SUBJECTIVE PROBABILITIES FOR STATE DEPENDENT CONTINUOUS UTILITY*

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For the expected utility model with state dependent utilities, Karni, Schmeidler and Vind (1983) have shown how to recover uniquely the involved subjective probabilities if the preferences, contingent on a hypothetical probability distribution over the state space, are known. This they do for consequence spaces, consisting of lotteries on sets of prizes. We adapt their work to consequence spaces that are connected topological spaces, without using lotteries on them. E.g. our consequences may be money, or commodity bundles.

Key words: Subjective probability; state dependent expected utility; additive representation.

1. Introduction

One of the restrictions for the applicability of the usual expected utility model is that in many cases the consequences are thus state-specific, that the usual state independence is inconceivable. Already Ramsey (1931), the first derivation of subjective expected utility, indicated that events may very well *not* be ethically neutral, which comes down to the same thing as state dependence. Some references, examples, and applications of state dependent expected utility are Eisner and Strotz (1961), Section IV in Yaari (1969), Arrow (1974), Cook and Graham (1977). For further references, and many results, see Karni (1985).

If state dependent utilities are permitted, then, without further information, the factors probability and utility can no longer be separated in a unique way. Any change in probabilities, not affecting the positivity of them, can be handled by a multiplication of utilities with appropriate factors. In Karni, Schmeidler and Vind (1983), hereafter abbreviated KSV, it is indicated how, with the further information of the preferences contingent on a hypothetical probability distribution, the factors probability and utility can be separated in a unique way, under a 'consistency

axiom'. Thus KSV obtain a derivation of expected utility *with* state dependent utility. This result has been derived for the case where consequences are lotteries on a finite set. It can without any problem be extended to mixture spaces (for a definition see Herstein and Milnor, 1953) as consequence spaces, as long as utility is linear. So their approach not only handles the case where consequences are lotteries, but also the case where consequences are commodity bundles (or money) with linear utility.

In this paper we adapt the work of KSV to the case where the consequence spaces are connected topological spaces, and utility only has to be continuous. Thus we handle for instance the case where consequences are commodity bundles (or money), and utility is not (necessarily) linear. The price for this generalization is that we must replace 'weak consistency', the appealing characterizing condition of KSV, by 'cardinal consistency', a more complicated condition. The latter condition is still necessary (and sufficient) though, and in the KSV set-up quickly can be derived from linearity and ordinal consistency (see Proposition 4.2).

2. Definitions, notations, and preliminary results

Let $S = \{s_1, \ldots, s_n\}$ be a finite *state space*. Let for every s_j be given a *consequence space* $\mathcal{C}_j$. $L = X_{j=1}^n \mathcal{C}_j$ is the *set of acts*. Act $x = (x_1, \ldots, x_n) \in L$ assigns x_j to s_j, $j = 1, \ldots, n$. The *(actual) preference relation* $\succcurlyeq$ of a decision maker is a binary relation on L. As usual we write $x \preccurlyeq y$ if $y \succcurlyeq x$, $x \succ y$ if $x \succcurlyeq y$ and not $y \succcurlyeq x$, $x \prec y$ if $y \succ x$, and $x \approx y$ if both $x \succcurlyeq y$ and $y \succcurlyeq x$. In Section 3 we shall furthermore introduce a *hypothetical preference relation* $\hat{\succcurlyeq}$ on L, from which $\hat{\prec}, \hat{\succ}, \hat{\prec}, \hat{\approx}$ are derived analogously. We say $\succcurlyeq$ is a *weak order* if it is transitive and complete (i.e. $x \succcurlyeq y$ or $y \succcurlyeq x$ for all x, y). Then it also is reflexive, and $\approx$ is an equivalence relation.

Notation 2.1. *For $x \in L$, $y_j \in \mathcal{C}_j$, $x_{-j}y_j$ is (x with x_j replaced by y_j).*

Definition 2.2. Coordinate j is *essential* w.r.t. $\succcurlyeq$ if there exist $x \in L$, $v_j \in \mathcal{C}_j$ s.t. $x \succ x_{-j}v_j$. Otherwise it is *inessential*.

KSV dealt with the following assumption, where a simple lottery on a set X is a probability measure on $(X, 2^X)$, assigning probability 1 to a finite subset of X (they in fact took X finite):

Assumption 2.3. (Lottery Assumption). *For every j, the set $\mathcal{C}_j$ is the set of simple lotteries on a non-empty set X_j of 'prizes'.*

Obviously L is endowed with a 'mixture operation', assigning, to every $\alpha \in [0, 1]$, and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in L, the simple lottery $\alpha x + (1 - \alpha)y = (\alpha x_1 + (1 - \alpha)y_1, \ldots, \alpha x_n + (1 - \alpha)y_n)$. Under the Lottery Assumption, we call $\succcurlyeq$ *continuous* if, for every x, y, z in L with $x \succ y \succ z$, there exist α and β in $(0, 1)$ such

that $\alpha z + (1 - \alpha)x > y > \beta x + (1 - \beta)z$; we call $\succcurlyeq$ *independent* if, for every x, y, z in L and α in $(0, 1)$, $x > y$ implies $\alpha x + (1 - \alpha)z > \alpha y + (1 - \alpha)z$. A function $V_j : \mathscr{C}_j \rightarrow \mathbb{R}$ is *affine* if $V_j(\alpha x_j + (1 - \alpha)y_j) = \alpha V_j(x_j) + (1 - \alpha)V_j(y_j)$, for all x_j, y_j in $\mathscr{C}_j$, and all $\alpha \in [0, 1]$.

Theorem 2.4. *Under the Lottery Assumption 2.3, the following two statements are equivalent:*

Statement 2.4.i: 'There exist affine $V_j : \mathscr{C}_j \rightarrow \mathbb{R}$, $j = 1, \ldots, n$, *s.t.:*

$$\left[x \succcurlyeq y \Leftrightarrow \sum_{j=1}^{n} V_j(x_j) \geq \sum_{j=1}^{n} V_j(y_j) \right] \quad \textit{for all } x, y \in L'.$$

Statement 2.4.ii: '$\succcurlyeq$ is a continuous independent weak order'.

Furthermore, if Statement 2.4.i holds, then $(V_j)_{j=1}^{n}$ can be replaced by $(V_j')_{j=1}^{n}$ if and only if there exist real $(\beta_j)_{j=1}^{n}$, and positive α, s.t. $V_j' = \beta_j + \alpha V_j$ for all j. $\square$

We shall adapt the work of KSV to the case where, instead of the Lottery Assumption, we have:

Assumption 2.5. (Topological Assumption). *For every j, $\mathscr{C}_j$ is a connected topological space. L is endowed with the product topology.*

In Kelley (1955) the reader can find definitions and basic results of topology. A reader, not interested in general topology, may simply assume that every $\mathscr{C}_j$ is a convex subset of a linear space, e.g. $\mathscr{C}_j = \mathbb{R}_+^m$, and consequences are commodity bundels, or $\mathscr{C} = \mathbb{R}$, and consequences are amounts of money. Under the Topological Assumption the preference relation $\succcurlyeq$ is *continous* if $\{x \in L : x \succcurlyeq y\}$ and $\{x \in L : x \preccurlyeq y\}$ are closed, for all $y \in L$.

The following property is a central tool under the Topological Assumption.

Definition 2.6. $\succcurlyeq$ is *coordinate independent* (CI) if

$$x_{-j}v_j \succcurlyeq y_{-j}v_j \Leftrightarrow x_{-j}w_j \succcurlyeq y_{-j}w_j \quad \text{for all } j, x, y, v_j, w_j.$$

CI means that the preference between x and y is independent of those coordinates where x and y are equal. It is known under various names as '(strong) separability', (part of the) 'sure-thing principle', '(preferential) independence'. Note that CI is implied by Statement 2.4.i, thus by Statement 2.4.ii as well. Under CI, the following definition of preference relations $\succcurlyeq_j$ on $\mathscr{C}_j$ is of use:

Definition 2.7. For every j, v_j and $w_j \in \mathscr{C}_j$ we write $v_j \succcurlyeq_j w_j$ if there exists $x \in L$ s.t. $x_{-j}v_j \succcurlyeq x_{-j}w_j$.

Under CI we have $v_j \succcurlyeq_j w_j \Leftrightarrow [x_{-j}v_j \succcurlyeq x_{-j}w_j$ for *all* $x \in L]$, and every $\succcurlyeq_j$ is a weak

order if $\succeq$ is; furthermore continous if $\succeq$ is. The following result, a slight strengthening of Theorem 3 in Debreu (1960), and of Theorem 14 in Section 6.11.1 of Krantz et al. (1971), is proved in Wakker (1986, Theorem 4.1).

Theorem 2.8. *Let the Topological Assumption 2.5 hold. Let at least three coordinates be essential w.r.t.* $\succeq$. *Then the following two statements are equivalent:*

Statement 2.8.i: '*There exist continuous* $V_j : \mathscr{C}_j \to \mathbb{R}$, $j = 1, \ldots, n$, *s.t.*

$$\left[x \succeq y \Leftrightarrow \sum_{j=1}^{n} V_j(x_j) \geq \sum_{j=1}^{n} V_j(y_j) \right] \quad \text{for all } x, y \in L\text{'.}$$

Statement 2.8.ii: '$\succeq$ *is a continous CI weak order'.*

Furthermore, if Statement 2.8.i applies, then $(V_j)_{j=1}^{n}$ *can be replaced by* $(V_j')_{j=1}^{n}$ *if and only if there exist real* $(\beta_j)_{j=1}^{n}$, *and positive* α, *s.t.* $V_j' = \beta_j + \alpha V_j$ *for all* j.

Theorems 2.4 and 2.8 can be interpreted as characterizations of state dependent expected utility. There can always be thought to exist 'subjective probabilities' $(p_j)_{j=1}^{n}$ and 'state dependent utility functions' $(U_j)_{j=1}^{n}$, s.t. $V_j = p_j U_j$ for all j. Note however that from merely the preference relation (all information of which is contained in $(V_j)_{j=1}^{n}$) one can never uncover uniquely what the p_j's and U_j's are. For that, further information is needed.

3. Auxiliary hypothetical probabilities

An example of further information, as mentioned above, is provided by KSV. Let $(\hat{p}_j)_{j=1}^{n}$, strictly positive, summing to one, be a hypothetical, to the decision maker conceivable, probability distribution on S. We suppose that we know $\hat{\succeq}$, the preference relation that the decision maker would have on L if his subjective probabilities *would be* equal to $(\hat{p}_j)_{j=1}^{n}$. And we are interested in the case where there exist $(p_j)_{j=1}^{n}$, $(U_j)_{j=1}^{n}$ s.t. both $x \succeq y \Leftrightarrow \sum_{j=1}^{n} p_j U_j(x_j) \geq \sum_{j=1}^{n} p_j U_j(y_j)$, and $x \hat{\succeq} y \Leftrightarrow \sum_{j=1}^{n} \hat{p}_j U_j(x_j) \geq \sum_{j=1}^{n} \hat{p}_j U_j(y_j)$, for all x, y. In such a case the decision maker could be called *consistent*. Under the hypothetical probabilities his 'tastes', quantified by $(U_j)_{j=1}^{n}$, have remained unaffected. Before continuing, for the sake of easy reference, we write out the analogue of Theorem 2.8 for $\hat{\succeq}$ instead of $\succeq$.

Theorem 3.1. *Let the Topological Assumption 2.5 hold. Let at least three coordinates be essential w.r.t.* $\hat{\succeq}$. *Then the following two statements are equivalent:*

Statement 3.1.i: '*There exist continuous* $\hat{V}_j : \mathscr{C}_j \to \mathbb{R}$, $j = 1, \ldots, n$, *s.t.*

$$\left[x \hat{\succeq} y \Leftrightarrow \sum_{j=1}^{n} \hat{V}_j(x_j) \geq \sum_{j=1}^{n} \hat{V}_j(y_j) \right] \quad \text{for all } x, y \in L\text{'.}$$

Statement 3.1.ii: '$\hat{\succeq}$ *is a continuous CI weak order'.* $\square$

In KSV the following condition ('Consistency axiom'; reformulated for our con-

text) was necessary and sufficient for consistency. Note for it that, by positivity of the $\hat{p}_j$'s, essentiality of j w.r.t. $\succcurlyeq$ under consistency implies essentiality of j w.r.t. $\hat{\succcurlyeq}$, but not the other way around. Some p_j's may be zero.

Definition 3.2. $\succcurlyeq$ is *ordinally consistent* with $\hat{\succcurlyeq}$ if the following two conditions hold:

  Condition 3.2.a: For every j, $\succcurlyeq_j \supset \hat{\succcurlyeq}_j$.
  Condition 3.2.b: For every j, essential w.r.t. $\succcurlyeq$, $\hat{\succcurlyeq}_j \supset \succcurlyeq_j$.

Obviously ordinal consistency holds if and only if $\succcurlyeq_j = \hat{\succcurlyeq}_j$ for all j essential w.r.t. $\succcurlyeq$. The following theorem gives the major part of the 'Main Theorem' of KSV; we omit their considerations concerning 'null states'.

Theorem 3.3. *Let the Lottery Assumption 2.3 hold. Let $\succ$ be non-empty. Let $\succcurlyeq$ and $\hat{\succcurlyeq}$ be continous independent weak orders. Let $\hat{p}_1, \ldots, \hat{p}_n$ be positive and sum to one. The following two statements are equivalent:*

  Statement 3.3.i: *'There exist affine $U_j : \mathscr{C}_j \to \mathbb{R}$ and $p_j \geq 0, j = 1, \ldots, n$, with $\sum p_j = 1$, s.t. $x \succcurlyeq y \Leftrightarrow \sum p_j U_j(x_j) \geq \sum p_j U_j(y_j)$ and $x \hat{\succcurlyeq} y \Leftrightarrow \sum \hat{p}_j U_j(x_j) \geq \sum \hat{p}_j U_j(y_j)$ for all x, y in L'.*

  Statement 3.3.ii: *'$\succcurlyeq$ is ordinally consistent with $\hat{\succcurlyeq}$'.*

Furthermore, the ratio of p_k and p_l in Statement 3.3.i is uniquely determined for all k, l, essential w.r.t. $\succcurlyeq$. $\square$

Obviously Statement 3.3.i implies Statement 2.4.i, with $V_j := p_j U_j$ for all j. The following Proposition shows that, under the Topological Assumption, ordinal consistency is too weak a requirement to imply the existence of the probabilities $p_1, \ldots, p_n$ as in Statement 3.3.i.

Proposition 3.4. *Let the Topological Assumption 2.5 hold. Let Statements 2.8.i and 3.1.i hold. Then are equivalent:*

  Statement 3.4.i: *'There exist continuous $\phi_j : \hat{V}_j(\mathscr{C}_j) \to \mathbb{R}$ s.t. $V_j = \phi_j \circ \hat{V}_j$, $j = 1, \ldots, n$, where every ϕ_j is strictly increasing or constant'.*

  Statement 3.4.ii: *'$\succcurlyeq$ is ordinally consistent with $\hat{\succcurlyeq}$'.*

Proof. If j is inessential w.r.t. $\succcurlyeq$, then V_j is constant, and everything follows. Next let j be essential w.r.t. $\succcurlyeq$. V_j represents $\succcurlyeq_j$, so is not constant. Thus under (i) ϕ_j is strictly increasing, and $\succcurlyeq_j = \hat{\succcurlyeq}_j$ follows. Under (ii), V_j and $\hat{V}_j$ represent the same $\succcurlyeq_j = \hat{\succcurlyeq}_j$, hence $V_j = \phi_j \circ \hat{V}_j$ for some strictly increasing ϕ_j. Since $V_j(\mathscr{C}_j)$ and $\hat{V}_j(\mathscr{C}_j)$ are connected, and the strictly increasing ϕ_j is onto $V_j(\mathscr{C}_j)$, ϕ_j cannot make 'jumps'. It must be continuous. $\square$

Under the Lottery Assumption and the conditions of the 'KSV Theorem' 2.4, one deals with 'vNM-utility functions', i.e. the V_j's and $\hat{V}_j$'s are affine. Then the ϕ_j's

as above also will be affine, and the probabilities can be derived. In Statement 3.4.i, the ϕ_j's may very well be non-affine, e.g. $\phi_j : \mu \mapsto \exp(\mu)$. Hence a stronger condition than ordinal consistency seems needed for the topological context. We shall first show that in the KSV-result a slight weakening of ordinal consistency would already have sufficed to obtain the desired result. The 'cardinal consistency condition', to be introduced in the next section, will be a strengthening of this weakening.

Lemma 3.5. *In Statement* 3.3.ii, *ordinal consistency may be replaced by the Condition* 3.2.a.

Proof. We show that, under the assumptions of Theorem 3.3, Condition 3.2.a implies Condition 3.2.b. So let for every i, $\succsim_i \supset \succsim_i$. And let j be essential w.r.t. $\succsim$, say $v_j \succ_j w_j$. Let further, for some $x_j, y_j, x_j \succsim_j y_j$. To show is that $x_j \succsim_j y_j$.

For any $a \in L$, we have $a_{-j}v_j > a_{-j}w_j$ and $a_{-j}x_j \succsim a_{-j}y_j$, hence $a_{-j}(\alpha v_j + (1 - \alpha)x_j) >$ $a_{-j}(\alpha w_j + (1 - \alpha)y_j)$ for every $\alpha \in (0, 1]$. So $\alpha v_j + (1 - \alpha)x_j \succsim_j \alpha w_j + (1 - \alpha)y_j$. By Condition 3.2.a. $\alpha v_j + (1 - \alpha)x_j \succsim_j \alpha w_j + (1 - \alpha)y_j$, for every $\alpha \in (0, 1]$. Thus $a_{-j}(\alpha v_j + (1 - \alpha)x_j) \succsim a_{-j}(\alpha w_j + (1 - \alpha)y_j)$. From continuity of $\succsim$ it can be derived that $a_{-j}x_j \succsim a_{-j}y_j$. It follows that $x_j \succsim_j y_j$. ⊔⊓

The idea in the above Lemma, to derive (3.2.b) from one-sided monotonicity as (3.2.a), has been inferred from the Proof of Theorem 1 in Peters (1985).

4. Cardinal consistency

The condition that in our set-up is the necessary and sufficient criterion for verification/falsification of consistency is the following:

Definition 4.1. $\succsim$ is *cardinally consistent* (CC) with $\succsim$ if $[x_{-i}a_i \succsim x_{-i}b_i, x_{-i}c_i \succsim y_{-i}d_i,$ and $v_{-i}a_i \succsim w_{-i}b_i]$ implies $[v_{-i}c_i \succsim w_{-i}d_i]$ for all $i, x, \dots, d_i$.

The idea is to conclude from the first two $\succsim$ preferences that, intuitively speaking, the intensity of preference of c_i over d_i is at least as large as that of a_i over b_i, for as far as $\succsim$ is concerned. Were now $v_{-i}c_i < w_{-i}d_i$, then a same reasoning would give an opposite result for $\succsim$. And that is forbidden by CC. The idea, leading to CC, can be recognized in the 'triple cancellation' condition in Krantz et al. (1971), the 'corresponding tradeoffs condition' in Keeney and Raiffa (1976), the 'Reidemeister condition' in Blaschke and Bol (1938), and in 'cardinal coordinate independence' of Wakker (1984).

Lemma 4.3.i will show that indeed cardinal consistency strengthens Condition 3.2.a of ordinal consistency. First we show that, under the Lottery Assumption, cardinal consistency is implied by ordinal consistency, in the context of the KSV-Theorem 2.4.

Proposition 4.2. *Let the Lottery Assumption 2.3 hold, and let Statement 2.4.ii hold. Let Condition (3.2.a) of ordinal consistency hold. Then* $\succ$ *is CC with* $\hat{\succ}$.

Proof. Say we had $x_{-i}a_i \hat{\precsim} y_{-i}b_i$, $x_{-i}c_i \hat{\succ} y_{-i}d_i$, $v_{-i}a_i \succ w_{-i}b_i$, and $v_{-i}c_i \prec w_{-i}d_i$. The first two preferences imply $(\frac{1}{2}x + \frac{1}{2}y)_{-i}(\frac{1}{2}a_i + \frac{1}{2}d_i) \hat{\precsim} (\frac{1}{2}x + \frac{1}{2}y)_{-i}(\frac{1}{2}b_i + \frac{1}{2}c_i)$, i.e. $\frac{1}{2}a_i + \frac{1}{2}d_i \hat{\precsim}_i \frac{1}{2}b_i + \frac{1}{2}c_i$. The third and fourth preferences in this proof imply

$$(\tfrac{1}{2}v + \tfrac{1}{2}w)_{-i}(\tfrac{1}{2}a_i + \tfrac{1}{2}d_i) \succ (\tfrac{1}{2}v + \tfrac{1}{2}w)_{-i}(\tfrac{1}{2}b_i + \tfrac{1}{2}c_i),$$

i.e. $\frac{1}{2}a_i + \frac{1}{2}d_i \succ_i \frac{1}{2}b_i + \frac{1}{2}c_i$. This and the $\hat{\precsim}_i$-preference above give violation of Condition 3.2.a in ordinal consistency. $\square$

Lemma 4.3. *Let* $\succ$ *and* $\hat{\succ}$ *be weak orders. let* $\succ$ *be CC with* $\hat{\succ}$. *Then follows:*
   (i) '$\hat{\succ}_j \subset \succ_j$ *for all j*'.
   (ii) '*If j is essential w.r.t.* $\succ$, *it is w.r.t.* $\hat{\succ}$'.
   (iii) '*If the Topological Assumption 2.5, and Statements 2.8.i and 3.1.i hold, then continuous non-decreasing* $\phi_j : \hat{V}_j(\mathscr{C}_j) \to \mathbb{R}$ *exist s.t.* $V_j = \phi_j \circ \hat{V}_j$, $j = 1, \dots, n$'.

Proof. Let $x_{-j}v_j \hat{\succ} x_{-j}w_j$. By $x_{-j}v_j \hat{\precsim} x_{-j}v_j$, $x_{-j}v_j \hat{\succ} x_{-j}w_j$, $x_{-j}v_j \hat{\succ} x_{-j}v_j$ and CC we get $x_{-j}v_j \succ x_{-j}w_j$. This implies (i) and (ii). For (iii), first suppose $\hat{V}_j(x_j) = \hat{V}_j(y_j)$. Then $x_j \hat{\approx}_j y_j$, by twofold application of (i) we get $x_j \approx_j y_j$. Thus $\hat{V}_j(x_j) = \hat{V}_j(y_j)$ implies $V_j(x_j) = V_j(y_j)$. There must exist ϕ_j s.t. $V_j = \phi_j \circ \hat{V}_j$. Nondecreasingness of ϕ_j is by (i), continuity is as in Proposition 3.4. $\square$

We shall need the following result from elementary analysis.

Lemma 4.4. *Let* $A \subset \mathbb{R}$ *be convex. Let* $\phi : A \to \mathbb{R}$ *be continuous. Let* $\varepsilon > 0$ *and* $\zeta > 0$ *exist s.t., for all* $\alpha, \beta, \gamma, \delta$ *in A with* $|\alpha - \beta| \le \varepsilon$ *and* $|\phi(\alpha) - \phi(\beta)| \le \zeta$, $[\alpha - \beta = \gamma - \delta]$ *implies* $[\phi(\alpha) - \phi(\beta) = \phi(\gamma) - \phi(\delta)]$. *Then* ϕ *is affine.*

Proof. We show ϕ has a second derivative 0 everywhere. Let $\mu \in A$. Let $\nu > 0$ be so small that $\nu \le \varepsilon/2$, and for all σ, μ in A, $|\sigma - \mu| \le \nu \Rightarrow |\phi(\sigma) - \phi(\mu)| \le \zeta/2$. Thus, with $\beta = \gamma = \alpha/2 + \delta/2$ above, we get $\phi(\alpha/2 + \delta/2) = \phi(\alpha)/2 + \phi(\delta)/2$ for all α, δ in $A \cap [\mu - \nu, \mu + \nu]$. The continuous ϕ must be affine on $A \cap [\mu - \nu, \mu + \nu]$. So second derivative exists in μ, and equals 0. $\square$

The following proposition captures the meaning of CC, without relating it yet to the hypothetical probabilities $(\hat{p}_j)_{j=1}^n$.

Proposition 4.5. *Let the Topological Assumption 2.5 hold. Let Statements 2.8.i and 3.1.i hold. Let two or more coordinates be essential w.r.t.* $\succ$. *Then are equivalent:*

   (i) '*There exist positive affine or constant* $\phi_j : \hat{V}_j(\mathscr{C}_j) \to \mathbb{R}$ *s.t.* $V_j = \phi_j \circ \hat{V}_j$, *for all j*'.
   (ii) '$\succ$ *is cardinally consistent with* $\hat{\succ}$'.

Proof. First (i)⇒(ii). Let $x_{-j}a_j \gtrsim y_{-j}b_j$, $x_{-j}c_j \gtrsim y_{-j}d_j$, $v_{-j}a_j \gtrsim w_{-j}b_j$. Substituting $\hat{V}_j$'s gives

$$\hat{V}_j(a_j) - \hat{V}_j(b_j) \le \sum_{i \ne j} [\hat{V}_i(y_i) - \hat{V}_i(x_i)] \le \hat{V}_j(c_j) - \hat{V}_j(d_j).$$

Thus

$$V_j(c_j) - V_j(d_j) \ge V_j(a_j) - V_j(b_j) \ge \sum_{i \ne j} [V_i(w_i) - V_i(v_i)],$$

the first inequality by constantness or positive affinity of ϕ_j, the second since $v_{-j}a_j \gtrsim w_{-j}b_j$. Now $v_{-j}c_j \gtrsim w_{-j}d_j$ follows.

Next (ii)⇒(i). Let $j \le n$ be arbitrary. Since there is another coordinate than j that is essential w.r.t. $\gtrsim$, and thus by Lemma 4.3.ii w.r.t. $\hat{\gtrsim}$, there exist x, y, v, w s.t.

$$\sum_{i \ne j} [\hat{V}_i(x_i) - \hat{V}_i(y_i)] =: \varepsilon > 0, \qquad \sum_{i \ne j} [V_i(v_i) - V_i(w_i)] =: \zeta > 0.$$

By continuity of $x \mapsto \sum_{i \ne j} \hat{V}_i(x_i)$ and of $x \mapsto \sum_{i \ne j} V_i(x_i)$, and connectedness of L, for every $-\varepsilon \le \bar{\varepsilon} \le \varepsilon$ and $-\zeta \le \bar{\zeta} \le \zeta$, $\bar{x}, \bar{y}, \bar{v}, \bar{w}$ exist to give

$$\sum_{i \ne j} [\hat{V}_i(\bar{x}_i) - \hat{V}_i(\bar{y}_i)] = \bar{\varepsilon}, \qquad \sum_{i \ne j} [V_i(\bar{v}_i) - V_i(\bar{w}_i)] = \bar{\zeta}.$$

Consider now the ϕ_j of Lemma 4.3.iii. Let in its convex domain $A := \hat{V}_j(\mathcal{C}_j)$ be given $\alpha, \beta, \gamma, \delta$ s.t. $-\varepsilon \le \alpha - \beta = \gamma - \delta \le \varepsilon$, and s.t. $-\zeta \le \phi_j(\alpha) - \phi_j(\beta) \le \zeta$. Let a_j, b_j, c_j, d_j be s.t. $\hat{V}_j(a_j) = \alpha$, $\hat{V}_j(b_j) = \beta$, $\hat{V}_j(c_j) = \gamma$, $\hat{V}_j(d_j) = \delta$. Take $\bar{x}, \bar{y}$ s.t. $\sum_{i \ne j} [\hat{V}_i(\bar{y}_i) - \hat{V}_i(\bar{x}_i)] = \alpha - \beta$. Then $\bar{x}_{-j}a_j \hat{\approx} \bar{y}_{-j}b_j$, $\bar{x}_{-j}c_j \hat{\approx} \bar{y}_{-j}d_j$. Take $\bar{v}, \bar{w}$ s.t. $\sum_{i \ne j} [V_i(\bar{v}_i) - V_i(\bar{w}_i)] = \phi_j(\beta) - \phi_j(\alpha)$. Then, since $V_j = \phi_j \cdot \hat{V}_j$, $\bar{v}_{-j}a_j \approx \bar{w}_{-j}b_j$. Twofold application of CC yields $\bar{v}_{-j}c_j \approx \bar{w}_{-j}d_j$, i.e. $\phi_j(\gamma) - \phi_j(\delta) = \sum_{i \ne j} [V_i(\bar{w}_i) - V_i(\bar{v}_i)] = \phi_j(\alpha) - \phi_j(\beta)$. Lemma 4.4 shows that ϕ_j is affine. From Lemma 4.3.i it can be derived that ϕ_j is non-decreasing, so either constant or positive affine. □

5. Main result and conclusions

Theorem 5.1. *Let* $n \in \mathbb{N}$, $(\mathcal{C}_j)_{j=1}^n$ *a sequence of connected topological spaces (e.g.* $\mathcal{C}_j = \mathbb{R}_+^{m_j}$), $\gtrsim$ *and* $\hat{\gtrsim}$ *two binary relations on* $L = X_{j=1}^n \mathcal{C}_j$, L *endowed with the product topology. Let* $(\hat{p}_j)_{j=1}^n$ *be a sequence of strictly positive real numbers, summing to one. Let at least three coordinates be essential w.r.t.* $\gtrsim$. *Equivalent are:*

(i)   *'There exist nonnegative* $(p_j)_{j=1}^n$, *summing to one, and continuous*

$$U_j : \mathcal{C}_j \to \mathbb{R}, \; j = 1, \ldots, n, \; s.t. \; x \gtrsim y \Leftrightarrow \sum_{j=1}^n p_j U_j(x_j) \ge \sum_{j=1}^n p_j U_j(y_j) \; and$$

$$x \hat{\gtrsim} y \Leftrightarrow \sum_{j=1}^n \hat{p}_j U_j(x_j) \ge \sum_{j=1}^n \hat{p}_j U_j(y_j) \; for \; all \; x, y \in L'.$$

(ii)   *'* $\gtrsim$ *and* $\hat{\gtrsim}$ *are continuous coordinate independent weak orders,* $\gtrsim$ *is cardinally consistent with* $\hat{\gtrsim}$ *'.*

Proof. That (i) implies (ii) is by the definitions $V_j := p_j U_j$, $\hat{V}_j := \hat{p}_j U_j$, $\phi_j(\alpha) := p_j \alpha / \hat{p}_j$, Proposition 4.5 and Theorems 2.8 and 3.1.

Next we assume (ii), and derive (i). By Lemma 4.3.ii $\gtrsim$ has at least three essential coordinates. Theorems 2.8 and 3.1 have statements (ii) satisfied, thus also statements (i). Now (i) in Proposition 4.5 follows. Hereby, for arbitrary fixed $z \in L$, there exist nonnegative $(\lambda_j)_{j=1}^{n}$ s.t. for all x and j we have $V_j(x_j) - V_j(z_j) = \lambda_j [\hat{V}_j(x_j) - \hat{V}_j(z_j)]$. At least three coordinates are essential w.r.t. $\gtrsim$, so at least one (even three) λ_j's are nonzero. Consequently $0 < v := \sum_{j=1}^{n} \hat{p}_j \lambda_j$. We define $U_j := [\hat{V}_j - \hat{V}_j(z_j)]/\hat{p}_j$, and $p_j = \lambda_j \hat{p}_j / v$ for all j. Thus $\hat{V}_j = \hat{p}_j U_j + \hat{V}_j(z_j)$, $V_j = v p_j U_j + V_j(z_j)$, for all j. $\square$

In (i) above $(U_j)_{j=1}^{n}$ can be replaced by $(\bar{U}_j)_{j=1}^{n}$ iff real $(\beta_j)_{j=1}^{n}$, and positive α, exist s.t. $\bar{U}_j = \beta_j + \alpha U_j$ for all j. The uniqueness result w.r.t. $(p_j)_{j=1}^{n}$ is not so simple. On the set of essential coordinates w.r.t. $\gtrsim$, the ratios of the p_j's are uniquely determined. For the coordinates, inessential w.r.t $\gtrsim$, so those with constant $\hat{V}_j$ and U_j, the p_j's are arbitrary. As long as the p_j's are nonnegative, and sum to one, of course. This can be seen in the above proof: after $(V_j)_{j=1}^{n}$ and $(\hat{V}_j)_{j=1}^{n}$ are chosen, for the constant $\hat{V}_j$'s the λ_j's can be arbitrarily chosen, for the nonconstant $\hat{V}_j$'s they are uniquely determined.

For interpretations of results as above the reader is referred to KSV. They also indicate interesting implications for statistics.

Schmeidler noted that in (ii) above CI for $\gtrsim$ may be derived from CC. This is by the choice $x = y$, $a_i = b_i$, $c_i = d_i$ in Definition 4.1. Also we could have weakened CC by requiring it only for $b_i = c_i$. Note that Lemma 4.3 did not yet give ordinal consistency. This now follows from (i) above. For the case of two essential coordinates in (ii) above we would have had to add in (ii) above for $\gtrsim$ and $\hat{\gtrsim}$ the 'Thomsen Condition'. For the definition of this the reader is referred to Krantz et al. (1971). The case of one essential coordinate for $\gtrsim$ is somewhat different. This refers in fact to the case where there is no uncertainty, hence is uninteresting for our purposes.

Let us finally remark that, for recovering from $\gtrsim$, via $(V_j)_{j=1}^{n}$, and $(\hat{p}_j)_{j=1}^{n}$, the probabilities $(p_j)_{j=1}^{n}$, it is not necessary to know all of $\hat{\gtrsim}$. For instance if (not $b_2 \hat{\approx}_2 a_2$) and $(x_{-1}a_1)_{-2}a_2 \hat{\approx} (x_{-1}b_1)_{-2}b_2$, then $\hat{p}_1[U_1(a_1) - U_1(b_1)] = \hat{p}_2[U_2(b_2) - U_2(a_2)]$ so $[U_1(a_1) - U_1(b_1)]/[U_2(b_2) - U_2(a_2)] = \hat{p}_2/\hat{p}_1$, and now from $\hat{p}_1 p_2[V_1(a_1) - V_1(b_1)] = p_1 \hat{p}_2[V_2(b_2) - V_2(a_2)]$ we can already uncover the ratio of p_1 and p_2, for nonconstant V_1 and V_2.

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