Characterizations of Quasilinear Representations and Specified Forms of These*

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ABSTRACT. In this paper two versions of 'equivalence independence' for binary relations on Cartesian products are introduced to characterize special kinds of representing functions. The obtained results are used to characterize quasilinear economic indexes, and several specified forms of these.

1. INTRODUCTION

Many problems from many fields of science come down to one and the same kind of mathematical problem: finding a function V of several variables to the reals that has certain desirable properties. These desirable properties can often be expressed in terms of functional equations, the mathematical theory of which is treated for instance in Aczél (1966). One example from economics is production theory, where variables are in- or outputs, and the function V is a production (efficiency) function. Another example is the theory of (statistical) price indexes. Here the variables are prices or consumed quantities of goods or services, at base or comparison times or locations, and the function V is a (statistical) price index or purchasing power parity. In the measurement of inequality the variables are the shares of total welfare, allocated to individuals, and V is a measure of inequality. In decision making under uncertainty the variables are consequences, contingent on the occurrence of some state of nature, and V describes or prescribes the decision making of an individual, facing uncertainty; e.g. V is an expected utility index.

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In some fields of science it is custom to take a binary relation ('preference relation') as primitive, and to search for characterizations (i.e. properties of the preference relation, necessary and sufficient) for the existence of a representing function V with desirable properties. Other fields take V as a primitive. We shall call such a V an **index**. One then searches for indexes that have desirable properties.

In sections 2, 3, and 4 we give some representation theorems for preference relations. In subsection 5.1 we indicate how such theorems can be reformulated for situations where indexes, instead of preference relations, are primitives, so that characterizations of the ordinal character of the indexes result. From these characterizations we derive, in subsection 5.2, characterizations of some well-known general kinds of indexes, all in an ordinal vein. Section 6 gives some conjectures. Finally, in subsection 6.3 we indicate that our results can also be used as a starting point for deriving 'non-ordinal' results, by the addition of a, usually weak, non-ordinal condition (i.e. 'test', 'axiom', 'property').

2. BASIC DEFINITIONS AND RESULTS

In this section we give definitions and state some preparatory results. Let Γ be a non-empty connected topological space. [For topological definitions, the reader is referred to Kelley (1955). Readers, not interested in topological details may simply take Γ to be a convex subset of $\mathbb{R}^m$. Such a Γ satisfies all the topological assumptions made in this paper (e.g. in Assumption 2.3 below.)] Let $n \in \mathbb{N}$. The Cartesian product Γ^n is endowed with the product topology. Its elements are called **alternatives**. An alternative $x \in \Gamma^n$ has i-th coordinate x_i.

A **preference relation** $\succeq$ is a binary relation on Γ^n. As usual we write $x \preceq y$ for $y \succeq x$, $x \approx y$ for $[x \succeq y \ \& \ y \succeq x]$, $x \succ y$ or $y \prec x$ for $[x \succeq y \ \& \text{ not } y \succeq x]$. The preference relation $\succeq$ is a **weak order** if it is **complete** ($x \succeq y$ or $y \succeq x$ for all $x,y \in \Gamma^n$) and transitive. If so, then $\approx$ is an equivalence relation. The preference relation $\succeq$ is **continuous** if $\{x \in \Gamma^n: x \succ y\}$ and $\{x \in \Gamma^n: x \prec y\}$ are open for all $y \in \Gamma^n$.

A function $V:\Gamma^n \to \mathbb{R}$ **represents** $\succeq$ if $[x \succeq y \Leftrightarrow V(x) \geq V(y)]$ for all $x,y \in \Gamma^n$. A representable preference relation is necessarily a weak order. In the literature a representing function is often called a 'utility function'. To settle terminology further, we call, for $A \subset \mathbb{R}$, a real-valued function $\varphi:A \to \mathbb{R}$ **increasing** if $\varphi(\alpha) > \varphi(\beta)$ whenever $\alpha > \beta$.

NOTATION 2.1. For $1 \leq i \leq n$, $\alpha \in \Gamma$, $x \in \Gamma^n$:

$x_{-i}\alpha$:= [x with x_i replaced by α].

DEFINITION 2.2. Coordinate (or index) i is **inessential** if $x_{-i}\alpha \approx x$ for all $\alpha \in \Gamma$, $x \in \Gamma^n$. If i is not inessential, then it is **essential**.

For weak orders $\succeq$ inessential coordinates do not have influence on the 'desirability' of an alternative. They may therefore be ignored and suppressed. That we shall do. For easy reference we state:

ASSUMPTION 2.3. All coordinates are essential. Γ is a connected topological space; Γ^n is endowed with the product topology. If n=1, Γ is topologically separable.

We shall be interested only in (increasing transforms of) functions of the following kind:

DEFINITION 2.4. A function $V:\Gamma^n \to \mathbb{R}$ is **additive** if there exist $V_j:\Gamma \to \mathbb{R}$ (j=1,...,n) such that $V:x \mapsto \sum_{j=1}^{n} V_j(x_j)$.

The following property goes under many names, such as (preferential) independence, additivity, strong (strict) separability, and 'sure-thing principle'. The present name abbreviates 'independent of equal coordinates'.

DEFINITION 2.5. The binary relation $\succeq$ is **coordinate independent** (CI) if for all i,x,y,α,β we have

$[x_{-i}\alpha \succeq y_{-i}\alpha] \Leftrightarrow [x_{-i}\beta \succeq y_{-i}\beta]$.

The idea is that, once two alternatives have the same i-th coordinate, the preference between them does not depend anymore upon this particular coordinate, be it α,β, or whatever. The idea can already be recognized in Fisher (1927, page 175).

The following theorem is mainly due to Debreu (1960). We give the slightly stronger result of Wakker (1986a, Theorem III.3.7), which supplies Krantz et al. (1971, Theorem 6.14) with continuity, thus shows that the assumption of topological separability, made in the original formulation of Debreu, can be omitted.

THEOREM 2.6 (Debreu). *Let $n \geq 3$. Under Assumption 2.3 the following two statements are equivalent for the binary relation $\succeq$ on Γ^n:*

(i) **There exists a continuous additive function on Γ^n which represents the binary relation $\succeq$.**

(ii) **The binary relation $\succeq$ is a continuous weak order which satisfies CI.**

□

The following property has been introduced in Wakker (1984), where an elucidation of its meaning is given. The chosen term is motivated, firstly, by the observation that under reflexivity of $\succeq$ the property implies CI (set x=y, $\alpha=\beta$, $\gamma=\delta$ below), and secondly by the observation that the condition adds to CI a preservation of first-order differences of utility, also independent of coordinates, which is known to imply cardinality in the presence of continuity w.r.t. a connected topology. See also Wakker (1984, section 3; 1986a, section 4.2; and 1986b, formula (3.3)) where reformulations are given in terms of a strength of preference relation, derived from the preference relation.

DEFINITION 2.7. The binary relation $\succeq$ satisfies <u>cardinal coordinate independence</u> (<u>CCI</u>) if for all $i,j,x,\ldots,\delta$ we have :

$$\begin{bmatrix} x_{-i}\alpha \preceq y_{-i}\beta & \& & v_{-j}\alpha \succeq w_{-j}\beta & \& \\ x_{-i}\gamma \succeq y_{-i}\delta \end{bmatrix}$$

$$\Rightarrow \qquad [v_{-j}\gamma \succeq w_{-j}\delta]$$

The idea of the condition is that the left two preferences reveal that the *strength of preference* of α over β is not larger than that of γ over δ. Further, by this the right preference above should imply the lower preference. The following theorem is the main result in Wakker (1986a; Theorem IV.3.3). Its main application lies in decision making under uncertainty where it characterizes subjective expected utility maximization, obtaining subjective probabilities and utilities simultaneously, as Savage (1954) did. Theorem V.6.1 in Wakker (1986a) extends the result to arbitrary, possibly infinite, state spaces, i.e. to possibly infinite Cartesian products. As compared to Savage's result, this theorem replaces restrictions concerning the state space by the restriction of continuity, a restriction usually satisfied in economic contexts.

THEOREM 2.8. *Under Assumption 2.3 the following two statements are equivalent for the binary relation $\succeq$ on Γ^n:*

(i) There exist positive $(p_j)_{j=1}^n$, summing to one, and a continuous function $U:\Gamma\to\mathbb{R}$, such that $x\mapsto\sum_{j=1}^n p_jU(x_j)$ represents the binary relation $\succeq$.

(ii) The binary relation $\succeq$ is a continuous CCI weak order.

□

For contexts other than decision making under uncertainty the restriction on the p_j's in (i) above may be undesirable. For instance in production theory one would want some p_j's (those associated with inputs x_j) to be negative. In the theory of price indexes one would want the weights p_j, associated with base prices x_j, to be negative. In section 4 we shall weaken the CCI condition for $\succeq$ in (ii) exactly far enough to obtain a characterization of (i) without restrictions on the p_j's. For this we first strengthen Debreu's Theorem 2.6 to a result which uses as much as possible the equivalence relation $\approx$ instead of the preference relation $\succeq$ (see Theorem 3.10 below). Because of the importance of Debreu's theorem, this result may have interest of its own.

3. <u>CHARACTERIZING ADDITIVE REPRESENTATIONS VIA EQUIVALENCE</u>

In this section we shall replace the CI condition in Debreu's Theorem 2.6 by a weaker condition, using the equivalence relation $\approx$ instead of the preference relation $\succeq$. Since it is easier to test for equivalence $\approx$ than for 'dominance' $\succeq$, this may be of use in the testing of additivity of representations and indexes.

DEFINITION 3.1. The binary relation $\succeq$ is <u>equivalence coordinate independent</u> (<u>ECI</u>) if for all i,x,y,α,β we have

$$[x_{-i}\alpha \approx y_{-i}\alpha] \iff [x_{-i}\beta \approx y_{-i}\beta].$$

LEMMA 3.2. *If $\succeq$ satisfies CI, then it satisfies ECI.*

PROOF. Let $\succeq$ satisfy CI.Then $[x_{-i}\alpha \approx y_{-i}\alpha]$ iff $[x_{-i}\alpha \succeq y_{-i}\alpha$ & $y_{-i}\alpha \succeq x_{-i}\alpha]$, which, by twofold application of CI, holds iff $[x_{-i}\beta \succeq y_{-i}\beta$ & $y_{-i}\beta \succeq x_{-i}\beta]$. The latter holds iff $[x_{-i}\beta \approx y_{-i}\beta]$.

□

We shall also need the following implication of CI:

DEFINITION 3.3. The binary relation $\succeq$ is <u>weakly separable</u> if, for all i,x,y,α,β, we have

$$[x_{-i}\alpha \succeq x_{-i}\beta] \iff [y_{-i}\alpha \succeq y_{-i}\beta].$$

LEMMA 3.4. *If $\succeq$ is a weak order, then CI implies weak separability.*

PROOF. CI allows one to replace in every preference any single pair of equal coordinates, thus by repetition, any arbitrary number of pairs of equal coordinates. Weak separability allows only the replacement of (n-1)-tuples of pairs of equal coordinates.

□

The following definition is useful for weakly separable weak orders.

DEFINITION 3.5. For any $1 \le i \le n$, $\succsim_i$ is defined by

$$[\alpha \succsim_i \beta] \iff [\text{there exists } x \text{ such that } x_{-i}\alpha \succsim x_{-i}\beta].$$

One easily verifies, with $\succ_i$ the asymmetric part of $\succsim_i$, $\approx_i$ the symmetric part, $\alpha \precsim_i \beta$ iff $\beta \succsim_i \alpha$, and $\alpha \prec_i \beta$ iff $\beta \succ_i \alpha$:

LEMMA 3.6. *If the binary relation $\succsim$ is a weakly separable weak order, then every $\succsim_i$ is a weak order; further*

$$[x_i \succsim_i y_i \textit{ for all } i] \implies [x \succsim y]$$

and

$$[x_i \succsim_i y_i \textit{ for all } i \;\&\; x_i \succ_i y_i \textit{ for some } i] \implies [x \succ y].$$

□

The following property, defined in Krantz et al. (1971), will be used in our analysis.

DEFINITION 3.7. The binary relation $\succsim$ satisfies restricted solvability if for all i, x, y, α, γ we have:

$$[x_{-i}\alpha \succsim y \succsim x_{-i}\gamma] \implies [\text{there exists } \beta \text{ such that } x_{-i}\beta \approx y].$$

LEMMA 3.8. *Under Assumption 2.3 a continuous weak order $\succsim$ satisfies restricted solvability.*

PROOF. See Krantz et al. (1971, section 6.12.3).

□

LEMMA 3.9. *Let the binary relation $\succsim$ be a weakly separable ECI weak order. Let Assumption 2.3 hold. Let $\succsim$ be continuous. Then $\succsim$ is CI.*

PROOF. Suppose $x_{-n}\alpha \succsim y_{-n}\alpha$. We must show that $x_{-n}\beta \succsim y_{-n}\beta$. (It is no restriction to take the i of Definition 2.5 equal to n.) Let $A := \{j<n: x_j \succ_j y_j\}$. Without loss of generality, suppose $A=\{1,\ldots,k\}$ for some $0 \le k < n$. Define $x^0 := x_{-n}\alpha$, $x^{\ell} := x^{\ell-1}_{-\ell}y_{\ell}$ for $1 \le \ell \le k$. Then, by Lemma 3.6, $x^0 \succsim x^1 \succsim \ldots \succsim x^k$. Further, since $(y_{-n}\alpha)_j \succsim_j (x^k)_j$ for all j, $y_{-n}\alpha \succsim x^k$. So $x^0 \succsim y_{-n}\alpha \succsim x^k$. Let ℓ be such that $x^{\ell-1} \succsim y_{-n}\alpha \succsim x^{\ell}$. By restricted solvability (Lemma 3.8) there exists z_{ℓ} such that $x^{\ell}_{-\ell}z_{\ell} \approx y_{-n}\alpha$. Since $(x^{\ell}_{-\ell}z_{\ell})_n = \alpha$, by ECI we obtain $(x^{\ell}_{-\ell}z_{\ell})_{-n}\beta \approx y_{-n}\beta$. From $x^{\ell}_{-\ell}x_{\ell} = x^{\ell-1} \succsim y_{-n}\alpha \approx x^{\ell}_{-\ell}z_{\ell}$ it follows that $x_{\ell} \succsim_{\ell} z_{\ell}$. Hence $(x_{-n}\beta)_j \succsim_j ((x^{\ell}_{-\ell}z_{\ell})_{-n}\beta)_j$ for all j. By Lemma 3.6, $x_{-n}\beta \succsim (x^{\ell}_{-\ell}z_{\ell})_{-n}\beta$.

The latter is equivalent to $y_{-n}\beta$.

□

THEOREM 3.10. *Let $n \ge 3$. Under Assumption 2.3 the following two statements are equivalent for the binary relation $\succsim$ on Γ^n:*

(i) **There exists a continuous additive function on Γ^n which represents the binary relation $\succsim$.**

(ii) **The binary relation $\succsim$ is a continuous weakly separable weak order which satisfies ECI.**

PROOF. The implication (i) => (ii) is straightforward. The implication (ii) => (i) is by Lemma 3.9 and Theorem 2.6.

□

The characterization in Theorem 3.10 is stronger than that in Theorem 2.6 in the sense that, firstly, statement (ii) in Theorem 3.10 follows elementarily from that in Theorem 2.6 (by Lemmas 3.2 and 3.4), whereas the converse is not true without essentially invoking continuity, as the following Example shows.

EXAMPLE 3.11. Let $n \ge 3$, $\Gamma = \mathbb{R}$. Let $W : x \mapsto \sum_{j=1}^{n} x_j + \min\{x_j : 1 \le j \le n\}$. Define $\succsim$ by $[x \succsim y] \iff [x=y \text{ or } (W(x), x_2, \ldots, x_n) >_L (W(y), y_2, \ldots, y_n)]$, with $>_L$ the lexicographic order. Since $x \approx y$ only if $x=y$, ECI is trivially satisfied. Further, for every i, $\succsim_i$ is the usual ordering of real numbers, from which weak separability follows. Still, $(1,9,1,9,\ldots,9) \succsim (1,5,5,9,\ldots,9)$ and $(9,9,1,9,\ldots,9) \prec (9,5,5,9,\ldots,9)$. This violates CI.

REMARK 3.12. In literature (see Debreu,1960, Theorem 3, or Krantz et al.,1971, Theorem 6.14) the Theorem 2.6 of Debreu in fact is formulated, more generally, for a Cartesian product $\Pi_{i=1}^{n}\Gamma_i$ of connected topological spaces Γ_i which do not have to be equal. All results of this section can also be formulated for such Cartesian products, nowhere did we use the equality of the coordinate sets.

Let us finally note that an alternative proof of Theorem 3.10 could have been obtained by simply going through all the proofs in literature contributing to Theorem 2.6, and by checking that nowhere there is made essential use of implications of CI other than weak separability or ECI. Because of the complicatedness of those proofs we chose our present way of proof.

4. EQUIVALENCE CCI

In this section we characterize representations of the form $x \mapsto \sum_{j=1}^{n} \lambda_j U(x_j)$ where U is continuous and there is no restriction on the λ_j's. Our characterization is an alternative for the one in Krantz et al. (1971, Theorem 6.15). The 'equivalence CCI' condition that we shall use is stronger than the 'standard sequence invariance' condition of Krantz et al. (1971). On the other hand, we have to impose only weak separability instead of the stronger coordinate independence, and we do not have to formulate separately the theorem for the case of two coordinates.

DEFINITION 4.1. The binary relation satisfies equivalence cardinal coordinate independence (ECCI) if for all $i,j,x,\ldots,\delta$ we have :

$$\left[\begin{array}{l} x_{-i}\alpha \approx y_{-i}\beta \quad \& \quad v_{-j}\alpha \approx w_{-j}\beta \quad \& \\ x_{-i}\gamma \approx y_{-i}\delta \end{array}\right] \Rightarrow [v_{-j}\gamma \approx w_{-j}\delta]$$

This is as cardinal coordinate independence, with all preferences replaced by equivalences.

LEMMA 4.2. *Let $\succcurlyeq$ be ECCI. If $\approx$ is reflexive (i.e. $x \approx x$ for all x) then $\succcurlyeq$ satisfies ECI.*

PROOF. Substitute $x=y$, $\alpha=\beta$, $\gamma=\delta$ in Definition 4.1. □

The following theorem generalizes Theorem 2.8 to the case of possibly non-positive weights. Motivation for it has been given below Theorem 2.8.

THEOREM 4.3. *Under Assumption 2.3 the following two statements are equivalent for the binary relation $\succcurlyeq$ on Γ^n:*

(i) **There exist real $(\lambda_j)_{j=1}^{n}$ and a continuous function $U:\Gamma \to \mathbb{R}$, such that $x \mapsto \sum_{j=1}^{n} \lambda_j U(x_j)$ represents the binary relation $\succcurlyeq$.**

(ii) **The binary relation $\succcurlyeq$ is a continuous weakly separable ECCI weak order.**

PROOF. The implication (i)=>(ii) is straightforward. So we assume (ii) and derive (i). If $n\geq 3$, then by Lemma 4.2 and Theorem 3.10 there exists an additive representation $V: x \mapsto \sum_{j=1}^{n} V_j(x_j)$ for $\succcurlyeq$. If $n\leq 2$ then such a representation also exists; elaboration is omitted here, and can be found in Wakker (1986a, section IV.4). If $n=1$ we take $U=V_1$, $\lambda_1=1$, and (i) is established. So from now on we assume $n\geq 2$. ECCI gives:

$$\left[\begin{array}{l} \quad V_i(\alpha)-V_i(\beta) = \sum_{k\neq i}[V_k(y_k)-V_k(x_k)] = V_i(\gamma)-V_i(\delta) \\ \& \quad V_j(\alpha)-V_j(\beta) = \sum_{k\neq j}[V_k(w_k)-V_k(v_k)] \end{array}\right] \qquad (4.1)$$
$$\Rightarrow [\ V_j(\alpha)-V_j(\beta) = V_j(\gamma)-V_j(\delta)\ ] \quad .$$

Setting $\alpha=\beta$, $x=y$, $v=w$, we have $[V_i(\gamma)=V_i(\delta)] \Rightarrow [V_j(\gamma)=V_j(\delta)]$. Hence there exists φ such that $V_j=\varphi\circ V_i$. Analogously there exists ψ such that $V_i=\psi\circ V_j$. Thus, $\varphi:V_i(\Gamma)\to V_j(\Gamma)$ is bijective. Since V_i and V_j are continuous, so is $\alpha\mapsto(V_i(\alpha),V_j(\alpha))$. Consequently the graph of φ, viz. $\{(V_i(\alpha),V_j(\alpha))\in\mathbb{R}^2: \alpha\in\Gamma\}$, is connected. This can be seen to imply that φ has the intermediate value property. Therefore φ, being bijective and having an interval as domain, must be continuous; it must be either increasing or decreasing.

Now let $V_i(\varsigma)$ be an arbitrary element of $V_i(\Gamma)$, the domain of φ. Recall that all k are essential; hence no V_k is constant, and all $V_k(\Gamma)$ are non-degenerate intervals. Since $n\geq 2$ and $V_j=\varphi\circ V_i$ with φ continuous, there exists an open interval S around $V_i(\varsigma)$ so small that for all $V_i(\alpha)$, $V_i(\beta)$ in S, there are x and y for which $V_i(\alpha)-V_i(\beta) = \sum_{k\neq i}[V_k(y_k)-V_k(x_k)]$ and there are v and w for which $[\varphi(V_i(\alpha))-\varphi(V_i(\beta))] = V_j(\alpha)-V_j(\beta) = \sum_{k\neq j}[V_k(w_k)-V_k(v_k)]$. Setting $\beta=\gamma$, and finding appropriate x,y,v,w, by (4.1) we get, for all $V_i(\alpha)$, $V_i(\beta)$, $V_i(\delta)$ in S:

$$[V_i(\alpha)-V_i(\beta) = V_i(\beta)-V_i(\delta)] \Rightarrow [V_j(\alpha)-V_j(\beta) = V_j(\beta)-V_j(\delta)].$$

Thus, on S, φ satisfies Jensen's equality: $\varphi((\sigma+\tau)/2) = [\varphi(\sigma)+\varphi(\tau)]/2$. By Theorem 1 of section 2.1.4 of Aczél (1966), or by (88) of section 3.7 of Hardy, Littlewood,&Pólya (1934), φ must be affine on S. Hence it has derivative zero at all $V_i(\varsigma)$. Consequently φ must be affine on $V_i(\Gamma)$.

We have shown that each V_j is an affine transform of V_1. The proof is completed by specifying U and the λ_j's as follows: For arbitrary fixed α and β in Γ with $V_1(\alpha)\neq V_1(\beta)$, set $U(.) := V_1(.)-V_1(\alpha)$ and set $\lambda_j := [V_j(\beta)-V_j(\alpha)]/[V_1(\beta)-V_1(\alpha)]$, $j=1,\ldots,n$. □

5. RESULTS FOR QUASILINEAR INDEXES

5.1. TRANSLATING RESULTS FOR PREFERENCE RELATIONS TO RESULTS FOR FUNCTIONS

We turn now to indicate how the theorems, formulated so far with a preference relation as primitive, can be reformulated in a simple way to give results when indexes are instead taken as primitives. Let $v:\Gamma\to\mathbb{R}$ be a continuous index. We may ascribe all properties, introduced before for binary relations, now to V by replacing all preferences such as $x \succeq y$ by the corresponding inequalities $V(x) \geq V(y)$, replacing all equivalences $x \approx y$ by equalities $V(x)=V(y)$, etc. For example:

DEFINITION 2.7-FOR-FUNCTIONS. The fuction V satisfies **cardinal coordinate independence** (CCI) if for all $i,j,x,\ldots,\delta$ we have :

$$\left[\begin{array}{ll} V(x_{-i}\alpha) \leq V(y_{-i}\beta) & \& \; V(v_{-j}\alpha) \geq V(w_{-j}\beta) \; \& \\ V(x_{-i}\gamma) \geq V(y_{-i}\delta) & \end{array}\right]$$
$$\Rightarrow \qquad [V(v_{-j}\gamma) \geq V(w_{-j}\delta)]$$

Compare this to Definition 2.7 of CCI for $\succeq$. We obtain new theorems, Theorem 2.6-for-functions, ..., Theorem 4.3-for-functions, from the corresponding theorems for binary relations, by replacing 'represents the binary relation $\succeq'$' by 'is a continuous increasing transform of V'', '$\succeq'$' by 'V', 'binary relation $\succeq'$' by 'function V', and 'weak order' by 'function'. These new theorems can be derived straightforwardly from the corresponding theorems for binary relations. For example:

THEOREM 2.8-FOR-FUNCTIONS. *Under Assumption 2.3 the following two statements are equivalent for the function* V *on* Γ^n*:*

(i) There exist positive $(p_j)_{j=1}^n$, summing to one, and a continuous function $U:\Gamma\to\mathbb{R}$, such that $x\mapsto\sum_{j=1}^n p_j U(x_j)$ is a continuous increasing transform of V.

(ii) The function V is a continuous CCI function.

□

When treating a continuous index V this way we use only the information of V contained in the preference relation represented by V. In other words, we are then studying the ordinal character of V. The major part of the present paper is written in this ordinal perspective. Such an ordinal approach is useful if one may, or wants to, use only ordinal information of indexes. If such ordinal information is combined with other information (for example the information that the index should be additive) then by the uniqueness results as in Debreu's Theorem 2.6 (see subsection 6.3) we may be able to determine the index up to a positive affine transformation.

5.2. QUASILINEAR INDEXES

Throughout this subsection we shall assume that $\Gamma\subset\mathbb{R}$ is an interval, and we shall study the properties of a continuous index $V:\Gamma^n\to\mathbb{R}$. The following definitions can be found in Hardy,Littlewood,&Pólya (1934), or Eichhorn (1978).

DEFINITIONS. Let $W:\Gamma^n\to\mathbb{R}$. Then W is:

(5.1) **generalized quasilinear** if there exist continuous strictly monotonic (i.e. increasing or decreasing) functions φ, $g_1,\ldots,g_n$ with appropriate domains, such that $W:x\mapsto\varphi(\sum_{j=1}^n g_j(x_j))$;

(5.2) **quasilinear** if there exist a real b, non-zero $\lambda_1,\ldots,\lambda_n$, and a continuous strictly monotonic U, such that $W:x\mapsto U^{-1}(\sum_{j=1}^n \lambda_j U(x_j)+b)$;

(5.3) a **quasilinear mean** if it is quasilinear with $b=0$, $\lambda_1>0,\ldots,\lambda_n>0$, $\sum_{j=1}^n\lambda_j=1$;

(5.4) a **quasiaddition** if it is quasilinear with $b=0$, $\lambda_1=\ldots=\lambda_n=1$.

(5.5) In (5.2) (see however the next sentence), (5.3) and (5.4) the adjective 'quasi' may be omitted if U is the identity function.

If in (5.2) U is the identity function, then in this paper the function W will be called **linear** only if $b=0$, and the term **affine** will be used for general b.

We shall show how the representation theorems of the previous sections can be used to characterize the ordinal character of the functions introduced above. We shall opt for 'additive' formulations. We might just as well have used multiplicative formulations. As an illustration, this is made explicit in (5.6.iv) below.

THEOREMS. *Let* $\Gamma\subset\mathbb{R}$ *be an interval, and* $V:\Gamma^n\to\mathbb{R}$. *Then:*

(5.6) *The following four statements are equivalent:*

(i) **V is generalized quasilinear.**

(ii) **V is continuous, strictly monotonic in each variable, and CI.**

(iii) **V is continuous, strictly monotonic in each variable, and ECI.**

(iv) **There exist continuous strictly monotonic functions ψ, $h_1,\ldots,h_n$ with appropriate domains, such that all h_j are positive and $V:x\mapsto\psi(\Pi_{j=1}^n h_j(x_j))$.**

(5.7) *The following three statements are equivalent:*

(i) **V is a continuous increasing transform of a quasilinear mean.**

(ii) **V is continuous, increasing in each variable, and CCI.**

(iii) **V is continuous, increasing in each variable, and ECCI.**

PROOFS. First note that the strict monotonicity of V in each variable implies weak separability of $\succeq$, and essentiality of all i. Further note that in Definition 5.1 one may always assume φ to be increasing. (For if φ is decreasing, we may replace $g_1,\ldots,g_n$ by $-g_1,\ldots,-g_n$, and φ by $\bar\varphi$ with $\bar\varphi:\mu\mapsto\varphi(-\mu)$.) Thus, in (5.6), (i)<=>(ii) follows from Theorem 2.6-for-functions, and (i)<=>(iii) from

Theorem 3.10-for-functions. Finally, (i)<=>(iv) in (5.6) follows by setting $h_j = \exp(g_j)$ for all j and $\psi = \varphi \circ \log$.

Next note that the function U in Definitions (5.2), (5.3) and (5.4) may be taken to be increasing. (For if U is decreasing we may replace U by -U and b by -b.) Thus, V in (5.7)(i) is increasing in each variable, and (i)=>(ii) and (i)=>(iii) in (5.7) follow. Now we suppose (ii) or (iii) in (5.7) holds, and we derive (i) there. By Theorem 2.8-for-functions or Theorem 4.3-for-functions, there exist real λ_j, j=1,...,n and a continuous function U such that $x \mapsto \sum_{j=1}^{n} \lambda_j U(x_j)$ is a continuous increasing transform of V. We may assume, because of the increasingness of V in each variable, that U is increasing and that all λ_j's are positive. (If not, replace U by -U, every λ_j by $-\lambda_j$.) Clearly, V is also a continuous increasing transform of $\sum_{j=1}^{n} p_j U(x_j)$ where $p_j = \lambda_j / (\sum_{j=1}^{n} \lambda_i)$ for all j. This gives $\sum_{j=1}^{n} p_j = 1$. The reason to replace $(\lambda_j)_{j=1}^{n}$ by $(p_j)_{j=1}^{n}$, is that $\sum_{j=1}^{n} p_j U(x_j)$ is in the (convex!) range of U for all x so that $U^{-1}[\sum_{j=1}^{n} p_j U(x_j)]$ is well defined. Now V must also be a continuous increasing transform of $U^{-1}[\sum_{j=1}^{n} p_j U(x_j)]$; thus (i) in (5.7) is established.

□

A complication in the proof of (ii)=>(i) and (iii)=>(i) in Theorem 5.7 above was that $U^{-1}[\sum_{j=1}^{n} \lambda_j U(x_j)]$ should be well defined. For characterizing quasilinear functions and quasiadditions, we shall need additional assumptions to solve the analogous complication.

DEFINITION 5.8. The function V satisfies <u>unrestricted solvability</u> if for all i,x,y there exist $\alpha, \beta \in \Gamma$ such that $V(x_{-i}\alpha) < y < V(x_{-i}\beta)$.

THEOREMS. *Let* $\Gamma \subset \mathbb{R}$ *be an interval, let* $V: \Gamma^n \to \mathbb{R}$. *Let either* Γ *be compact, or let* V *satisfy unrestricted solvability.*

(5.9). *The following two statements are equivalent:*

(i) **V is a continuous increasing transform of a quasilinear function.**

(ii) **V is continuous, stricly monotonic in each variable, and ECCI.**

(5.10). *The following three statements are equivalent:*

(i) **V is a continuous increasing transform of a quasiaddition.**

(ii) **V is continuous, increasing in each variable, CCI, and** $\alpha \neq \beta$ **exist such that** $V((\alpha,\dots,\alpha)_{-1}\beta) = V((\alpha,\dots,\alpha)_{-j}\beta)$ **for all j.**

(iii) **V is continuous, increasing in each variable, ECCI, and** $\alpha \neq \beta$ **exist such that, for all j,** $V((\alpha,\dots,\alpha)_{-1}\beta) = V((\alpha,\dots,\alpha)_{-j}\beta)$.

PROOFS. Since U in Definitions 5.2 and 5.4 can be made increasing (see previous proofs), (i)=>(ii) in (5.9) and (5.10), and (i)=>(iii) in (5.10) are immediate; these implications do not need the additional compactness or unrestricted solvability condition. Next we suppose (ii) holds in (5.9), or (ii) or (iii) in (5.10), and we derive (i). The beginning of our reasoning will apply both to 5.9 and 5.10. If n=1, we may simply set U=identity, $\lambda_1 = 1$ if V is increasing and $\lambda_1 = -1$ if V is decreasing. So from now on we suppose $n \geq 2$. By Theorem 4.3-for-functions, V is a continuous increasing transform of $x \mapsto \sum_{j=1}^{n} \lambda_j U(x_j)$, with the λ_j's real, and U continuous. We may assume, by the strict monotonicity of V, that U is actually increasing (otherwise replace U by -U, λ_j by $-\lambda_j$ for all j) and that no λ_j is zero. If V satisfies unrestricted solvability, then, n being ≥ 2, U can be seen to be unbounded from above and below. If Γ is compact, then we can extend U to a continuous increasing $U': \mathbb{R} \to \mathbb{R}$ that is unbounded from above and below. We then write U for U'. Anyway, now $U^{-1}[\sum_{j=1}^{n} \lambda_j U(x_j)]$ is always defined. And V is a continuous increasing transform of this; thus (i) in (5.9) follows.

Finally suppose (ii) or (iii) in (5.10) holds. We derive (i). V is a continuous increasing transform of $x \mapsto [\sum_{j=1}^{n} \lambda_j U(x_j)]$ where U is increasing and unbounded from above and below as we saw above. By the strict increasingness of V in each variable, all the λ_j's are positive. The 'α,β-condition' in (ii) and (iii) implies that $\lambda_j = \lambda_1$ for all j. So V is an increasing transform of $\sum_{j=1}^{n} U(x_j)$, hence of the well-defined $U^{-1}[\sum_{j=1}^{n} \lambda_j U(x_j)]$.

□

The following example demonstrates why in Theorem 5.9 we needed an extra assumption for the implication (ii)=>(i).

EXAMPLE 5.11. Let $V(x) = \exp(x_1) - \exp(x_2)$ for all $x \in \mathbb{R}^2$. If, for some $b, \lambda_1, \lambda_2, U$, this V is a increasing transform of $x \mapsto \sum_{j=1}^{2} \lambda_j U(x_j) + b$, then, as is easily verified, $\lambda_1 = -\lambda_2$ and there exist σ, τ such that $U = \tau + \sigma\exp$ with $\lambda_1 \sigma > 0$. Suppose $\lambda_1 > 0$. (The case of $\lambda_1 < 0$ is analogous, instead of $(0, x_2)$ take $(x_1, 0)$ with x_1 large below.) Then $\sigma > 0$, and U is bounded below, unbounded above. Further there exists x_2 so large that for $x = (0, x_2)$ the value $\sum_{j=1}^{2} \lambda_j U(x_j) + b$ is too negative to be in the range of U. Thus the inverse U^{-1} of $\sum_{j=1}^{2} \lambda_j U(x_j) + b$ cannot be defined. That is, although (ii) in Theorem 5.9 is satisfied, (i) is not.

THEOREM 5.12. *Let* $\Gamma \subset \mathbb{R}$ *be an interval. A necessary and sufficient condition for the function U in Theorems 2.8-for-functions and 4.3-for-functions to be linear, and for the omission of 'quasi' in (i) in (5.7), (5.9), (5.10), is that for all (or one) coordinates i, and for all* $\alpha, \beta, \alpha+\varepsilon, \beta+\varepsilon$ *in* Γ, *and all x,y, the implication* $[V(x_{-i}\alpha) = V(y_{-i}(\alpha+\varepsilon))] \Rightarrow [V(x_{-i}\beta) = V(y_{-i}(\beta+\varepsilon))]$ *holds.*

PROOF. Obvious for n=1. For n≥2 the necessary and sufficient condition can be seen to be equivalent to the requirement that U 'locally' satisfies $U(\alpha+\varepsilon)-U(\alpha) = U(\beta+\varepsilon)-U(\beta)$ for all α,β,ε. Since U is continuous on a convex domain, this means that U must be affine. Therefore U (and also its extension in the proof of Theorems 5.9 and 5.10) may furthermore be taken to be linear.

□

REMARK 5.13. In Theorem 5.7 we may replace the term 'quasilinear' by 'quasiarithmetic' (i.e. the λ_j's of Definition 5.3 all equal 1/n) if in (ii) and (iii) we add the same 'α,β condition' as in (5.10.ii). This characterizes (5.10(ii)) and (5.10(iii)) without compactness or unrestricted solvability restriction.

6. FURTHER RESULTS AND FUTURE RESEARCH

6.1. SPECIFIED FORMS OF QUASILINEAR INDEXES

Several indexes that have attracted interest in literature are specified forms of quasilinear indexes. We can use Theorem 5.9 or 4.3-for-functions, add to (ii) there one extra condition which is weaker than those customary in the literature, and obtain from (i) the desired specified form. For instance, if n=2, then the case where V is a continuous increasing transform of $x \mapsto U(x_1)-U(x_2)$, interpreted as an index for 'intensity of preference' of x_1 over x_2, can be characterized by adding an extra condition such as $V(\alpha,\alpha)=V(\beta,\beta)$ for all α,β (or even just for some α and β with $V(\alpha,\beta) \neq V(\beta,\beta)$). The case where V is a continuous increasing transform of $x \mapsto \sum_{j=1}^{n} \lambda^j U(x_j)$, with $0<\lambda\leq1$, is useful in dynamic utility theory (see Koopmans, 1972), where λ is the 'discount factor'. We can characterize such a V by an additional 'weak stationarity' assumption such as: there exist α,β,γ with $V(\alpha,\ldots,\alpha) \geq V(\beta,\ldots,\beta) > V(\gamma,\ldots,\gamma)$, and $V((\gamma,\ldots,\gamma)_{-1}\beta) = V((\gamma,\ldots,\gamma)_{-(i+1)}\alpha)$ for all $i \leq n-1$.

6.2. FUTURE RESEARCH

It is a large task, not taken up in this paper, to catalogue the indexes, used in literature, which are specified forms of quasilinear functions, and to determine which non-ordinal conditions should be added in the statements, numbered (ii) or (iii) in this paper, to characterize these specified forms. Results along this line may be obtained from Aczél&Alsina (1984), Stehling (1975), Eichhorn (1978, in particular Chapter 2).

We conjecture that our 'equivalence-separability condition' ECI in Theorem 3.10 can be weakened in the spirit of Gorman (1968). He showed that coordinate independence in Theorem 2.6 of Debreu (allowing the replacement in preferences of any one pair of equal coordinates, and thus by repetition of any arbitrary number of pairs of equal coordinates), can be weakened to allow for replacement of only certain subgroups of equal coordinates; see also Murphy (1981).

Probably in Theorem 5.10 the assumption of compactness of Γ or unrestricted solvability of V could have been omitted. The demonstration of that seems to be tedious.

6.3. UNIQUENESS; ADAPTING OUR RESULTS TO NON-ORDINAL CONTEXTS

We have not formulated uniqueness results for the several functions that we derived. These follow straightforwardly from the uniqueness result obtained by Debreu (1960) and Krantz et al. (1971), who showed that the additive representation V of Theorem 2.6 is cardinal, i.e. it is unique up to a positive affine transformation.

If one wants to specify V further than (as in most of our results) unique up to a continuous increasing transformation, then our results may still be useful. This is because weak 'non-ordinal' conditions will suffice, in addition to the ordinal conditions, to give non-ordinal results. A condition such as $[V(x_{-i}\alpha)+V(y_{-i}\beta) = V(x_{-i}\beta)+V(y_{-i}\alpha)]$ will enable one to replace 'V is a continuous increasing transform of' in the theorems above by 'V is', or 'V is a positive affine transform of'. Multiplicative forms, with positive factors, are characterized by conditions such as $[V(x_{-i}\alpha) \times V(y_{-i}\beta) = [V(x_{-i}\beta) \times V(y_{-i}\alpha)]$.

Analogously one may replace ECCI by

$$[V(x_{-i}\alpha) - V(y_{-i}\beta) = V(x_{-i}\gamma) - V(y_{-i}\delta)] \Rightarrow$$
$$[V(v_{-j}\alpha) - V(w_{-j}\beta) = V(v_{-j}\gamma) - V(w_{-j}\delta)]$$

to derive cardinal additive indexes, or, for multiplicative indexes, by the condition

$$[V(x_{-i}\alpha)/V(y_{-i}\beta) = [V(x_{-i}\gamma)/V(y_{-i}\delta)] \Rightarrow$$
$$[V(v_{-j}\alpha)/V(w_{-j}\beta) = [V(v_{-j}\gamma)/V(w_{-j}\delta)] .$$

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