Transforming Ordinal Riskless Utility into Cardinal Risky Utility: A Comment on Chung, Glimcher, & Tymula (2019)

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Chung, Glimcher, & Tymula (2019, this journal) observed both consumers' choices over commodity bundles and choices under risk. They assumed a cardinal riskless utility function V representing consumer choices, and a cardinal risky utility function U. The two were inconsistent. This note shows that the two functions can be reconciled if we assume that V is ordinal. Then one utility function U can accommodate both risky and riskless choices.

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The revealed preference paradigm has prevailed in economics since the ordinal revolution of the 1930s, when neoclassical cardinal utility was discarded and utility became strictly ordinal (Moscati 2018). Samuelson (1938 p. 65) wrote: "It is not only that we can get along without this cardinal concept, but literally nothing is added by its assumption." The cardinal risky utility function that appeared in expected utility (Zeuthen 1937; von Neumann & Morgenstern 1947, 1953) and, later, in prospect theory (Tversky & Kahneman 1992) then should be distinguished from neoclassical cardinal utility (Baumol 1958).

A minority of authors argued that ordinal utility is too narrow and that neoclassical cardinal (riskless) utility should not be discarded.[1] Then risky and riskless utility may be cardinally different (Keeney & Raiffa 1976; Dyer & Sarin 1982). Some authors favored equating them, based on intuitive and theoretical tractability arguments (Loomes & Sugden 1982; Harsanyi 1988 p. 127; Wakker 1994; Luce 1996 §1.3).

Many empirical studies have compared risky and riskless utility. Then cardinal riskless utility is mostly derived from preference intensity comparisons (Hutton Barron, von Winterfeldt, & Fischer 1984; Keller 1985; Smidts 1997; Stalmeier & Bezembinder 1999). However, such comparisons are based on introspective judgments rather than on revealed preferences, and strict ordinalists will not accept them. Some studies have compared cardinal risky utility with the cardinal intertemporal utility function that appears in discounted utility, and that is also riskless (Abdellaoui et al. 2013; Epper & Fehr-Duda 2015). However, when introducing discounted utility, the strict ordinalist Samuelson (1937 p. 161)

[1] See Loomes & Sugden (1982), Harsanyi (1988 p. 127), Kahneman (1994), Wakker (1994), Luce (1996 §1.3), van Praag & Ferrer-i-Carbonell (2004), and Chung, Glimcher, & Tymula (2019).

immediately argued that the intertemporal cardinal utility function should be distinguished from neoclassical cardinal utility.

Chung, Glimcher, & Tymula (2019), CGT henceforth, discussed the above literature (pp. 36-37) and introduced a new approach. Henceforth, U denotes the cardinal[2] utility of prospect theory (comprising EU as a special case), and V denotes a riskless utility function that may or may not be cardinal. CGT derived V from riskless choices (consumer choices) between two-dimensional commodity bundles (x_1, x_2). CGT implicitly made what I call the *cardinal-V assumption*: they assumed that a particular representing function V that they had chosen was plausible as a cardinal riskless function (their Eq. 1). With this assumed, the discussion of cardinal properties in their theoretical §1, such as concavity/convexity and part (ii) in their Assumption 1, can be justified. Their V exhibited cardinal properties different from risky utility U, which CGT also measured. In particular, V did not exhibit reflection (concavity for gains versus convexity for losses) whereas U did. Instead, V was concave throughout. CGT concluded that the cardinal utility function of prospect theory cannot play the role of cardinal riskless neoclassical utility, contrary to suggestions by some authors cited above.

This note will present an alternative analysis, so as to reconcile risky and riskless utility after all. The essential difference is that this note does not make the cardinal-V assumption, but instead an *ordinal-V assumption*: the consumer choices only provide ordinal information and, hence, CGT's V can be replaced by any strictly increasing

[2] For simplicity, we assume a fixed 0-level of utility henceforth, so that we need not distinguish between cardinal (interval) and ratio scales. That distinction is not relevant for the purposes of this paper.

transformation. This assumption requires no commitment to the existence or non-existence of cardinal riskless utility, a point further discussed below.

To reconcile the consumer choices observed by CGT with prospect theory, it now suffices to find a strictly increasing transformation φ such that

(1) $U = \varphi o V$.

Example A1 in the appendix shows that this is possible. Thus, the additional parameter φ makes it possible to reconcile all risky and riskless phenomena of U and V found by CGT.

Ordinalists will be satisfied at this stage, and will leave it at that. Advocates of cardinal riskless utility, and of equating that with cardinal risky utility, will also be satisfied. For them, Eq. 1 provides the right cardinal function, both risky and riskless. Finally, advocates of cardinal riskless utility different from cardinal risky utility can also be satisfied. For them, the proper cardinal class for the riskless V remains to be determined, and further data is required to do so. Such data may consist of preference intensity comparisons, measurements of discounted utility, or whatever the favored interpretation of cardinal riskless utility is.

Two further comments:

- Tversky and Kahneman's (1991) formal analysis was not restricted to constant marginal sensitivities, contrary to CGT (p. 35; p. 58).
- Kahneman and Tversky did not generally commit to status quo/current wealth as the reference point, neither implicitly nor explicitly, contrary to CGT (p. 41).

CONCLUSION. Chung, Glimcher, & Tymula (2019) found quasiconvexity throughout for riskless two-dimensional utility, and reflection for risky utility[3] (concavity for gains and convexity for losses). These findings are not inconsistent with any of the current theories under our ordinal-V assumption. In particular, they can be accommodated by prospect theory with one utility function (U) for both risky and riskless choices.

[3] This is taken in a one-dimensional sense explained in the appendix.

APPENDIX. RECONCILING THE RISKY AND RISKLESS CHOICES IN CGT

This appendix makes the ordinal-V assumption, deviating from the cardinal-V assumption of CGT. In the following example, the preferences are the most plausible empirical ones, and they exhibit all the characteristics found in CGT's experiment. Riskless preferences are quasi-convex (called bowed-in by CGT) throughout, both for gains and for losses. Because of the ordinal-V assumption, this does not preclude the common empirical findings of prospect theory, and in this regard we deviate from claims made throughout CGT (e.g., p. 53 bottom). If we, following CGT, take utility[4] U below as a one-dimensional function by fixing one of the two commodities, then it satisfies reflection, exactly as found by CGT. Finally, if we take U as a one-dimensional function of money, then it is exactly as in Tversky & Kahneman (1992). It, again, exhibits reflection.

EXAMPLE A1. As in CGT, we assume two-dimensional outcomes (x_1, x_2). We assume $x_1 \geq 1, x_2 \geq 1$. An interpretation could be that x_1 is the duration of a time interval during which one receives money, free of charge, without having to do anything for it, but then with 10 subtracted, and x_2 is the money received per time unit. One then receives a total of $x_1 \times x_2 - 10$. We assume that discounting can be ignored, e.g., by restricting x_1 to short durations, and that only the final sumtotal matters. That is, riskless preferences maximize $x_1 \times x_2$. Take $V(x_1, x_2) = \ln(x_1) + \ln(x_2)$. It represents riskless preference. The mixed derivatives are 0 so that they, in CGT's (p. 39) terminology, will surely not "overwhelm" marginal utilities, and the

[4] Kahneman and Tversky used the term value function.

marginal utilities of both goods (again using CGT's cardinal concept) are diminishing. The function is Cobb-Douglas, a common utility function in consumer theory and included as a special case by CGT (p. 47 footnote 3; their ρ is 0 then).

(x_1, x_2) is a *gain* if $x_1 \times x_2 > 10$, and a *loss* if $x_1 \times x_2 < 10$. V is concave and riskless preferences satisfy quasi-convexity throughout, for gains as well as for losses.

To capture risk attitude, we have to specify φ in Eq. 1 where we take V as defined before:

(2) $v > ln(10) \Rightarrow \varphi(v) = (exp(v) - 10)^{0.88}$;

(3) $v = ln(10) \Rightarrow \varphi(v) = 0$;

(4) $v < ln(10) \Rightarrow \varphi(v) = -2.25 \times \big(10 - exp(v)\big)^{0.88}$.

This gives

(5) $x_1 \times x_2 - 10 > 0 \Rightarrow U(x_1, x_2) = (x_1 \times x_2 - 10)^{0.88}$;

(6) $x_1 \times x_2 - 10 = 0 \Rightarrow U(x_1, x_2) = 0$;

(7) $x_1 \times x_2 - 10 < 0 \Rightarrow U(x_1, x_2) = -2.25 \times (10 - x_1 \times x_2)^{0.88}$.

Because the only relevant aspect of (x_1, x_2) for the decision maker is $x_1 \times x_2 - 10$, being the total money amount obtained, U is exactly the utility function obtained by Tversky & Kahneman (1992). It is concave for gains and convex for losses. It belongs to the CRRA parametric family for both gains and losses.

In their experiment, CGT considered fixing one variable of x_1, x_2 and varying only the other. Then, in terms of U, the second derivative of utility is positive for losses, reflecting convexity there and enhancing risk seeking. However, in terms of V, the second derivative is negative, also for losses. These two phenomena are not inconsistent but are explained by the intervening transformation φ. In particular, prospect theory is not contradicted here. CGT's (p. 58) cardinal-V assumption led to the opposite conclusion. The two phenomena illustrate that concavity/convexity, the

sign of the second derivative, and marginal utility, concepts central in CGT's analyses of V throughout, are not meaningful under our ordinal-V assumption.[5] □

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[5] CGW (p. 60) point this out when writing: "Of course a traditional theorist who treats utility inferred from riskless choice as an ordinal object would not be tempted to make statements about the curvature of the elicited utility function and indifference curves. ... The principle of decreasing marginal utility as well as the definitions of complementarity and substitution between the goods are not unique up to positive affine [ordinal] transformations and, hence, are meaningless under ordinal utility." For completeness I add here that rates of substitution, (curvature of) indifference curves, and complementarity and substitution between goods are meaningful under ordinal utility, whereas curvature of utility and decreasing marginal utility are not.

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