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A GRAPH-THEORETIC APPROACH TO REVEALED PREFERENCE

by

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§ 1 *Introduction*

Social choice theory studies the aggregation of individual preference relations into one social preference relation. Thus preference relations play here a central role, as they do in many theoretical decision sciences. For applications, observations, and discussions choice functions are more suited to represent the attitude of the individual decision makers and/or the society. Thus Arrow (1963), in discussions and examples, often considers choice behavior instead of preference relations. Wakker (1986) argues that one of the reasons for mislead intuitions, and discussions on examples as in section III.3 of Arrow (1963), lies in the problems about the derivations of choice functions from preference relations, and vice versa. For discussions of Arrow's examples, and further references, see Wakker, 1986, or Ray, 1973.

The study of these derivations is done in revealed preference theory. It has originated from consumer demand theory. There the choice function is called 'demand function'; it indicates, for every possible 'budget set' (i.e. set of available (alternatives =) commodity bundles) what the consumer would choose from this set. Early contributions were Samuelson (1938), Ville (1951-1952, earlier 1946), Houthakker (1950). *The main results of interest to our present work are Arrow (1959) and Richter (1966)*. These concern conditions for choice functions, necessary and sufficient for the

existence of 'weak orders' from which the choice functions can be considered to be derived.

Recently new impulses have come from social choice theory. See Sen (1970, 1971), Plott (1971, 1973), Fishburn (1973), Kelly (1978). One tried way out of Arrow's impossibility theorem is to relax the requirement of transitivity of a social preference relation. Cf. Blair & Pollak (1979), Hansson (1969), (Storcken, 1987). These works show that non-drastic relaxations will not suffice to avoid impossibility. This induced the study of conditions for choice functions, weaker than those characterizing the existence of transitive preference relations.

In § 2 some standard results in the field of revealed preference theory will be presented. § 3 introduces the graph-theoretic approach to revealed preference. This approach makes possible a visual presentation by which the subject becomes easily accessible. We have illustrated this visual presentation in the proofs of Lemmas 3.1, 3.4, and 3.5. The reader may wish to construct visual presentations for several of the other results. We use this approach to relate several results to what we consider the main result: Theorem 2.2, *the theorem of Richter*. In § 4 a duality approach in revealed preference theory is related to duality in graph-theory, as occurring when the roles of vertices and arcs are interchanged. This enables us to relate results of Weddepohl (1970) and Hansson (1968) to the result of Richter. § 5 finally considers some conditions, essentially weaker than those characterizing representing weak orders.

§2. *Revealed Preference*

As usual A denotes a non-empty set of alternatives. D is a non-empty collection of non-empty subsets of A , so-called *choice situations*. The (dispositional) choice behavior of a decision maker (person, society, ...) is represented by a *choice function* C . This assigns to every choice situation D from D a non-empty subset $C(D) \subset D$. If the decision maker would have to choose one element from D as his 'best', then he would be willing to take any element of $C(D)$ for this. The idea to consider not just the choice behavior of the decision maker at one actual choice situation, but to consider his choice behavior at several (potential) choice situations, and to compare these several choices with each other through properties of the choice function such as the weak axiom of revealed preference (see (2.2)),

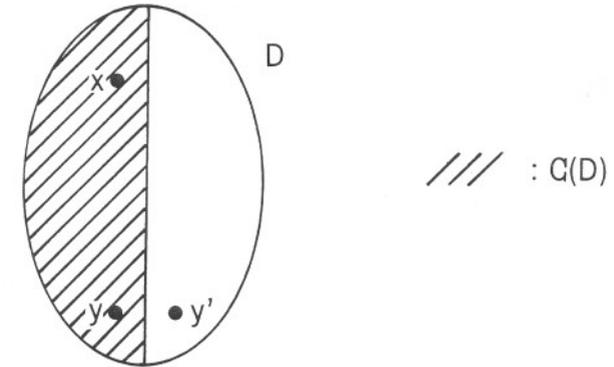


Illustration 2.1. Choice situation D shows that xRy , xRy' , and $xP'y'$. It does not show that $xP'y$. There may possibly exist another choice situation E to show $xP'y$; or to show $y'Rx$, or $y'Ry$.

is one of the other stones of decision theory. From this preference relations and utility functions can be derived.

A preference relation \geq represents C if C assigns to every choice situation D the set of \geq -best elements of D , i.e. $C(D) = \{x \in D : x \geq y \text{ for all } y \in D\}$. In the search for preference relations which represent choice functions, we start by the well-known ways to derive 'revealed' preference relations from choice functions, as follows (see Illustration 2.1).

Alternative x is *directly revealed preferred* to y , notation xRy , if there is a choice situation $D \in D$ s.t. $x \in C(D)$ and $y \in D$, or if $x = y$. Alternative x is *directly strictly revealed preferred* to y , notation $xP'y$, if there is a choice situation $d \in D$ s.t. $x \in C(D)$ and $y \in D/C(D)$. It may very well happen that both xRy and $yP'x$, hence P' is not necessarily the asymmetric part of R , and hence we added the prime on top of P' . Note that an observation of the choice behavior of the decision maker in one choice situation may already lead to the conclusion $xP'y$. This is not the case for the asymmetric part of R .

LEMMA 2.1. A transitive binary relation \geq represents C if and only if for all $x, y \in A$ [$\{xRy \rightarrow x \geq y\}$ & $\{xP'y \rightarrow x > y\}$]

PROOF. Let \geq represent C . [$xRy \rightarrow x \geq y$] is obvious. Now let $xP'y$, say $x \in C(D)$ and $y \in D/C(D)$. Obviously $x \geq y$. And not $y \geq x$, for $y \geq x$ and transitivity of \geq and [$x \geq z$ for all $z \in D$] would imply [$y \geq z$ for all $z \in D$], i.e. $y \in C(D)$. We conclude $x > y$.
Next let [$xRy \rightarrow \geq y$, $xP'y \rightarrow x > y$, for all alternatives x, y]. If $x \in C(D)$, then xRy , so $x \geq y$, for all $y \in$

D. Consequently $C(D) \subset \{x \in D : x \succeq y \text{ for all } y \in D\}$. And if $x \in D/C(D)$, then we start by taking $y \in C(D)$ ($\neq \emptyset!$). Then $y P' x$, so $y > x$, whence not $x \succeq y$. Consequently $C(D) \supset \{x \in D : x \succeq y \text{ for all } y \in D\}$

We say that alternative x is *indirectly revealed preferred to* alternative y ($x \bar{R} y$) if there exist x^0, x^1, \dots, x^n such that $x = x^0$, $y = x^n$, and $x^0 R x^1, x^1 R x^2, \dots, x^{n-1} R x^n$. So \bar{R} is the transitive closure of R . Further, x is *indirectly strictly revealed preferred to* y ($x \bar{P}' y$) if there exist x^0, x^1, \dots, x^n such that $x = x^0$, $y = x^n$, $x^0 R x^1, x^1 R x^2, \dots, x^{n-1} R x^n$ (so $x^0 \bar{R} x^n$) and furthermore $x^{i-1} P' x^i$ for at least one j .

We shall now list some well-known conditions for choice functions. After that we shall end this section by giving some standard representation theorems, involving the given well-known conditions. A closer (graph-theoretic) look at the conditions, and proofs of the theorems, are given in the next section. The choice function C satisfies:

Congruency if: $x \bar{R} y \rightarrow \text{not } y P' x$ (2.1)

WARP (the weak axiom of revealed preference) if: $x R y \rightarrow \text{not } y P' x$ (2.2)

IIIA (individual independence of irrelevant alternatives) if: (2.3)

$[S \subset L \text{ and } S \cap C(L) \neq \emptyset \rightarrow C(S) = S \cap C(L)]$.

The word 'individual' in (2.3) is not used in literature. We added it to express clearly that (2.3) is different from the condition 'independence of irrelevant alternatives' as used in social choice theory, e.g. in Arrow's impossibility theorem.

The following main result of revealed preference theory has first been obtained in Richter (1966, Theorem 1). Cf. also Richter (1971, Theorems 5 and 8).

THEOREM 2.2. The choice function C satisfies congruency, if and only if there exists a transitive preference relation representing C , which is if and only if there exists a weak order representing C .

The next result was essentially first published in Arrow (1959).

THEOREM 2.3. Let every two- and three-point subset of A be in the domain D of C . Then there exists a weak order, representing C , if and only if C satisfies IIIA, which is if and only if C satisfies WARP.

§ 3. A Graph-theoretic Approach

In this § we present the graph-theoretic approach to reformulate and prove the results of the previous section. See Illustration 3.1.

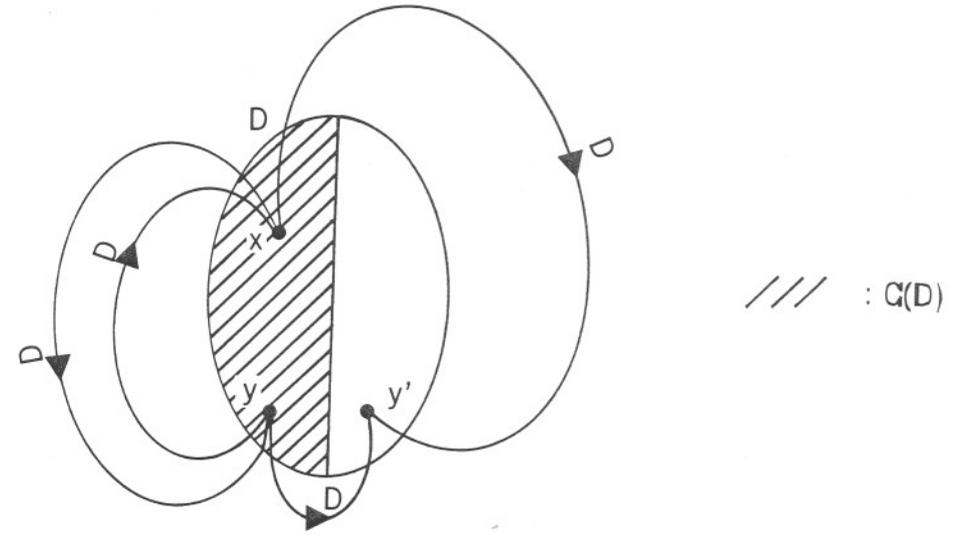


Illustration 3.1. Choice situation D induces (reversible) arcs $x D y$ and $y D x$, and (irreversible) arcs $x D y'$ and $y' D x$.

Alternatives are vertices. Every $D \in D$ yields a (directed) arc $x D y$ between any alternatives x, y with $x \in C(D)$, $y \in D$. So between alternatives x, y there may be several different arcs $x D_j y$, deriving from different choice situations D_j . The function 'r' remembers, for every arc, from which choice situation the arc derives, so assigns D to the arc $x D y$. The graph-theoretic approach is most useful if the role of this function r is small.

A *directed path*, or *dipath* (of length n) (from x^0 to x^n) is a finite sequence $(x^0 D_1 x^1, x^1 D_2 x^2, \dots, x^{n-1} D_n x^n)$, abbreviated $x^0 D_1 x^1 D_2 x^2 \dots x^{n-1} D_n x^n$. If $x^0 = x^n$, then we also call the dipath a *directed circuit*, or *dicircuit*.

An arc $x D y$ is *reversible* if also $y D x$ is an arc, the *reversed* of $x D y$. A dipath is *reversible* if every arc in it is reversible. Obviously we have:

- $x R y$ ↔ there exists an arc $x D y$, or $x = y$
- $x P' y$ ↔ there exists an irreversibly arc $x D y$
- $x \bar{R} y$ ↔ there exists a dipath from x to y , or $x = y$

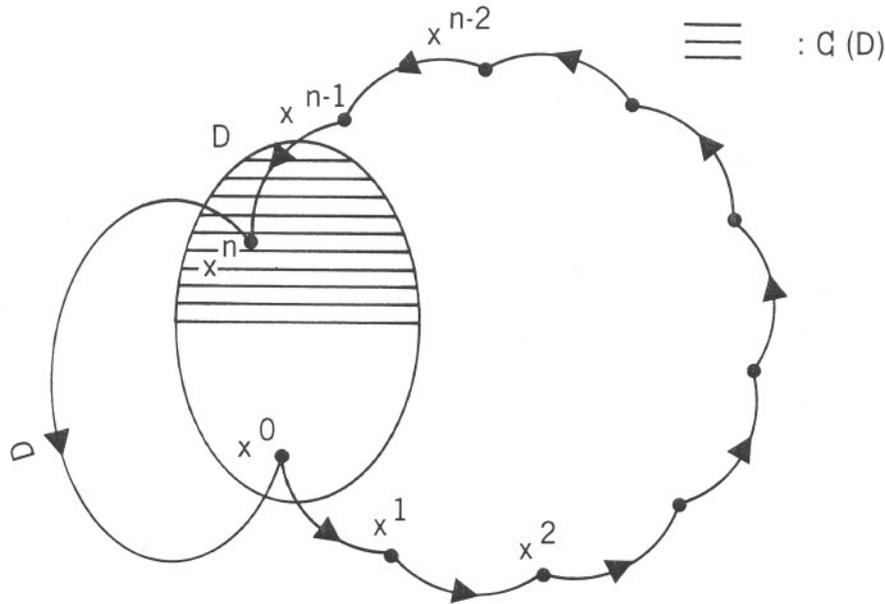


Illustration 3.2.a. A violation of congruency occurs iff there are alternatives x^0, x^n s.t. $x^0 R x^n$ and $x^n P' x^0$, which is iff x^0, x^n are on a common dicircuit, with irreversible arc $x^n D x^0$.

$x P' y \iff$ there exists an irreversible dipath from x to y
Further we have:

LEMMA 3.1. The choice function C satisfies congruency if and only if every dicircuit is reversible, which is if and only if, for all alternatives x, y on a common dicircuit, every dipath from x to y is reversible.

PROOF. See Illustrations 3.2.a and 3.2.b.

LEMMA 3.2. Every dicircuit is reversible if and only if there exists a transitive preference relation \geq that represents C , which is if and only if \bar{R} represents C .

PROOF. Let a transitive \geq , representing C , exist. By Lemma 2.1, and transitivity of \geq , for all x, y on a common dicircuit $x \approx y$, so not $x > y$, neither $x P' y$ according to Lemma 2.1. The dicircuit is reversible.

Next let every dicircuit be reversible. Define $\geq = \bar{R}$. So $x \geq y$ iff: A dipath from x to y exists or $x = y$. Then \geq is transitive.

Obviously $x R y \rightarrow x \geq y$. Further $x P' y$ and $y \geq x$ would give an irreversible dicircuit, which cannot be, so $x P' y \rightarrow x > y$. Ny Lemma 2.1, $\geq (= \bar{R})$ represents C .

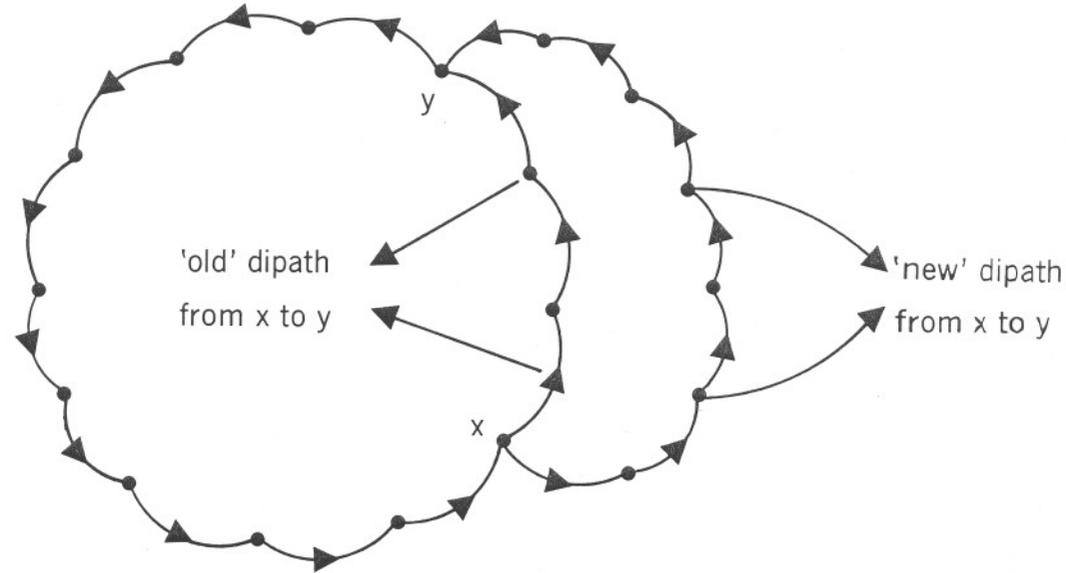


Illustration 3.2.b. If every dicircuit is reversible, then every ('new') dipath from alternative x to alternative y is reversible, for x and y on a common dicircuit. To see this, let the 'new' dipath participate in a dicircuit containing x and y , obtained by replacing the 'old' dipath by the new dipath.

LEMMA 3.3. If there exists a transitive binary relation, representing C , then there exists a weak order, representing C .

PROOF. By the previous Lemma \bar{R} represents C . The idea is to obtain a complete transitive extension \geq of \bar{R} by extending the directed graph, derived from C , through an addition of 'artificial' arcs, and by then defining $x \geq y$ whenever there is a dipath from x to y . Obviously:

$$\text{This } \geq \text{ will be transitive} \tag{3.1}$$

And obviously:

$$x R y \rightarrow x \geq y. \tag{3.2}$$

By Lemma 2.1 we must take care that $x P' y \rightarrow x > y$. By (3.2) and $p' \subset R$, it suffices to guarantee $x P' y \rightarrow$ not $y \geq x$. In the original directed graph $[x P' y]$, by Lemma 3.1, implies that x and y are not on a common dicircuit. Hence, if after the addition of some artificial arcs, we find $y \geq x$ and $x P' y$, then the dicircuit on which both x and y now can be seen to be, is a new one not present in the original graph.

So we can ensure:

$$x P' y \rightarrow x > y \tag{3.3}$$

by adding artificial arcs in such a way that never a new dicircuit occurs. Further we must have \geq complete, so as long as there exist x, y such that no dipath from x to y or from y to x exists yet, we add an artificial arc between x and y . Note that the addition of such an arc indeed does not induce a new dicircuit. That indeed this extension process can be continued till the final stage, where there is a dipath from x to y or from y to x for all alternatives x, y , can be shown by means of the Lemma of Zorn; elaboration is omitted here. Finally:

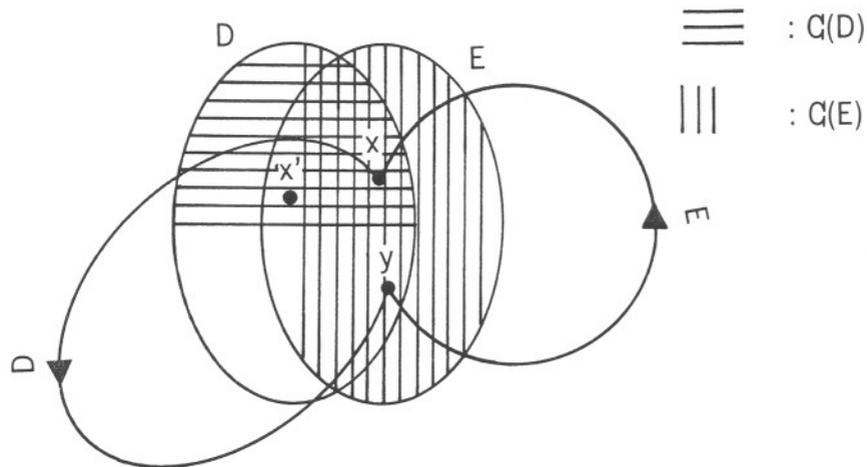


Illustration 3.3. A violation of WARP occurs iff there are alternatives y and x (or x' ; say x) with yRx and $xP'y$, which is iff there is a dicircuit $xDyEx$ of length 2 with irreversible arc xDy .

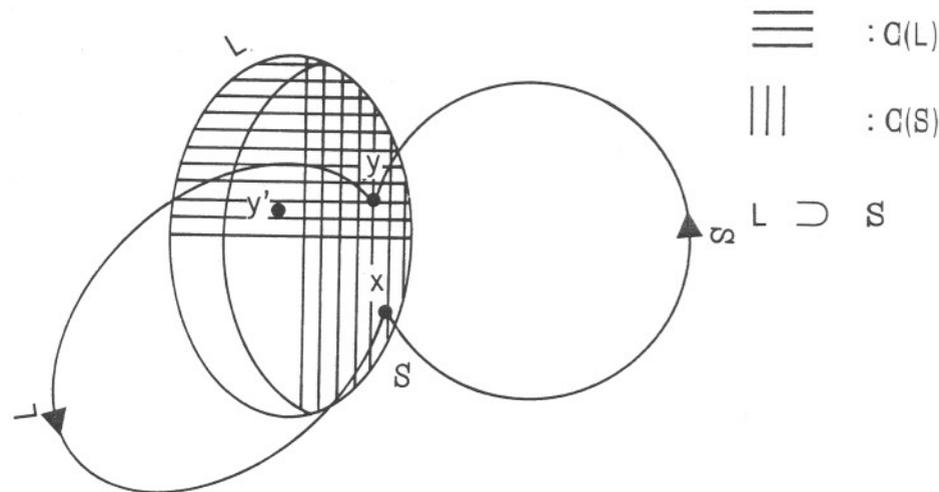


Illustration 3.4.b. The second type of violation of IIIA. There is an alternative x in $C(S)$, which is not in $S \cap C(L)$, whereas there does exist a y (or y' ; say y) which is in $S \cap C(L)$. This corresponds with a dicircuit $xSyLx$ of length 2, with irreversible second arc yLx .

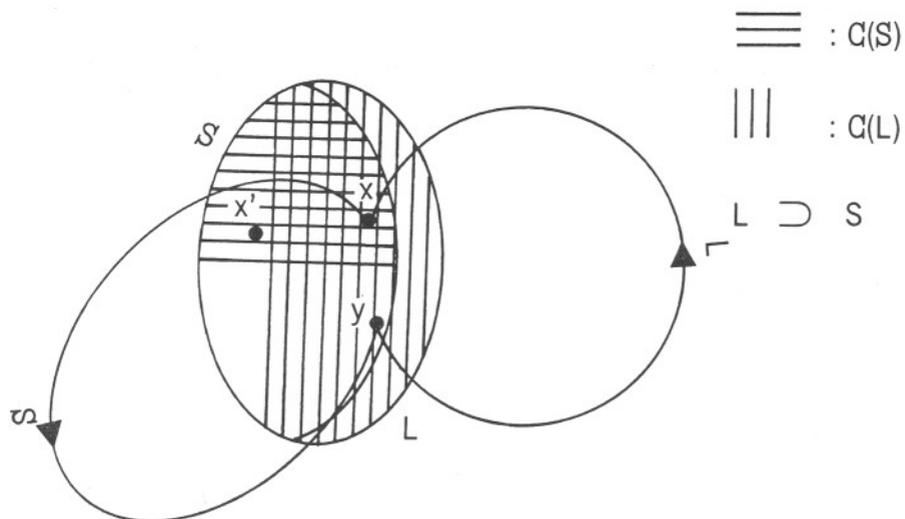


Illustration 3.4.a. One type of violation of IIIA. There is an alternative y in $S \cap C(L)$, which is not in $C(S)$. Taking an x (or x' ; say x) in $C(S)$, one sees that this corresponds with a dicircuit $xSyLx$ of length 2, with irreversible first arc xSy .

\geq is complete (3.4)
 Now (3.4) and (3.1) show that \geq is a weak order, (3.2), (3.3) and Lemma 2.1 show that \geq represents C .
 From the above three Lemmas Theorem 2.2, the Theorem of Richter, follows.

LEMMA 3.4. The choice function C satisfies WARP if and only if every dicircuit of length 2 is reversible.

PROOF. See Illustration 3.3.

LEMMA 3.5. The choice function C satisfies IIIA, if and only if every dicircuit $xSyLx$ with $S \subset L$ is reversible.

PROOF. See Illustrations 3.4.a and 3.4.b.

COROLLARY 3.6. Congruency implies WARP, WARP implies IIIA.

LEMMA 3.7. If D contains every two- and three-point subset of A , then IIIA implies congruency.

PROOF. Let IIIA be satisfied. Let $x^0 D_1 x^1 D_2 x^2 \dots x^{n-1} D_n x^n D_0 x^0$ be a dicircuit. We shall show that $x^n D_0 x^0$ is reversible. First we show: If there is a dipath $x D y D' z$ of length 3 from x to z , (3.5) then there is a 'short-cut' $x\{x,z\}z$.

If $x D y$, then by IIIA we have $x \in C\{x,y\}$; $y D' z$ implies $y \in C\{y,z\}$. Next consider $E := \{x,y,z\}$. We must have $x \in C(E)$ by a 'domino'-argument: If $z \in C(E)$ then IIIA via $y \in C\{y,z\}$ gives $y \in C(E)$, and if $y \in C(E)$ then IIIA via $x \in C\{x,y\}$ gives $x \in C(E) \neq \emptyset$! Now $x \in C(E)$ by IIIA implies $x \in C\{x,z\}$. So (3.5) holds. Thus we can repeatedly short-cut $x^0 D_1 x^1 D_2 x^2 \dots x^{n-1} D_n x^n D_0 x^0$ to end up with $x^0\{x^0,x^n\}x^n D_0 x^0$. By IIIA and Lemma 3.5, $x^n D_0 x^0$ is reversible.

The above Lemma, Corollary 3.6, and Theorem 2.2 now prove Theorem 2.3. A further condition for choice functions, met in literature, is the 'strong axiom of revealed preference' (SARP), introduced by Houthakker (1950). In our graph-theoretic approach, it means that there do not exist dicircuits (apart from trivial ones, containing no more than one alternative). Obviously this implies congruency, hence representability by a weak order. The weak order can always be taken antisymmetric as can be derived from the observation that in the proof of Lemma 3.3 no new 'artificial' dicircuits were added.

§ 4. Duality

The following definitions of Weddepohl (1970) will be central in this section. For choice situations D_1, D_2, D_1 is *directly revealed as favorable as* D_2 , notation $D_1 R^* D_2$, if $D_1 \cap C(D_2) \neq \emptyset$.

Further choice situation E is *indirectly revealed as favorable as* choice situation F , notation $E (\tilde{R}^* F$, if there exist $E = D_0, D_1, \dots, D_n = F$ s.t. $D_{j-1} R^* D_j$ for all j .

The interest in the above definitions stems from duality considerations in consumer demand theory. There, for a fixed income M , say $M = 1$, there is a one-to-one correspondence between price vectors and (choice sets =) budget sets. A consumer with income M will consider a price vector p^1 at least as favorable as p^2 if under p^1 he can buy a commodity bundle which he considers best under p^2 . This coincides with the revealed favorability condition for the involved budget sets.

Again we introduce a graph-theoretic terminology, dual to the 'primal' terminology of the previous section. This time choice situations are considered to be vertices; an alternative x yields a (directed) arc $E x F$ between any two choice situations E, F for which $x \in E \cap C(F)$. Dipaths and dicircuits are defined and denoted analogously to the previous section, e.g. $x^0 D_1 x^1 \dots x^{n-1} D_n x^n D_0 x^0$ is a typical dual dicircuit. Note that the middle of this notation, e.g. $\dots x^2 D_3 x^3 D_4 \dots$ does not reveal if one is in a primal or a dual dipath. This has been done deliberately, since $x^2 D_3 x^3 D_4 \dots$ means: $x^2 \in C(D_3), x^3 \in D_3, x^3 \in C(D_4)$, both in the primal and in the dual case. And any primal

dicircuit $x^0 D_1 x^1 \dots x^{n-1} D_n x^n D_0 x^0$ corresponds with a dual dicircuit $D_0 x^0 D_1 \dots D_{n-1} x^{n-1} D_n x^n D_0$.

Weddepohl (1970) considers the following condition, named 'strong axiom of revealed preference' there, but renamed here: C satisfies the *strong axiom of revealed favorability* (SARF) if $E \tilde{R}^* F \rightarrow \text{not } F a(R^*) E$. (a : asymmetric part of)

LEMMA 4.1. C satisfies SARF if and only if, for every dual dicircuit $D_0 x^0 D_1 \dots D_{n-1} x^{n-1} D_n$ (with $D_n = D_0$), there exists an arc $D_j y^{j-1} D_{j-1}$, for every j .

PROOF. SARF is violated if and only if there are E, F with $E \tilde{R}^* F, F R^* E$, and not $E R^* F$. This is if (with, for some $j, D_j = E, D_{j-1} = F$, and $D_0 = D_n$) there is a dual dicircuit $D_0 x^0 D_1 \dots D_{n-1} x^{n-1} D_n$, for which no arc $D_j y^{j-1} D_{j-1}$ exists.

Note that congruency implies SARF, under congruency $y^{j-1} = x^{j-1}$ can be taken in the above Lemma. Both SARF does not imply congruency. It may happen, in the above Lemma, that for some j , it is impossible to take $x^{j-1} = y^{j-1}$. Then we are faced with a dual dicircuit $D_{j-1} x^{j-1} D_j y^{j-1} D_{j-1}$ in which the first (dual) arc $D_{j-1} x^{j-1} D_j$ is irreversible. This implies the occurrence of a primal dicircuit, of length 2, $x^{j-1} D_j y^{j-1} D_{j-1} x^{j-1}$, with irreversible second arc. So if SARF holds, and congruency is violated, then WARP is violated. We conclude (see also Corollary 3.6):

THEOREM 4.2. The choice function C satisfies SARF and WARP, if and only if it satisfies congruency, which is if and only if it is represented by a weak order.

The above theorem, in combination with Richter's Theorem 2.2, shows an alternative way to derive results as in Chapter 3 of Weddepohl (1970).

Next we demonstrate how to derive a result of Hansson (1968) from Richter (1966). Hansson calls an n -tuple D_0, D_1, \dots, D_n ($D_n = D_0$) *C-connected* if $D_{j-1} \cap C(D_j) \neq \emptyset$ for all j . In our terminology this means that D_0, \dots, D_{n-1}, D_n are on one dual dicircuit. The 'HARP'-condition of Hansson says that, for all $j, D_{j-1} \cap C(D_j) = C(D_{j-1}) \cap D_j$ should hold for D_0, \dots, D_n as above. This implies congruency, because, for any primal dicircuit $x^0 D_1 x^1 \dots D_n x^n$ ($x^n = x^0$), we have a corresponding dual dicircuit $D_1 x^1 \dots D_n x^n D_1$, which is C -connected. By HARP, $x^j \in C(D_j)$ for all j . Thus the primal dicircuit is reversible, and C is congruent.

Conversely, congruency implies that C can be represented by a weak order. This straightforwardly implies HARP. We conclude that Hansson's characterization of representability of C by a weak order, by means of the

HARP condition, can be derived from Richter's characterization by means of the congruency condition.

§ 5. *Conditions, Necessary But Not Sufficient For Representability By A Transitive Binary Relation*

The following properties, weakenings of IIIA, are called 'property α ', respectively 'property β ', in Sen (1971). Property α already occurred in Arrow (1959), as 'C3'; also in Chernoff (1954) it occurred. We propose here alternative terms:

We say that the choice function C exhibits *non-increasing eligibility* if $S \subset L \rightarrow C(S) \supset S \cap C(L)$. This is violated iff we are in the situation of Illustration 3.4.a, i.e. there is an alternative y in the 'small' S , which is chosen in the 'large' L , but not in the small S . It seems then that the eligibility of the alternative has increased with the size of the choice situation.

We say that the choice function C has *non-decreasing eligibility* if $S \subset L$ & $S \cap C(L) \neq \emptyset \rightarrow C(S) \subset S \cap C(L)$. Then, if not S is inferior to L (i.e. $S \cap C(L) = \emptyset$), every alternative which is eligible in S , remains eligible in the 'larger-size' L . (A violation occurs if and only if the situation of illustration 3.4.b occurs.)

LEMMA 5.1. The choice function C satisfies IIIA, if and only if it satisfies both non-increasing and non-decreasing eligibility.

PROOF. Obvious.

Several variations of the above two properties are studied in literature, Sen introduces properties, γ , δ , $\beta+$, etc. We shall not consider all these. Rather shall we relate the above two properties to the following appealing condition, introduced by Plott cf., e.g., Plott, 1973. The work in the sequel requires some structural assumptions, such as that for any $E, F \in D$, also $E \cup C(F)$ is in D . We do not want to pay attention to the complications because of this, so we assume:

Assumption D is the collection of all finite non-empty subsets of A .

We say that C satisfies *Path Independence (PI)* if $C(E \cup F) = C(C(E) \cup F)$ for all $E, F \in D$.

LEMMA 5.2. C has non-increasing eligibility, if and only if $C(E \cup F) \subset C(C(E) \cup F)$ for all E, F .

PROOF. Suppose C has non-increasing eligibility. Set $L = E \cup F, S = E$, to obtain $C(E \cup F) \cap E \subset C(E)$. Hence: $C(E \cup F) \subset C(E) \cup F$. (5.1)

Next set $S = C(E) \cup F, L = E \cup F$. By non-increasing eligibility, $C(E \cup F) \cap C(E) \cup F \subset C(C(E) \cup F)$ (5.2)

By (5.1) we may rewrite (5.2) as: $C(E \cup F) \subset C(C(E) \cup F)$, which is what should be proved. Next suppose that C does not have non-increasing eligibility. Then there are L, S, x with $L \supset S, x \in S \cap C(L)$, and not $x \in C(S)$. Set $E = S, F = L/S$. Then $x \in C(E \cup F)$, but not $x \in C(E) \cup F$, so $x \notin C(C(E) \cup F)$.

LEMMA 5.3. Non-decreasing eligibility implies that $C(E \cup F) \supset C(C(E) \cup F)$.

PROOF. First suppose: $C(E \cup F) \cap (C(E) \cup F) = \emptyset$. (5.3)

Contradiction will follow. Since $C(E \cup F) \subset E \cup F$, by (5.3) we get $C(E \cup F) \subset E/C(E)$. Hence $C(E \cup F) \cap E \neq \emptyset$, by non-decreasing eligibility $C(E \cup F) \cap E \supset C(E)$ follows. This contradicts (5.3). So we may suppose that (5.3) does not hold. Then, setting $L = E \cup F, S = C(E) \cup F$, by non-decreasing eligibility we get $C(E \cup F) \cap (C(E) \cup F) \supset C(C(E) \cup F)$, so $C(E \cup F) \supset C(C(E) \cup F)$.

The above Lemmas show that non-increasing and non-decreasing eligibility together (so IIIA) imply path independence, and that path independence comes close to imply IIIA. Still, path independence does not imply non-decreasing eligibility:

Let $A = \{x, y, z\}, D = 2^A/\{\emptyset\}, C(D) = D$ whenever $\|D\| \leq 2, C(A) = \{x, y\}$.

APPENDIX, ELEMENTARITIES CONCERNING BINARY RELATIONS

A binary relation \geq on a set A is a *weak order* if it is *complete* (i.e., for all x, y in $A, x \geq y$ or $y \geq x$) and *transitive* (for all x, y, z in A , if $x \geq y$ and $y \geq z$, then $x \geq z$). By \approx we denote the *symmetric part* of \geq , i.e. $x \approx y$ iff both $x \geq y$ and $y \geq x$. The *asymmetric part* of \geq may be denoted by $>$, so $x > y$ if $x \geq y$ and not $y \geq x$. Also the asymmetric part of \geq may be denoted as a (\geq) .

§ 6. *Summary*

One of the issues in the impossibility theorem of Arrow is the difference between choice behavior, as considered by Arrow in most of the illustrations for the conditions in his theorem, and binary relations as dealt with in Arrow's theorem. The relations between choice behavior and binary relations are studied in revealed preference theory, a theory which originates from consumer demand theory. This paper presents a graph-theoretic approach to revealed preference theory. This is done by considering al-

ternatives as vertices, and choice situations as arcs. By means of this method alternative proofs are obtained for some known results. In particular it is shown that many results from literature can be derived from what may be the main result from revealed preference theory, a theorem of Richter (1966). Next a duality approach is sketched, where vertices and arcs are interchanged as done in dual graph theory. Finally some results are given for non-transitive binary relations. For these there is an increasing interest because of Arrow's theorem.

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