1998, ‘event commutativity’). An appealing variation on bisymmetry was used by Gul (1992, Assumption 2) and Chew and Karni (1994, ‘act independence’). A complete logical analysis of these specific axioms and their relations to our axioms is beyond the scope of this paper.

Finally, d’Aspremont and Gevers (1990) is discussed. They use two *equally-likely* states s_1, s_2 to derive subjective expected utility. That is, they assume that $(\alpha,\beta,z,\dots,z) \sim (\beta,\alpha,z,\dots,z)$ for all outcomes α,β,z. They derive a strength of preference relation $\succcurlyeq^{**}$ similarly as in Eq. (14), i.e., $\alpha\beta \succcurlyeq^{**} \gamma\delta$ if $(\alpha,\delta,z,\dots,z) \succcurlyeq (\beta,\gamma,z,\dots,z)$ for some outcome z. They introduce a new preference condition, *difference-scale neutrality*, a condition that is equivalent to constant absolute risk aversion in the special case of real outcomes with linear utility: Let f,g,f',g' be four acts. Then $f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g'$ whenever, for some state t, $f(s)f(t) \sim^{**} f'(s)f'(t)$ and $g(s)f(t) \sim^{**} g'(s)f'(t)$ for all states s. A detailed analysis of the logical relations between their condition and tradeoff consistency would take too much space, therefore only an illustrating example is presented.

Example 13. Assume a preference relation $\succcurlyeq$ on IR^n, weak ordering, continuity, strong monotonicity, the sure-thing principle, equally likely states s_1 and s_2, difference-scale neutrality, and $n=3$. Assume

$$(z,x,\alpha) \succcurlyeq (z,y,\beta) \tag{15}$$

and

$$(z,x,\gamma) \preccurlyeq (z,y,\delta), \tag{16}$$

hence $\alpha\beta \succcurlyeq^{*} \gamma\delta$. Assume that z',x',y',β' can be found such that

$$\alpha\gamma \sim^{**} zz' \sim^{**} xx' \sim^{**} yy' \sim^{**} \beta\beta'. \tag{17}$$

We show that $\alpha\beta \succcurlyeq^{**} \gamma\delta$.

By Debreu (1960), there exists an additive representation $\sum_{j=1}^{n} V_j(x_j)$. Due to equal-likelihood of s_1 and s_2, $V_1 = V_2$ may be assumed. By substitution in the definition of $\succcurlyeq^{**}$, it follows that differences of $(V_1 =)$ V_2 represent $\succcurlyeq^{**}$. Because of difference-scale neutrality, Eq. (15) together with Eq. (17) imply

$(z', x', \gamma) \succcurlyeq (z', y', \beta')$. As Eq. (17) implies $V_2(x') - V_2(y') = V_2(x) - V_2(y)$, the additive representation implies $(z, x, \gamma) \succcurlyeq (z, y, \beta')$. That, Eq. (16), transitivity, and strong monotonicity imply $\beta' \leq \delta$. By Eq. (17), $\alpha\gamma \sim^{**} \beta\beta' \succcurlyeq^{**} \beta\delta$, i.e., $\alpha\beta \succcurlyeq^{**} \gamma\delta$. A similar reasoning with strict preference in Eq. (16) yields $\alpha\beta \succ^{**} \gamma\delta$.

These reasonings show how $\succcurlyeq^{*}$ and $\succ^{*}$ revelations for state s_3 correspond to $\succcurlyeq^{**}$ and $\succ^{**}$, hence to V_1 differences when Eq. (17) can be satisfied; Eq. (17) can always be satisfied 'locally'. Therefore, such $\succcurlyeq^{*}$ and $\succ^{*}$ revelations cannot contain inconsistencies, and utility differences for s_3 must be the same as for the first two states, first locally, then as a consequence also globally. In other words, the additive representation is a subjective expected utility representation.

For more than three states, similar reasonings can be applied to any state s_j for $j \geq 3$, by keeping the outcomes for other states than s_1, s_2, s_j fixed. □

Appendix B. Proofs

Deriving linearity of utility in **Theorem 2**: Assume Statement (ii) and hence, as derived in the text, assume SEU with continuous utility U and positive probabilities. It is proved here that additivity of $\succcurlyeq$ implies linearity of U.

Consider an indifference $\alpha_1 f \sim \beta_1 g$. Adding up $(\varepsilon, 0, \ldots, 0)$ for $\varepsilon \in IR$ implies

$$(\alpha + \varepsilon, f_2, \ldots, f_n) \sim (\beta + \varepsilon, g_2, \ldots, g_n)$$

because of additivity. Substitution of SEU in both indifferences implies

$$U(\alpha + \varepsilon) - U(\beta + \varepsilon) = U(\alpha) - U(\beta). \tag{18}$$

This holds for all α, β for which f and g can be found such that $\alpha_1 f \sim \beta_1 g$. That is, for each $z \in IR$ an open neighborhood can be found in which the equality (18) holds. This implies local linearity of U, thus global linearity. U can be taken equal to the identity. □

Proof of Lemma 4. In order to prove that $\succcurlyeq$ is continuous on Γ^n it has to be shown that the sets

$$\{g \in \Gamma^n | g \prec h\} \text{ and } \{g \in \Gamma^n | g \succ h\}$$

are open in Γ^n for each $h \in \Gamma^n$.

The set of prizes $X = \{x_1, \ldots, x_m\}$ is finite. We prove openness of $A := \{g \in \Gamma^n | g \prec h\}$. Take any $g \in A$, hence $g \prec h$. Because the set of prizes is finite, there exists a 'best' prize $y \in X$, i.e., $y \succcurlyeq x$ for each prize x. Because of J-continuity, there exists a $0 < \mu < 1$ such that $\mu y + (1-\mu) g \prec h$. Because of monotonicity, $y \succcurlyeq f$ for every act f. By repeated application of vNM-independence, for each $0 \leq \mu' \leq \mu$ we have $h \succ \mu y + (1-\mu) g \succcurlyeq \mu' y + (1-\mu') g \succcurlyeq \mu' f + (1-\mu') g$. Hence, if $g \in A$, then we have an open neighborhood $\{f' \in \Gamma^n: f' = \mu' f + (1 -$

$\mu')g$ for some $0 \le \mu' < \mu$ and $f \in \Gamma^n\}$ of g in A. This implies that A is an open subset of Γ^n. Similarly, $\{g \in \Gamma^n | g \succ h\}$ is an open subset of Γ^n. $\succcurlyeq$ is continuous. □

Deriving linearity of utility in Theorem 6: First the following implication is derived:

$$\alpha_1 f \sim \left(\frac{\alpha+\beta}{2}\right)_1 g \Rightarrow \left(\frac{\alpha+\beta}{2}\right)_1 f \sim \beta_1 g. \tag{19}$$

Assume, for contradiction, that

$$\alpha_1 f \sim \left(\frac{\alpha+\beta}{2}\right)_1 g \text{ and } \beta_1 g \succ \left(\frac{\alpha+\beta}{2}\right)_1 f. \tag{20}$$

Taking $(1/2)/(1/2)$ mixtures of the left-hand sides and of the right-hand sides we get, by repeated application of vNM-independence and transitivity,

$$\left(\frac{\alpha+\beta}{2}\right)_1 \frac{f+g}{2} \succ \left(\frac{\alpha+\beta}{2}\right)_1 \frac{f+g}{2},$$

contradicting reflexivity. Similarly, Eq. (20) with $\prec$ instead of $\succ$ also implies a contradiction of reflexivity. Hence Eq. (19) must hold true.

Eq. (19) implies, for $z = (\alpha+\beta)/2$, $\alpha z \succcurlyeq^* z\beta$ and $\alpha z \preccurlyeq^* z\beta$ hence, by Eq. (1),

$$U(\alpha) - U\left(\frac{\alpha+\beta}{2}\right) = U\left(\frac{\alpha+\beta}{2}\right) - U(\beta)$$

which is equivalent to

$$U\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\left[U(\alpha) + U(\beta)\right]. \tag{21}$$

Because of continuity of U in the SEU model, for each γ in the convex set Γ there exists an open neighborhood in which, for each α and β, f and g can be found to imply the first indifference in Eq. (19). Eq. (21) follows in the open neighborhood of γ, hence the continuous U is linear there. If it is locally linear, then it is globally linear. □

Proposition 14. *Theorem 6 can be modified by letting X be infinite and Γ the set of all simple probability distributions over X.*

Proof. If all prizes are equivalent then, because of monotonicity, the theorem is trivial; U then is constant. Next suppose there are two prizes α, β such that $a \succ b$. Set $U(\beta) = 0, U(\alpha) = 1$. Consider any finite subset Y of X containing α and β. If we restrict attention to prizes from Y, then Theorem 6 can be invoked, giving an

SEU model with probabilities p_j^Y and utility U^Y. We may set $U^Y(\beta)=0$ and $U^Y(\alpha)=1$. For another finite subset Z of X containing α and β, we get an SEU model with probabilities p_j^Z and utility U^Z with, again, $U^Z(\beta)=0$ and $U^Z(\alpha)=1$. Now consider the finite set $Y \cup Z$. Because of uniqueness, the two SEU models coincide when attention is restricted to prizes from $Y \cap Z$. That is, any two such SEU models coincide on overlapping domain. Therefore the superscripts Y, Z can be dropped and probabilities p_j and utility U result such that SEU represents preferences when restricted to any finite subset of prizes. That suffices to describe all preferences between acts involving only simple lotteries over prizes. □

References

Abdellaoui, M., 1998. Eliciting Utilities and Decision Weights under Cumulative Prospect Theory. Working paper. GRID, ENS, Cachan, France.

Anscombe, F.J., Aumann, R.J., 1963. A definition of subjective probability. Annals of Mathematical Statistics 34, 199–205.

Blackorby, C., Davidson, R., Donaldson, D., 1977. A homiletic exposition of the expected utility hypothesis. Economica 44, 351–358.

Bleichrodt, H., Luis Pinto, J., 1998. The Validity of QALYs when Expected Utility Theory no Longer Holds. iMTA, Erasmus University, Rotterdam, the Netherlands.

Bouzit, A.M., Gleyses, G., 1996. Empirical Estimation of RDEU Preference Functional in Agricultural Production. GRID, ENS, Cachan, France.

Chew, S.H., 1989. The Rank-Dependent Quasilinear Mean. Unpublished manuscript, Department of Economics, University of California, Irvine, USA.

Chew, S.H., Epstein, L.G., 1989. A unifying approach to axiomatic non-expected utility theories. Journal of Economic Theory 49, 207–240.

Chew, S.H., Karni, E., 1994. Choquet expected utility with a finite state space: Commutativity and act-independence. Journal of Economic Theory 62, 469–479.

d'Aspremont, C., Gevers, L., 1990. Invariance, neutrality and weakly continuous expected utility. In: Gabszewicz, J.J., Richard, J.F., Wolsey, L.A. (Eds.), Economic Decision-making: Games, Econometrics and Optimisation: Contributions in honour of Jacques H. Drèze. North-Holland, Amsterdam, pp. 87–100.

Debreu, G., 1960. Topological methods in cardinal utility theory. In: Arrow, K.J., Karlin, S., Suppes, P., (Eds.), Mathematical Methods in the Social Sciences. Stanford University Press, Stanford, CA, 1959, pp. 16–26.

de Finetti, B., 1931. Sul significato soggettivo della probilità. Fundamenta Mathematicae 17, 298–329.

de Finetti, B., 1937. La prévision: ses lois logiques, ses sources subjectives. Annales de l'Institut Henri Poincaré 7, 1–68. Translated into English by Kyburg, H.E., Foresight: Its Logical Laws, its Subjective Sources. In: Kyburg, H.E., Smokler, H.E. (Eds.), 1964. Studies in Subjective Probability. Wiley, New York, pp. 53–118; 2nd edn. 1980, Krieger, New York.

Drèze, J.H., 1987. Essays on Economic Decision under Uncertainty. Cambridge University Press, London.

Fennema, H., van Assen, M.A.L.M., 1997. Measuring the Utility of Losses by Means of the Tradeoff Method. NICI, University of Nijmegen.

Fishburn, P.C., 1970. Utility Theory for Decision Making. Wiley, New York.

Fishburn, P.C., 1982. Foundations of risk measurement: II. Effects of gains on risk. Journal of Mathematical Psychology 22, 226–242.

Fishburn, P.C., 1986. The axioms of subjective probability. Statistical Science 1, 335–358.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with a non-unique prior. Journal of Mathematical Economics 18, 141–153.

Gul, F., 1992. Savage's theorem with a finite number of states. Journal of Economic Theory 57, 99–110, Erratum, 61, 1993, 184.

Hazen, G.B., 1989. Ambiguity aversion and ambiguity content in decision making under uncertainty. Annals of Operations Research 19, 415–434.

Jensen, N.E., 1967. An Introduction to Bernoullian Utility Theory, I, II. Swedish Journal of Economics 69, 163–183, 229–247.

Keeney, R.L., Raiffa, H., 1976. Decisions with Multiple Objectives. Wiley, New York. (Second edn., Cambridge University Press, Cambridge, 1993).

Klibanoff, P., 1995. Dynamic Choice with Uncertainty Aversion. Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A., 1971. Foundations of Measurement, Vol. I. Additive and Polynomial Representations. Academic Press, New York.

Luce, R.D., 1988. Rank-dependent, subjective expected-utility representations. Journal of Risk and Uncertainty 1, 305–332.

Luce, R.D., 1998. Coalescing, Event Commutativity, and Theories of Utility. Journal of Risk and Uncertainty 16, 87–114.

Münnich, Á., Maksa, G., Mokken, R.J., 1997. Multi-Attribute Aggregation: A Generalisation of Bisymmetry to *n* Variables. Institute of Psychology, Kossuth Lajos University, Debrecen, Hungary.

Nakamura, Y., 1990. Subjective expected utility with non-additive probabilities on finite state spaces. Journal of Economic Theory 51, 346–366.

Nakamura, Y., 1992. Multi-symmetric structures and non-expected utility. Journal of Mathematical Psychology 36, 375–395.

Nakamura, Y., 1995. Rank dependent utility for arbitrary consequence spaces. Mathematical Social Sciences 29, 103–129.

Pfanzagl, J., 1968. Theory of Measurement. Physica-Verlag, Vienna.

Pollak, R.A., 1967. Additive von Neumann–Morgenstern utility functions. Econometrica 35, 485–494.

Quiggin, J., 1981. Risk perception and risk aversion among Australian farmers. Australian Journal of Agricultural Economics 25, 160–169.

Quiggin, J., Wakker, P.P., 1994. The axiomatic basis of anticipated utility; A clarification. Journal of Economic Theory 64, 486–499.

Ramsey, F.P., 1931. Truth and probability. In: The Foundations of Mathematics and other Logical Essays. Routledge and Kegan Paul, London, pp. 156–198. Reprinted in: Kyburg, H.E., Smokler, H.E. (Eds.), 1964. Studies in Subjective Probability. Wiley, New York, pp. 61–92.

Sarin, R.K., Wakker, P.P., 1992. A simple axiomatization of nonadditive expected utility. Econometrica 60, 1255–1272.

Sarin, R.K., Wakker, P.P., 1997. A single-stage approach to Anscombe and Aumann's expected utility. Review of Economic Studies 64, 399–409.

Savage, L.J., 1954. The Foundations of Statistics. Wiley, New York. (Second edn., Dover, New York, 1972).

Schmeidler, D., 1989. Subjective probability and expected utility without additivity. Econometrica 57, 571–587.

Segal, U., 1993. Order indifference and rank-dependent probabilities. Journal of Mathematical Economics 22, 373–397.

Vickrey, W., 1945. Measuring marginal utility by reactions to risk. Econometrica 13, 319–333.

von Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton University Press, Princeton, NJ.

Wakker, P.P., 1984. Cardinal coordinate independence for expected utility. Journal of Mathematical Psychology 28, 110–117.

Wakker, P.P., 1989a. Continuous subjective expected utility with nonadditive probabilities. Journal of Mathematical Economics 18, 1–27.
Wakker, P.P., 1989b. Additive Representations of Preferences, A New Foundation of Decision Analysis. Kluwer Academic Publishers, Dordrecht.
Wakker, P.P., 1993. Unbounded utility for Savages' foundations of statistics', and other models. Mathematics of Operations Research 18, 446–485.
Wakker, P.P., 1994. Separating marginal utility and probabilistic risk aversion. Theory and Decision 36, 1–44.
Wakker, P.P., Deneffe, D., 1996. Eliciting von Neumann–Morgenstern utilities when probabilities are distorted or unknown. Management Science 42, 1131–1150.
Wakker, P.P., Tversky, A., 1993. An axiomatization of cumulative prospect theory. Journal of Risk and Uncertainty 7, 147–176.