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NOTE

RISK SENSITIVITY, INDEPENDENCE OF IRRELEVANT ALTERNATIVES AND CONTINUITY OF BARGAINING SOLUTIONS

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Bargaining solutions are considered which have the following four properties: individual rationality, Pareto optimality, independence of equivalent utility representations, and independence of irrelevant alternatives. A main result of this paper is a simple proof of the fact that all such bargaining solutions are risk sensitive. Further a description is given of all bargaining solutions satisfying the four mentioned properties. Finally, a continuous bargaining solution, satisfying the first three properties, is given which is not risk sensitive.

1. Introduction

A *bargaining game* or *bargaining pair* (S,d) consists of a compact convex subset S of $\mathbb{R}^2$ and a point $d \in S$ (*disagreement point*) such that there exists an $x \in S$ with $x_1 > d_1$ and $x_2 > d_2$. The set of all bargaining pairs is denoted by $\underline{B}$. An element (S,d) in $\underline{B}$ corresponds to the following game situation: Two players may cooperate and agree upon choosing a point $s \in S$, which has utility s_i for player i, or they may not cooperate. In the latter case they are punished by getting point d, which has utility d_i for player i $(i=1,2)$.

A map $\phi : \underline{B} \to \mathbb{R}^2$ will be called a *bargaining solution*. In this paper we mainly consider bargaining solutions satisfying the following four properties:

Property (P.1) (*Individual Rationality*) $\phi(S,d) \geq d$ for each $(S,d) \in \underline{B}$.

Property (P.2) (*Pareto Optimality*) $\phi(S,d) \in P(S)$ for each $(S,d) \in \underline{B}$, where

$$P(S) := \{p \in S;\ \text{for each } s \in S \text{ with } s \geq p \text{ we have } s = p\}$$

is the Pareto set of S.

Property (P.3) (*Independence of Equivalent Utility Representations*) For

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each $(S,d)\in \underline{B}$ and each transformation $A:\mathbb{R}^2\to\mathbb{R}^2$ of the form $A(x_1,x_2)=(a_1x_1+b_2,a_2x_2+b_2)$ for all $(x_1,x_2)\in\mathbb{R}^2$, where $a_1>0$, $a_2>0$, $b_1,b_2\in\mathbb{R}$, we have $\phi(A(S),A(d))=A(\phi(S,d))$.

Property (P.4) (*Independence of Irrelevant Alternatives*) For all (S,d), $(T,e)\in\underline{B}$ we have $\phi(S,d)=\phi(T,e)$, if $d=e$, $S\subset T$ and $\phi(T,e)\in S$.

The theory of bargaining solutions started with a paper of Nash (1950). For more background information we refer to Roth (1979).

The organization of this paper is as follows. In Section 2 we describe all bargaining solutions satisfying (P.1)–(P.4), and also the solutions, satisfying these properties and the following continuity property:

Property (P.5) (*Continuity*) For each $(S,d)\in\underline{B}$ and each sequence (S^1,d^1), $(S^2,d^2),\ldots,$ in $\underline{B}$ with $\lim_{n\to\infty}\|d^n-d\|_2=0$ and $\lim_{n\to\infty} d_{\mathrm{H}}(S^n,S)=0$ (where d_{H} is the Hausdorff metric), we have $\lim_{n\to\infty}\phi(S^n,d^n)=\phi(S,d)$.

In Section 3 the risk sensitivity of these solutions is considered. A simple proof is given of the fact that all (P.1)–(P.4) solutions are risk sensitive, i.e., that they have the following property:

Property (P.6) (*Risk Sensitivity*) For each increasing concave transformation $k:\mathbb{R}\to\mathbb{R}$ and each $(S,d)\in\underline{B}$ we have

$$\phi_1(K^2(S),K^2(d)\geq\phi_1(S,d),\qquad \phi_2(K^1(S),K^1(d)\geq\phi_2(S,d),$$

where $K^1(s):=(k(s_1),s_2)$, $K^2(s):=(s_1,k(s_2))$ for each $s\in S$, and $K^i(S):=\{K^i(s);\ s\in S\}$ for $i\in\{1,2\}$.

In Section 4 we give an example of a bargaining solution, which has Properties (P.1), (P.2), (P.3) and (P.5) and not (P.6).

2. Solutions with Properties (P.1)–(P.4)

Harsanyi and Selten (1972), Kalai (1977) and also Roth (1979) pay attention to the bargaining solutions $F^t:\underline{B}\to\mathbb{R}^2$ $(0<t<1)$, which are defined as follows. For each $(S,d)\in\underline{B}$, $F^t(S,d)$ is the unique point of $\{x\in P(S);\ x\geq d\}$, where the function $(x_1,x_2)\to(x_1-d_1)^t\ (x_2-d_2)^{1-t}$ attains its maximum. All these solutions satisfy (P.1)–(P.4). In the following proposition a characterization of F^t is given. For a proof we refer to Roth (1979, p. 16, Theorem 3). By $\triangle$ we mean the triangle in $\mathbb{R}^2$ with vertices (0,0), (1,0) and (0,1).

Proposition 2.1. *Let ϕ be a bargaining solution which has Properties* (P.1)–(P.4). *Then $\phi=F^t$ iff $\phi(\triangle,0)=(t,1-t)$ for all $t\in(0,1)$.*

Now $\{F^t;\ t\in(0,1)\}$ does not exhaust the family of bargaining solutions satisfying (P.1)–(P.4). These properties also hold for the bargaining solutions F^0 and F^1, defined as follows. For $(S,d)\in\underline{B}$, $F^0(S,d)$ is the point in $\{x\in P(S);\ x\geq d\}$ with maximal second coordinate, and $F^1(S,d)$ is the point in $\{x\in P(S);\ x\geq d\}$ with maximal first coordinate. A characterization of F^0 and F^1 is given in the following proposition.

Proposition 2.2. *Let* $\phi:\underline{B}\to\mathbb{R}^2$ *be a bargaining solution with Properties* (P.1)–(P.4). *Then*

(i) $\phi=F^0$ *iff* $\phi(\triangle,0)=(0,1)$,

(ii) $\phi=F^1$ *iff* $\phi(\triangle,0)=(1,0)$.

Proof. We only prove (i). It is clear that $\phi(\triangle,0)=(0,1)$ if $\phi=F^0$. Now suppose $\phi(\triangle,0)=(0,1)$. Let $(S,d)\in\underline{B}$, $S_1:=\{x\in\mathbb{R}^2; x\geq d,\ \exists_{y\in S}[y\geq x]\}$ and S_2 be the convex hull of $\{d\}\cup\{s\in P(S);\ s\geq d\}$.

By (P.1), (P.2) and (P.4) we see that $\phi(S_2,d)=\phi(S,d)=\phi(S_1,d)$ and $F^0(S_2,d)=F^0(S,d)=F^0(S_1,d)$. Let $s\in P(S)$ with $s\geq d$ and $s_2<F_2^0(S,d)$. Let $v:=(d_1,F_2^0(S,d))$ and let z be the unique point on the straight line through v and s with $z_2=d_2$. Let T be the triangle with vertices d, v and z and $T_1:=T\cap S_1$. Since $\phi(\triangle,0)=(0,1)$, Property (P.3) gives $\phi(T,d)=v$. So (P.4) gives $\phi(T_1,d)=v\neq s$.

Again (P.4) gives $s\neq\phi(S_1,d)=\phi(S,d)$. We conclude: $\phi_2(S,d)=F_2^0(S,d)$, so $\phi=F^0$. □

Our main result in this section is the following.

Theorem 2.3. $\{F^t;\ t\in[0,1]\}$ *is the family of all bargaining solutions satisfying Properties* (P.1)–(P.4).

Proof. We have already seen that each of the maps F^t has Properties (P.1)–(P.4). Now let ϕ be a bargaining solution, satisfying (P.1)–(P.4). Then $\phi(\triangle,0)=(s,1-s)$ for some $s\in[0,1]$. If $s=0$ ($s=1$), then, by Proposition 2.2, $\phi=F^0\in\{F^t;\ t\in[0,1]\}$ ($\phi=F^1\in\{F^t; t\in[0,1]\}$). If $s\in(0,1)$, then, by Proposition 2.1, $\phi=F^s\in\{F^t;\ t\in[0,1]\}$. □

In Jansen and Tijs (1980) it is proved that F^t is continuous for each $t\in(0,1)$, and that F^0 and F^1 are discontinuous bargaining solutions. This leads to the following corollary.

Corollary 2.4. $\{F^t;\ t\in(0,1)\}$ *is the family of all continuous bargaining solutions, satisfying Properties* (P.1)–(P.4).

3. Risk sensitivity of bargaining solutions with Properties (P.1)–(P.4)

Kihlstrom, Roth and Schmeidler (1981) proved that the Nash solution $F^{1/2}$ satisfies (P.6). It is also easy to show that F^0 and F^1 are risk sensitive. Peters and Tijs (1981) proved that, for $t\in(0,1)$, the bargaining solution F^t is risk sensitive. In their proof use is made of the fact that, for $(S,d)\in \underline{B}$, there exists a supporting line for S through $F^t(S,d)$ with slope $t(t-1)^{-1}\,(F_2^t(S,d)-d_2)(F_1^t(S,d)-d_1)^{-1}$. Now a simple proof of the above results can be given, by applying Theorem 3.1 below and Theorem 2.3. Theorem 3.1 can be seen as the main result of this paper. In proving Theorem 3.1 no use is made of Theorem 2.3 but merely of Properties (P.1)–(P.4).

Theorem 3.1. *Let $\phi:\underline{B}\to\mathbb{R}^2$ be a bargaining solution, satisfying Properties* (P.1)–(P.4). *Then ϕ is risk sensitive.*

Proof. Suppose ϕ is not risk sensitive, i.e., there exists an increasing concave transformation k and an $(S,d)\in\underline{B}$ such that

$$\phi_i(K^j(S),K^j(d))<\phi_i(S,d) \quad \text{for some } i\in\{1,2\} \text{ and } j\in\{1,2\},\ j\neq i.$$

We assume $i=1$, $j=2$ (the reverse case is similar), so we have

$$\phi_1(K^2(S),K^2(d))<\phi_1(S,d). \tag{3.1}$$

Furthermore, we may w.l.o.g. assume in view of (P.4) that

$$y\in S \quad \text{if } d\leq y\leq x \text{ for some } x\in S. \tag{3.2}$$

Let (q_1,q_2) be the point in $P(S)$ with $q_1=\phi_1(K^2(S),K^2(d))$. Then $q_2>\phi_2(S,d)\geq d_2$, by (P.1), (P.2) and (3.1).

In view of (P.3) we may suppose w.l.o.g. that

$$d=(0,0), \qquad k(0)=0, \qquad k(q_2)=q_2. \tag{3.3}$$

Then $K^2(d)=(d_1,k(d_2))=(0,0)$ and $\phi(K^2(S),K^2(d))=(q_1,q_2)$. The concavity of k and (3.3) imply

$$k(x)\geq x \quad \text{for all } x\in[0,q_2]. \tag{3.4}$$

Now let $\tilde{P}:=\{p\in P(S);\ q_1\leq p_1\leq\phi_1(S,0)\}$ which set has $\phi(S,0)$ and $\phi(K^2(S),0)$ as elements. Let D be the convex hull of (0,0) and $\tilde{P}$. Then, in view of (3.2) and (3.4), $D\subset K^2(S)$. Hence, (P.4) implies

$$\phi(D,0)=\phi(K^2(S),0). \tag{3.5}$$

On the other hand, $D\subset S$. By applying (P.4) again we obtain

$$\phi(D,0)=\phi(S,0). \tag{3.6}$$

Finally, combining (3.1), (3.5) and (3.6) yields a contradiction, which completes the proof. □

We conclude this section with a remark made by Kalai. Let $\tilde{B}$ be the subset of bargaining pairs $(S,0)$ with the following properties:

(i) $s\geq 0$ for all $s\in S$,

(ii) if for an $x\in\mathbb{R}^2$ there is an $y\in S$ with $0\leq x\leq y$, then $x\in S$.

Let $\text{TR} := \{x \in \mathbb{R}^2;\ x_1 \geq 0,\ x_2 \leq 0\}$. (Elements of TR can be seen in our context as utility transfers from player 2 to player 1.) Let us consider the following two properties for bargaining solutions ϕ:

Property (P.7) (*Transfer Property*) For all $(S,0),(T,0) \in \tilde{B}$ with $\phi(T,0) \in S$ we have

$$T \setminus S \subset (\phi(T,0) - \text{TR}),\ S \setminus T \subset (\phi(T,0) + \text{TR}) \Rightarrow \phi(T,0) \in (\phi(S,0) - ,$$

$$T \setminus S \subset (\phi(T,0) + \text{TR}),\ S \setminus T \subset (\phi(T,0) - \text{TR}) \Rightarrow \phi(T,0) \in (\phi(S,0) + \text{TR}).$$

Property (P.8) (*Comprehensiveness Property*) $\phi(\tilde{S},d) = \phi(S,d)$ for all $(S,d) \in \underline{B}$, where

$$\tilde{S} := \{x \in \mathbb{R}^2;\ x \geq d,\ \exists_{y \in S}[y \geq x]\}.$$

Kalai noted that by using the same ideas as in the proof of Theorem 3.1, one can prove that a solution ϕ, having Properties (P.1), (P.2), (P.3), (P.7) and (P.8), is risk sensitive. Furthermore, it is not difficult to show that Properties (P.1)–(P.4) imply (P.7) and (P.8).

4. A continuous bargaining solution, satisfying (P.1)–(P.3), which is not risk sensitive

Kihlstrom, Roth and Schmeidler (1981) proved the following theorem (cf. Roth (1979), p. 47).

Theorem 4.1. *Let $\phi : \underline{B} \to \mathbb{R}^2$ be a map, satisfying* (P.2). *Then, if ϕ satisfies* (P.6), *ϕ satisfies* (P.3).

Peters and Tijs (1981) showed that the converse of this theorem does not hold. But they conjectured that (P.3) and (P.5) imply (P.6). The following example is a counterexample for this conjecture: The bargaining solution W in this example is continuous and not risk sensitive.

Example 4.2. Let the map $W : \underline{B} \to \mathbb{R}^2$ be defined as follows:

(i) For $(S,d) \in \underline{B}$ with $d = (0,0)$ and $u_i(S,d) := \max\{x_i \in \mathbb{R};\ x \in S,\ x \geq d\}$ equal to 1 for each $i \in \{1,2\}$, let $r(S)$ be the unique point on $P(S)$ with $r_1(S) = r_2(S)$. Now put $W(S,d) := r(S)$ if $r_1(S) \in [\frac{7}{10}, 1]$, and let $W(S,d)$ be the unique Pareto point of S with first coordinate $1\frac{4}{10} - r_1(S)$ if $r_1(S) \in [\frac{1}{2}, \frac{7}{10}]$.

(ii) If, for $(S,d) \in \underline{B}$, $d \neq (0,0)$ or $u(S,d) \neq (1,1)$, construct a map A as in (P.3), such that $A(d) = (0,0)$ and $u(A(S),0) = (1,1)$, and define $W(S,d) := A^{-1}(W(A(S), A(d)))$.

Obviously, (P.1), (P.2) and (P.3) hold for W.

It is straightforward to show that W is continuous. But W is not risk sensitive, as we will show now. Let S be the triangle in $\mathbb{R}^2$ with vertices (0,0), (1,0) and (0,1), and $d := (0,0)$. Then $r_1(S) = \frac{1}{2}$ and $W(S,d) = (\frac{9}{10}, \frac{1}{10})$. Now let $k: \mathbb{R} \to \mathbb{R}$ be the increasing concave transformation with $k(x) = \frac{7}{3}x$ for $x \leq \frac{3}{10}$ and $k(x) = \frac{3}{7}x + \frac{4}{7}$ for $x \geq \frac{3}{10}$. Then $K^2(S)$ is the convex hull of the points (0,0), (1,0), $(\frac{7}{10}, \frac{7}{10})$ and (0,1), and $K^2(d) = (0,0)$. Now $r_1(K^2(S)) = \frac{7}{10}$ and $W(K^2(S), K^2(d)) = (\frac{7}{10}, \frac{7}{10})$. Then $W_1(K^2(S), K^2(d)) = \frac{7}{10} < \frac{9}{10} = W_1(S,d)$, which implies that W is not risk sensitive.

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