OPERATIONS RESEARCH
Vol. 63, No. 6, November–December 2015, pp. 1420–1430
ISSN 0030-364X (print) | ISSN 1526-5463 (online)

http://dx.doi.org/10.1287/opre.2015.1433

Discounted Utility and Present Value—A Close Relation

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We introduce a new type of preference condition for intertemporal choice, which requires present values to be independent of various other variables. The new conditions are more concise and more transparent than traditional ones. They are directly related to applications because present values are widely used tools in intertemporal choice. Our conditions give more general behavioral axiomatizations, which facilitate normative debates and empirical tests of time inconsistencies and related phenomena. Like other preference conditions, our conditions can be tested qualitatively. Unlike other preference conditions, our conditions can also be directly tested quantitatively, and we can verify the required independence of present values from predictors in regressions. We show how similar types of preference conditions, imposing independence conditions between directly observable quantities, can be developed for decision contexts other than intertemporal choice and can simplify behavioral axiomatizations there. Our preference conditions are especially efficient if several types of aggregation are relevant because we can handle them in one stroke. We thus give an efficient axiomatization of a market pricing system that is (i) arbitrage-free for hedging uncertainties and (ii) time consistent.

Keywords: intertemporal optimization; present value; discounted utility; time inconsistency; arbitrage-free; preference axiomatization; decision analysis: sequential; utility: multi-attribute; mathematics: functions.
Subject classifications: decision analysis: sequential; utility/preference: multiattribute; mathematics: functions.
Area of review: Decision Analysis.
History: Received February 2015; revision received June 2015; accepted August 2015. Published online in *Articles in Advance* November 5, 2015.

1. Introduction

Debates about appropriate models of intertemporal choice, both for the normative purpose of making optimal decisions and for the descriptive purpose of fitting decisions, usually focus on the critical preference conditions that distinguish these models. The two most discussed conditions are time consistency, which plays a role in distinguishing constant and hyperbolic discounting, and intertemporal separability, which pertains to habit formation, satiation, addiction, and sequencing effects.[1] Both time consistency and intertemporal separability are assumed in the classical models but they are usually violated empirically. Their normative status has also been questioned.[2]

To shed new light on the appropriateness of intertemporal decision models, we introduce a new kind of preference conditions to distinguish them, stated directly in terms of present values (PVs). PVs are simple and tractable and have been widely used in intertemporal choice, both when reflecting the preferences of the financial market[3] and when reflecting subjective preferences of individuals.[4] They relate to the indifferences most commonly encountered in everyday life. We often have to decide on whether to pay up front for goods consumed later, whether to pay a price now for a financial product with future financial consequences, or whether to choose a savings plan that requires the money to be delivered now. For these reasons, present values are widely used in experimental measurements of intertemporal preferences.

People can more easily relate to independence conditions imposed on present values than to independence preference conditions. "For your present value of this extra payment on day 10, the payments on the other days do (not) matter" is easier to relate to for most people than the usual preference conditions (see Equation (16) in §7, for instance). In general, PVs can depend on many variables, such as the periods of the receipt of future outcomes, the initial wealth levels at those periods, and the wealth levels at other periods. Our preference conditions will impose independence of PVs from some of those other variables. We show that many models can be characterized by the appropriate independencies.

Like all preference conditions, our conditions can be tested qualitatively. Unlike other preference conditions, our conditions can also be directly tested quantitatively.

We can, for instance, carry out regressions with PV as the dependent variable and the other relevant variables as predictors.[5] We can then test which of those other variables are significantly associated with PV and whether the variables claimed to be independent in our conditions really are so. Such tests are more widely known and better understood than qualitative tests of preference conditions.

To illustrate our new PV conditions, we apply them to some well-known models. Table 1 gives a concise presentation of these models and their representations. Details of the table will be explained in the following sections. Table 1 is presented here because it illustrates the organization of the models in the first four sections.

We provide the most concise and most general preference axiomatizations presently available in the literature for (a) constant discounted value as commonly used by financial markets (Hull 2013); (b) constant discounted utility (Samuelson 1937); and (c) general discounted utility, which includes hyperbolic discounting. We also provide results that are relevant to multi-attribute optimization problems other than intertemporal: (d) no-bookmaking and no-arbitrage for uncertainty, which are commonly used for financial markets, and (e) additive separability for general multi-attribute aggregations (Debreu 1960, Gorman 1968). In such other contexts we should find a quantitative index that can play a role similar to present value for intertemporal choice. Section 6 considers aggregation over two dimensions, for instance, time and uncertainty. Here our technique is particularly efficient because it can handle both aggregations in one blow. We derive the most common pricing model used in finance: as-if risk neutrality together with constant discounting, which avoids arbitrage for both uncertainty and time.

2. Preferences and Subjective PVs

We derive appropriateness of an intertemporal goal function V from the decisions that it implies, modeled through a binary preference relation $\succcurlyeq$ over *outcome sequences* $x = (x_0, \ldots, x_T) \in \mathbb{R}^{T+1}$. The preferences can, for instance, concern (i) observed consumer choices in descriptive applications or (ii) pension savings plans or market prices with the financial market taken as decision maker in prescriptive applications. The outcome sequence yields *outcome* x_t in *period* t, for each t; $t = 0$ denotes the present. We assume $T \geqslant 2$ to avoid trivialities and keep all other aspects of our analysis as simple as possible (assuming one fixed $T \in \mathbb{N}$) so as to focus on the novelty of our conditions. We use indexed Roman letters x_t to specify the period t of receipt of outcome x_t and Greek letters $\alpha, \beta, \ldots$ to refer to outcomes (real numbers) when no period of receipt needs to be specified. By $\alpha_t x$ we denote x with x_t replaced by α.

The goal function V *represents* $\succcurlyeq$ if $V\colon \mathbb{R}^{T+1} \to \mathbb{R}$ and $x \succcurlyeq y \Leftrightarrow V(x) \geqslant V(y)$ for all $x, y \in \mathbb{R}^{T+1}$. The existence of a representing V implies that $\succcurlyeq$ is a *weak order*; i.e., $\succcurlyeq$ is *complete* ($x \succcurlyeq y$ or $y \succcurlyeq x$ for all x, y) and transitive. We therefore assume throughout that $\succcurlyeq$ is a weak order. Strict preference $\succ$, indifference $\sim$, reversed preference $\preccurlyeq$, and strict reversed preference ($\prec$) are as usual. We also assume *monotonicity* (strictly improving an outcome

Table 1. The column "functional" gives the functional forms evaluating $(x_0, \ldots, x_T)$.

Model	Functional	Pref. conditions	PV condition	PV formula
Nondiscounted value	$\sum_{t=0}^{T} x_t$	Additive + Symmetric	$\pi(\varphi)$	φ
Constant discounted value	$\sum_{t=0}^{T} \lambda^t x_t$	Additive + Stationary	$\pi(\varphi, t) = \tau(\varphi, t+1)$	$\lambda^t \varphi$
Time-dependent discounted value	$\sum_{t=0}^{T} \lambda_t x_t$	Additive	$\pi(\varphi, t)$	$\lambda_t \varphi$
Nondiscounted utility	$\sum_{t=0}^{T} U(x_t)$	Separable + Symmetric	$\pi(\varphi, e_0, e_t)$	$U^{-1}(U(e_t+\varphi) - U(e_t) + U(e_0)) - e_0$
Constant discounted utility	$\sum_{t=0}^{T} \lambda^t U(x_t)$	Separable + Stationary	$\pi(\varphi, t, e_0, e_t) = \tau(\varphi, t+1, e'_1 = e_0, e'_{t+1} = e_t)$	$U^{-1}(\lambda^t U(e_t+\varphi) - \lambda^t U(e_t) + U(e_0)) - e_0$
Time-dependent discounted utility	$\sum_{t=0}^{T} U_t(x_t)$	Separable	$\pi(\varphi, t, e_0, e_t)$	$U_0^{-1}(U_t(e_t+\varphi) - U_t(e_t) + U_0(e_0)) - e_0$

Notes. Here, $0 < \lambda$ is a discount factor, $0 < \lambda_t$ is a period-dependent discount factor with $\lambda_0 = 1$, U is a strictly increasing continuous utility function, and U_t is a period-dependent strictly increasing continuous utility function. The column "Pref. conditions" gives preference conditions traditionally used in preference axiomatizations of the functional forms, and defined in §7. The column "PV condition" gives the PV condition used in our preference axiomatizations indicating how $\pi(\varphi, t, e)$ can be rewritten. Here, $\pi(\varphi, t, e)$ denotes the present value of receiving φ extra in period t if the endowment is e. Constant discounting has an extra equality involving τ, tomorrow's value. Note that these cells contain complete definitions. The column "PV formula" gives the formula of PV under each model.

strictly improves the outcome sequence) and continuity of $\succcurlyeq$ throughout. The conditions imply that all discount weights in this paper are positive and that all utility functions are continuous and strictly increasing.

The following condition considers sums of outcomes $x_t = \varphi + e_t$. Here we call e_t an *initial endowment*. The specification of the initial endowment only serves to facilitate interpretations and does not refer to any type of reference dependence. Formally, our analysis is entirely in terms of final wealth and is classical in this respect.

Definition. $\pi(\varphi, t, e)$ is the *present value* (*PV*) of outcome φ received in period t with (*initial*) *endowment* $e = (e_0, \ldots, e_T)$ if $(e_0 + \pi(\varphi, t, e))_0 e \sim (e_t + \varphi)_t e$; i.e.,

$$(e_0 + \pi(\varphi, t, e), e_1, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_T) \sim (e_0, \ldots, e_{t-1}, e_t + \varphi, e_{t+1}, \ldots, e_T). \quad \square \tag{1}$$

Equation (1) means that with e the current endowment, receiving an additional future outcome φ in period t is exactly offset by receiving an additional present outcome π. That is, the PV of a future outcome φ in period t is π. For simplicity, we assume in this paper that a PV always exists. Generalizations are discussed in §8.

By monotonicity, the PV is unique. In applications, PVs (denoted π here) are often used for general outcome sequences x with endowment e: $(e_0 + \pi)_0 e \sim e + x$. However, we will not need this general concept in this paper.

The PV, π, can in general depend on all of φ, t, and e, and π is a function $\pi(\varphi, t, e)$. As a convention, if we write π without its arguments, then it designates the general function depending on all its arguments. For $t = 0$, it trivially follows that $\pi(\varphi, 0, e) = \varphi$. Note that π can be subjective, depending on $\succcurlyeq$, and thus it reflects the tastes and attitudes of the decision maker. The preference conditions presented in the following sections amount to independence of PV, π, from some of the variables (φ, t, e). We express this independence by writing only the arguments that π depends on. For example, if $\pi(\varphi, t, e)$ depends only on φ (i.e., is independent of t and e), then we write

$$\pi = \pi(\varphi). \tag{2}$$

Similarly, if π depends on e only through e_0 and e_t, then we write

$$\pi = \pi(\varphi, t, e_0, e_t). \tag{3}$$

Preference conditions should be directly verifiable from preferences, the observable primitives in the revealed preference paradigm, without invoking theoretical constructs such as utilities. In Equation (16) in §7, we illustrate how our PV conditions can be rewritten in terms of preferences. They are therefore genuine preference conditions, and our conditions can be tested in the same way as all qualitative preference conditions.

3. Linear utility

This section considers models with linear utility, as commonly used in financial markets. Such models can serve as approximations for subjective individual choices if the stakes are moderate (Epper et al. 2011; Luce 2000, p. 86; Pigou 1920, p. 785). The first model in Table 1 maximizes the sum of outcomes:

$$\textit{Nondiscounted Value}: \quad \sum_{t=0}^{T} x_t. \tag{4}$$

This model does not involve subjective parameters, is directly observable, and does therefore not need a preference axiomatization. But it serves well as a first illustration of the nature of PV conditions.

Theorem 1. *The following two statements are equivalent*:
(i) *Nondiscounted value holds.*
(ii) $\pi = \pi(\varphi)$. $\square$

Throughout this paper, Condition n(ii) refers to Statement (ii) of Theorem n. Theorem 1 shows that if π depends only on φ (Condition 1(ii)) in whatever general sense one might think of, then it must be through the identity function $\pi(\varphi) = \varphi$. This implication may seem surprising at first, the more so as Condition 1(ii) in addition implies the summation operation in nondiscounted value in Equation (4). To illustrate the strength of Condition 1(ii), first note that substituting $t = 0$ already implies that φ can only be the identity. The following informal proof further illustrates the condition: according to Condition 1(ii), the extra value of any extra future outcome is always the same and can therefore be added to today's wealth. Then all that matters is the sum of all future outcomes, which may as well be received immediately today. The implication $\pi(\varphi) = \varphi$ can also be inferred from the last two columns of the corresponding row of Table 1.

Our next model involves a subjective parameter, the discount factor λ:

$$\textit{Constant Discounted Value}: \quad \sum_{t=0}^{T} \lambda^t x_t \quad \text{for } \lambda > 0. \tag{5}$$

By monotonicity, $\lambda > 0$. Under the usual assumption that the decision maker is impatient, we have $\lambda \leqslant 1$. In PV calculations of cash flows, constant discounted value is commonly used, setting $\lambda = 1/(1+r)$ with r the interest or discount rate of the market. In this case, if the decision maker is, say, an individual financial trader, the discount factor λ is not a subjective parameter reflecting the attitude of the decision maker but it is a given constant, publicly known and determined by the market. The following theorem, then, does not apply to the financial trader in the role of decision maker.

The following theorem is still relevant for market pricings if the financial market is the decision maker who determines (rational) PVs. Then λ reflects the market attitude, which may, for instance, be determined by the attitude of a central bank choosing a goal function for its optimal control problem, and which in this sense is subjective. Condition 2(ii) rationalizes this common evaluation system. In other contexts where the parameter λ reflects the attitude of an individual decision maker, it will probably be influenced by the market interest rate but need not be identical to it, for instance, because it may incorporate extra risks borne (Smith and McCardle 1999, §2.1; Smith 1998).

In the following theorem, we use tomorrow's value as an analogue to PV, defined as follows:

DEFINITION. τ is *tomorrow's value* of an outcome φ received in period t $(t \geqslant 1)$ with *endowment* e, denoted $\tau = \tau(\varphi, t, e)$, if $(e_1+\tau)_1 e \sim (e_t+\varphi)_t e$; i.e.,

$$(e_0, e_1+\tau, e_2, \dots, e_t, \dots, e_T) \sim (e_0, e_1, \dots, e_t+\varphi, \dots, e_T). \quad \square \tag{6}$$

Here τ is the extra outcome in period 1 that exactly offsets the extra outcome φ in period t. Thus τ is tomorrow's PV. Such "future" present values are central tools in recursive intertemporal models (Campbell and Shiller 1987; Ju and Miao 2012 and their references; Maccheroni et al. 2006). Experimental measurements of subjective individual discounting in studies often compare present values with tomorrow's values. To measure the latter, so-called front-end delays are then added (Ahlbrecht and Weber 1997, Luhmann 2013). The main violations of time consistency occur when present value is changed into tomorrow's value (immediacy effect), and this effect is central in the popular quasi-hyperbolic discount model (Laibson 1997). Section 8 discusses preference conditions for the following theorem entirely in terms of present values. We prefer using tomorrow's value here because it leads to the most appealing condition that we have been able to find. As with π, if we write τ without its arguments, then it designates the general function depending on all its arguments.

THEOREM 2. *The following two statements are equivalent*:
(i) *Constant discounted value holds.*
(ii) $\pi = \pi(\varphi, t) = \tau(\varphi, t+1) = \tau$.[6]
In Statement (i), *the discount factor* λ (λ *as in Equation* (5)) *is uniquely determined.* $\square$

Condition 2(ii) entails that π and τ are independent of the endowments and that tomorrow's perception of future income is the same as today's. The condition implies that PV depends only on stopwatch time (time differences) and not on calendar time (absolute time).

Statement 2(ii) formulates the common stationarity in a simplified manner for the case of linear utility. Only one future outcome φ and one present value today (π) or tomorrow (τ) are involved, rather than involving general preferences between general outcome sequences as in common formulations. Most tests of stationarity in the literature are, in fact, tests of our simplified condition (see Takeuchi 2010 and his extensive survey), which captures the essence of the condition.

Many studies have shown that constant discounting is violated empirically. Hence the following generalization is of interest:

Time-Dependent Discounted Value

$$\sum_{t=0}^{T} \lambda_t x_t \quad (\textit{with } \lambda_0 = 1). \tag{7}$$

The weights λ_t are all positive by monotonicity. This model allows for general discount weights with unrestricted time dependence. Many special cases of such discount weights have been studied in the literature, the best known being hyperbolic discounting. The representation in Equation (7) is not affected if all λ_t's are multiplied by the same positive factor. The common scaling $\lambda_0 = 1$ is therefore always possible.

THEOREM 3. *The following two statements are equivalent*:
(i) *Time-dependent discounted value holds.*
(ii) $\pi = \pi(\varphi, t)$.
In Statement (i), *the discount factors* λ_t (λ_t *as in Equation* (7)) *are uniquely determined.* $\square$

An implication that can be inferred from the last two columns of Table 1 is that if π is any function of φ and t, then it must be the function $\lambda_t \times \varphi$.

4. Nonlinear Utility

The models presented in the preceding section take a weighted or unweighted sum of the outcomes. They assume constant marginal utility in the sense that an extra euro received in a particular period gives the same utility increment regardless of the endowment of that period. In individual choice, unlike market pricing, this condition is often violated empirically and it is not normative. More realistic and more popular models allow for nonlinear utility; then marginal utility depends on the endowment, and the models of the preceding section become

$$\textit{Nondiscounted Utility}: \quad \sum_{t=0}^{T} U(x_t); \tag{8}$$

$$\textit{Constant Discounted Utility}: \quad \sum_{t=0}^{T} \lambda^t U(x_t); \tag{9}$$

$$\textit{Time-Dependent Discounted Utility}: \quad \sum_{t=0}^{T} U_t(x_t). \tag{10}$$

Continuity and monotonicity of $\succcurlyeq$ readily imply $\lambda > 0$ and strict increasingness and continuity of U and all U_t's. Equation (9) is Samuelson's (1937) discounted utility, the most popular model for intertemporal choice. Each utility model reduces to the corresponding value model if utility is linear. In particular, in time-dependent discounted utility, if the $U_t(\alpha)$ are linear, they can be written as $\lambda_t \times \alpha$ and we can renormalize them such that $\lambda_0 = 1$, resulting in time-dependent discounted value results.

The mathematics underlying the preference axiomatizations of the utility models in Equations (8)–(10) is more advanced than for Theorems 1–3. Whereas these theorems solved linear equalities, we now have to deal with nonlinear equalities, with nonlinear utilities intervening. Fortunately, this increased mathematical complexity does not show up in the preference conditions and, consequently, in the empirical tests of the models. The relevant PV preference conditions are obtained directly from those defined in §3 by adding dependence on the endowment levels e_0, e_t. This way we readily obtain Theorems 4–6 from Theorems 1–3, respectively.

Theorem 4. *The following two statements are equivalent*:
(i) *Nondiscounted utility holds.*
(ii) $\pi = \pi(\varphi, e_0, e_t)$.[7]
The following uniqueness result holds for Statement (i): *A real-valued time-independent constant μ can be added to U (U as in Equation* (8)), *and U can be multiplied by a positive constant ν.* □

An implication, displayed in the last two columns of Table 1, is that if π is any function of φ, e_0, and e_t, then it must be of the form displayed there.

Several authors have argued that any discounting, even if consistent over time, is irrational and have thus recommended using nondiscounted utility for intertemporal choice (Jevons 1871, pp. 72–73; Ramsey 1928; Rawls 1971). Condition 4(ii) characterizes this proposal. Studies providing preference axiomatizations for nondiscounted utility (sums of utilities) include Kopylov (2010), Krantz et al. (1971), Marinacci (1998), Pivato (2014), and Wakker (1986). We now turn to discounting.

Theorem 5. *The following two statements are equivalent*:
(i) *Constant discounted utility holds.*
(ii) $\pi = \pi(\varphi, t, e_0, e_t) = \pi(\varphi, t+1, e_1' = e_0, e_{t+1}' = e_t)$.[8]
The following uniqueness result holds for Statement (i): *A real-valued time-independent constant μ can be added to U (U and λ as in Equation* (9)), *and U can be multiplied by a positive constant ν. λ is uniquely determined.* □

The first preference axiomatization of constant discounted utility was in Koopmans (1960), with generalizations in Harvey (1995), Bleichrodt et al. (2008), and Kopylov (2010). Condition 5(ii) requires that the same trade-offs are made tomorrow as today. Such requirements have sometimes been taken as rationality requirements (see the introduction). The following theorem generalizes Theorem 3.

Theorem 6. *The following two statements are equivalent*:
(i) *Time-dependent discounted utility holds.*
(ii) $\pi = \pi(\varphi, t, e_0, e_t)$.
The following uniqueness result holds for Statement (i): *A time-dependent real constant μ_t can be added to every U_t (U_t as in Equation* (10)), *and all U_t's can be multiplied by a joint positive constant ν.* □

Statement 6(ii) expresses that the trade-offs between periods 0 and t are independent of what happens in the other periods, reflecting a kind of separability. Time-dependent discounted utility is a general additive representation, which has been axiomatized several times before.[9] It implies intertemporal separability, which is arguably the most questionable assumption of most intertemporal choice models (Baucells and Sarin 2007, Dolan and Kahneman 2008).

Another generalization of time-dependent discounted value (Equation (7)) can be considered that is intermediate between Equations (9) and (10), being

$$\sum_{t=0}^{T} \lambda_t U(x_t). \tag{11}$$

We have not yet succeeded in finding an appealing present value condition for this representation.

5. Applications to Contexts Other Than Intertemporal Choice

The mathematical results of the previous sections and the preference conditions used can be applied in contexts other than intertemporal choice. For instance, Theorem 3 is of special interest for decision under uncertainty, capturing nonarbitrage in finance. To see this point, we reinterpret the periods t as *states of nature*. Exactly one state obtains, but it is uncertain which one (Savage 1954). Now $x = (x_0, \ldots, x_T)$ refers to an uncertain prospect yielding outcomes x_t if state of nature t obtains. For simplicity, we focus on a single time for all outcomes here so that discounting plays no role. The next section will consider both uncertainty and time. If we divide the discount weights λ_t by their sum $\sum_{t=0}^{T} \lambda_t$ (relaxing the requirement of $\lambda_0 = 1$), they sum to 1, and the representation becomes subjective expected value.

Subjective expected value was first axiomatized by de Finetti (1937) using a no-book argument, which is equivalent to the no-arbitrage condition of finance. In finance, the representation is as-if risk neutral, and the decision maker is the market that sets rational prices for state-contingent assets. For state j, a state-contingent asset $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ yields outcome 1 if j happens and nothing otherwise. In this interpretation, λ_j becomes the market price of this state-contingent asset, and PV is the offsetting quantity of state-0-contingent assets. Condition 3(ii) provides the most concise formulation of the no-book and the no-arbitrage principle presently available in the literature.

For decision under uncertainty, certainty equivalents are more natural quantities than state-contingent prices. Reformulating our conditions in terms of certainty equivalents is a topic for future research. For decision under risk, Equation (8) can be interpreted as von Neumann-Morgenstern expected utility for equal-probability lotteries, which essentially covers all lotteries with rational probabilities (writing every probability i/j as i probabilities $1/j$). Equation (8) can also be interpreted as ambiguity under complete absence of information (Gravel et al. 2012). Equation (10) is Debreu's (1960) additively separable utility. Here again, our Statement 6(ii) provides the most concise preference axiomatization presently available in the literature.

6. Time and Uncertainty: Aggregating Over Two Dimensions

This section applies our technique to aggregations over two dimensions. We consider the special case where one dimension refers to time and the other refers to uncertainty. In applications, usually both time and uncertainty play a role (Smith and McCardle 1999). We assume periods $0, \ldots, T$ and states of nature $0, \ldots, n$. Exactly one state is true but the decision maker is uncertain which one. We consider $(T+1) \times (n+1)$ tuples $(x_{0,0}, \ldots, x_{T,n})$ yielding outcome $x_{t,s}$ in period t if state of nature s is true. Such tuples are called *act sequences*. Thus, every period yields an *act* (map from states to $\mathbb{R}$), and every state of nature yields an outcome sequence. *Constant discounted expected value* is

$$\sum_{t=0}^{T} \sum_{s=0}^{n} \lambda^t p_s x_{t,s} \tag{12}$$

with $\lambda > 0$, $p_s > 0$ for all s, and $\sum p_s = 1$. Constant discounted expected value is the common evaluation system used in cost-effectiveness studies and by financial markets. In the latter case, the p_js and λ are the parameters. They are subjective from the market perspective. The evaluation formula is both arbitrage-free and time consistent (under the common time invariance).

We use state-contingent present values, defined as follows, and using payoffs in state 0 and period 0 for calibration: $\pi = \pi(\varphi, t, s, e_{0,0}, e_{0,1}, \ldots, e_{T,n})$ is such that

$$\begin{aligned}&(e_{0,0} + \pi, e_{0,1}, \ldots, e_{t,s}, \ldots, e_{T,n})\\ &\quad \sim (e_{0,0}, e_{0,1}, \ldots, e_{t,s} + \varphi, \ldots, e_{T,n}).\end{aligned} \tag{13}$$

The following reinforcement of monotonicity is common in decision under uncertainty. First, we identify a sure outcome sequence $(x_0, \ldots, x_T)$ with the act sequence that assigns x_t to each (t, s), and we induce preferences over outcome sequences ($T+1$ tuples) this way. Second, we define *dominance* to hold if (a) preferences over outcome sequences satisfy monotonicity and (b) replacing an outcome sequence contingent on a state of nature s by a weakly (strictly) preferred outcome sequence leads to a weakly (strictly) preferred $(T+1) \times (n+1)$ tuple. In the next theorem, we use the same notation for tomorrow's value τ as in §3, but now it is state contingent. That is, we now use payoffs in state 0 and period 1 (tomorrow) for calibration: $\tau = \tau(\varphi, t, s, e_{0,0}, e_{0,1}, \ldots, e_{T,n})$ is such that

$$\begin{aligned}&(e_{0,0}, \ldots, e_{0,n}, e_{1,0} + \tau, e_{1,1} \ldots, e_{t,s}, \ldots, e_{T,n})\\ &\quad \sim (e_{0,0}, e_{0,1}, \ldots, e_{t,s} + \varphi, \ldots, e_{T,n}).\end{aligned} \tag{14}$$

Theorem 7. *Assume that $\succcurlyeq$ is a binary relation on $\mathbb{R}^{(T+1)\times(n+1)}$. It is represented by constant discounted expected value if and only if it is a continuous weak order satisfying dominance and*

$$\pi(\varphi, t, s) = \tau(\varphi, t+1, s). \tag{15}$$

The parameters $\lambda, p_1, \ldots, p_n$ (as in Equation (12)) are uniquely determined. □

For extending this result to nonlinear utility, expected utility for the aggregation over the states of nature (using an analog of Equation (11)) is of special interest. We leave this as a topic for future research. There is currently much interest in models with both risk and time and their interactions. Baucells and Heukamp (2012) proposed a general decision model. As in the preceding section, it is also desirable to obtain results in terms of present certainty equivalents rather than in terms of present contingent payments here. For example, Smith (1998) considered a combination of risk and time in a theoretical study, combining present values with certainty equivalents, Ahlbrecht and Weber (1997) did the same in an experimental study, and Pelsser and Stadje (2014) considered market pricings as in Theorem 7.

7. Proofs and Clarification of the Empirical Status of PV Conditions

We first present the proofs of Theorems 1–6. We present them from most to least general because this approach is most clarifying and most efficient. The presentation of the proofs clarifies the relationship between our PV conditions and well-known preference conditions, showing that PV conditions indeed are preference conditions. In each proof, we start from our PV condition, which is always weaker than the conditions that are derived and that are commonly used in the literature. We thus show that our PV conditions give stronger results. Because each Statement (ii) is immediately implied by substitution of the functional, we throughout assume Statement (ii) and derive Statement (i) and the uniqueness results.

Proof of Theorem 6. The uniqueness results for Statement 6(i), which uses Equation (10), follow from well-known uniqueness results in the literature (Krantz et al. 1971, Theorem 6.13; Wakker 1989, Observation III.6.6). We next derive Statement 6(i).

The equality $\pi(\varphi, t, e) = \pi(\varphi, t, e_0, e_t)$ in Condition 6(ii) means that π is independent of e_j, $j \neq 0, t$. This holds if and only if

$$\begin{aligned}&(e_0 + \pi, e_1, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_T)\\ &\quad \sim (e_0, e_1, \ldots, e_{t-1}, e_t + \varphi, e_{t+1}, \ldots, e_T)\\ &\Rightarrow (e_0 + \pi, e'_1, \ldots, e'_{t-1}, e_t, e'_{t+1}, \ldots, e'_T)\\ &\quad \sim (e_0, e'_1, \ldots, e'_{t-1}, e_t + \varphi, e'_{t+1}, \ldots, e'_T).\end{aligned} \tag{16}$$

This holds if and only if the implication holds with twice preference $\succcurlyeq$ instead of indifference $\sim$.[10] Equation (16) with preference instead of indifference is known as *separability* of $\{0, t\}$ (Gorman 1968). By repeated application of Gorman (1968), separability of every set $\{0, t\}$ holds if and only if $\succcurlyeq$ is *separable*; i.e., every subset of $\{0, \ldots, T\}$ is *separable* (preferences are independent of the levels where outcomes outside this subset are kept fixed, as with separability of $\{0, t\}$). This holds if and only if an additively decomposable representation holds,[11] which we call time-dependent discounted utility in the main text. □

Proof of Theorem 5. The uniqueness results for Statement 5(i), which uses Equation (9), follow from those in Theorem 6, where, in terms of Equation (10), λ is the proportion U_{t+1}/U_t for any t. It is useful to note that the sum of weights, $\sum \lambda^t$, is the same for each outcome sequence, implying that there is no special role for utility value 0. We next derive Statement 5(i).

The equality $\pi(\varphi, t, e_0, e_t) = \tau(\varphi, t+1, e'_1 = e_0, e'_{t+1} = e_t)$ in Condition 5(ii) means that π is independent of e_j, $j \neq 0$, t (as in Theorem 6, but now only for $t < T$) but also of whether it is measured in period 0 or period 1. This holds if and only if, writing α for $e_0 = e'_1$ and β for $e_t = e'_{t+1}$,

$$\begin{aligned}
&(\alpha+\pi, e_1, \ldots, e_{t-1}, \beta, e_{t+1}, \ldots, e_T)\\
&\quad\sim(\alpha, e_1, \ldots, e_{t-1}, \beta+\varphi, e_{t+1}, \ldots, e_T)\\
&\Leftrightarrow (e'_0, \alpha+\pi, \ldots, e'_{t-1}, e'_t, \beta, \ldots, e'_T)\\
&\quad\sim(e'_0, \alpha, \ldots, e'_{t-1}, e'_t, \beta+\varphi, \ldots, e'_T). \qquad (17)
\end{aligned}$$

It implies separability of $\{0, t\}$ for all $t < T$, as in Theorem 6, but now instead of separability of $\{0, T\}$ we have separability of all $\{1, t+1\}$ for all $t < T$. The latter separability can, for instance, be seen by replacing all primes in Equation (17) by double primes, which should not affect the second indifference because of the maintained equivalence with the first indifference. By repeated application of Gorman (1968), we still get separability of $\succcurlyeq$. Hence the above condition holds if and only if: time-dependent discounted utility holds with, further, U_0, U_t additively representing the same preference relation over $\mathbb{R}^2$ as U_1, U_{t+1} do. We can set $U_t(0) = 0$ for all t. Then by standard uniqueness results (Wakker 1989, Observation III.6.6′), $U_1/U_0 = U_{t+1}/U_t = \lambda$ for a positive constant λ. This proves the equivalence in Theorem 5. Because Condition 5(ii) implies constant discounted utility, it implies *stationarity*, used by Koopmans (1960) to axiomatize the model. The latter condition is defined as follows:

$$\begin{aligned}
&(x_0, x_1, \ldots, x_{T-1}, c_T) \succcurlyeq (y_0, y_1, \ldots, y_{T-1}, c_T)\\
&\quad\Leftrightarrow\; (c_0, x_0, \ldots, x_{T-1}) \succcurlyeq (c_0, y_0, \ldots, y_{T-1}). \qquad (18)
\end{aligned}$$

Our condition is weaker by considering trade-offs between two periods, keeping the outcomes in all other periods fixed. □

Proof of Theorem 4. The uniqueness results for Statement 4(i), which uses Equation (8), follow from those in Theorem 5. We next derive Statement 4(i).

The equality $\pi(\varphi, t, e) = \pi(\varphi, e_0, e_t)$ in Condition 4(ii) means that π is not only independent of e_j, $j \neq 0$, t, as in Theorem 6, but also of t. This holds if and only if[12]

$$\begin{aligned}
&(e_0+\pi, e_1, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_T)\\
&\quad\sim(e_0, e_1, \ldots, e_{t-1}, e_t+\varphi, e_{t+1}, \ldots, e_T)\\
&\Rightarrow\; (e_0+\pi, e'_1, \ldots, e'_{t'-1}, (e_t)_{t'}, e'_{t'+1}, \ldots, e'_T)\\
&\quad\sim(e_0, e'_1, \ldots, e'_{t'-1}, (e_t)_{t'}+\varphi, e'_{t'+1}, \ldots, e'_T). \qquad (19)
\end{aligned}$$

It readily follows that the above condition holds if and only if we have all the conditions of Theorem 6 and its representation, with the extra condition $U_t(e_t+\varphi) - U_t(e_t) = U_{t'}(e_t+\varphi) - U_{t'}(e_t)$, implying that we can take all functions U_t the same, independent of t. It implies symmetry, the condition commonly used in the literature to axiomatize nondiscounted utility. *Symmetry* requires invariance of preference under every permutation of the outcomes. Symmetry immediately implies that π is independent of t, which is what Condition 4(ii) adds to Condition 6(ii). This shows once again that the PV conditions are weak compared to conditions commonly used in the literature. □

Proof of Theorems 1–3. The uniqueness of the discount parameters in Theorems 2 and 3, based on Equations (5) and (7), follows from the uniqueness results of Theorems 6 and 5. We next derive the Statements (i).

The proof of Theorem 3 [2, 1] readily follows from Theorem 6 [5, 4] as follows. The theorems to be proved are the linear counterparts of the theorems from which they follow. The preference conditions are always the same except that dependence of the endowment levels e_0, e_t has been dropped. In the notation of Theorem 6 this means that $U_t(e_t+\varphi) - U_t(e_t)$ is independent of e_t, which implies linearity of U_t. Similarly, the utility functions in Theorems 5 and 4 are linear. Then Theorems 3, 2, and 1 follow. □

For completeness, we show how the conditions of Theorems 1–3 can be restated directly in terms of preferences:

The equality $\pi = \pi(\varphi, t)$ in Condition 3(ii) holds if and only if

$$\begin{aligned}
&(e_0+\pi, e_1, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_T)\\
&\quad\sim(e_0, e_1, \ldots, e_{t-1}, e_t+\varphi, e_{t+1}, \ldots, e_T)\\
&\Rightarrow\; (e'_0+\pi, e'_1, \ldots, e'_{t-1}, e'_t, e'_{t+1}, \ldots, e'_T)\\
&\quad\sim(e'_0, e'_1, \ldots, e'_{t-1}, e'_t+\varphi, e'_{t+1}, \ldots, e'_T). \qquad (20)
\end{aligned}$$

The equality $\pi = \pi(\varphi, t) = \tau(\varphi, t+1)$ in Condition 2(ii) holds if and only if

$$\begin{aligned}
&(e_0+\pi, e_1, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_T)\\
&\quad\sim(e_0, e_1, \ldots, e_{t-1}, e_t+\varphi, e_{t+1}, \ldots, e_T)\\
&\Leftrightarrow\; (e'_0, e'_1+\pi, \ldots, e'_{t-1}, e'_t, e'_{t+1}, \ldots, e'_T)\\
&\quad\sim(e'_0, e'_1, \ldots, e'_{t-1}, e'_t, e'_{t+1}+\varphi, \ldots, e'_T). \qquad (21)
\end{aligned}$$

The equality $\pi = \pi(\varphi)$ in Condition 1(ii) holds if and only if

$$\begin{aligned}
&(e_0+\pi, e_1, \ldots, e_{t-1}, e_t, e_{t+1}, \ldots, e_T)\\
&\quad\sim(e_0, e_1, \ldots, e_{t-1}, e_t+\varphi, e_{t+1}, \ldots, e_T)\\
&\Rightarrow\; (e'_0+\pi, e'_1, \ldots, e'_{t'-1}, e_{t'}, e'_{t'+1}, \ldots, e'_T)\\
&\quad\sim(e'_0, e'_1, \ldots, e'_{t'-1}, e_{t'}+\varphi, e'_{t'+1}, \ldots, e'_T). \qquad (22)
\end{aligned}$$

We next compare the PV conditions in Theorems 1–3 with other conditions used in the literature to axiomatize the models in question. Condition 3(ii) implies that the extra value of an extra outcome φ is independent of the level e_t to which it is added. This is implied by the well known *additivity* condition, requiring that a preference $x \succcurlyeq y$ is not affected by adding the same constant α to x_t and y_t. By adding the same constant to e_t and $e_t + \varphi$, we can change them into e'_t and $e'_t + \varphi$, implying our preference condition. Additivity is necessary and sufficient for time-dependent discounted utility (Wakker 2010, Theorem 1.6.1). It is more restrictive than our condition because we only consider trade-offs between two periods, keeping the outcomes in all other periods fixed. Similar observations apply to the elementary Theorem 2 and the trivial Theorem 1.

Proof of Theorem 7. As before, necessity of the preference conditions is obvious, so we assume the preference conditions and derive constant discounted expected value and the uniqueness results.

Statement (ii) implies that π can be written as $\pi(\varphi, t, s)$ and is independent of the endowments. By Theorem 3, treating the paired indexes (t, s) as one index with index $(0, 0)$ here playing the role of index 0 in Theorem 3, we obtain a linear representation $\sum_{t=0}^{T}\sum_{s=0}^{n}\mu_{t,s}x_{t,s}$. We do not impose the restriction that $\mu_{0,0} = 1$ here and, hence, the weights are uniquely determined up to one common positive factor.

By dominance, for every fixed s we have the same preference relation over outcome sequences. Hence, each such preference relation is represented by a positive constant (depending on s) times $\sum_{t=0}^{T}\mu_{t,0}x_{t,s}$. This follows from the uniqueness result of Theorem 3, now applied with s kept fixed and with the requirement $\lambda_0 = 1$ (here $\mu_{0,0}$) dropped. We can rewrite the representation as $\sum_{t=0}^{T}\sum_{s=0}^{n}\lambda_t p_s x_{t,s}$ with the p_s's summing to 1 and, hence, uniquely determined. We can renormalize further so that $\lambda_0 = 1$, after which all weights are uniquely determined. By Theorem 2, $\lambda_t = \lambda^t$ for $\lambda = \lambda_1$. Thus constant discounted expected value holds and uniqueness of the weights has also been established. □

8. Discussion

The primary purpose of preference axiomatizations is to make decision models with theoretical constructs directly observable, by restating their existence (Statements (i) in our theorems) in terms of preference conditions (Statements (ii) in our theorems). The simpler the preference conditions, the better they clarify the empirical meaning of the decision models. Similarity between the preference conditions and the functional helps to clarify the empirical meaning of the decision model. Hence this paper has introduced preference axiomatizations that are as simple as possible and that reflect the corresponding decision models as well and transparently as possible.

The conditions in our Statements (ii) use fewer words and characters than any conditions previously proposed in the literature, which provides an objective criterion for our claim that they are the most concise conditions presently existing. Further, we think that our conditions are easy to understand and test because present values are familiar objects. PVs can be used as the goal functions to be optimized in intertemporal choice. At the same time, they are directly defined in terms of preferences and, hence, the subjective and behavioral character of preference axiomatizations is not lost by using PVs. Our efficient results are based on this dual nature of PVs.

We used tomorrow's values in the characterizations of constant discounting, but conditions entirely in terms of present values are also possible. For example, $\pi(\varphi, t) = \sqrt{\pi(\varphi, t-1)\pi(\varphi, t+1)}$ characterizes constant discounted value.[13] We were unable to find an easy way to extend this condition to nonlinear utility. An alternative condition is $\pi(\varphi, t) = \pi(\pi(\varphi, 1), t-1)$, reflecting that the recursive structure at period t should be the same as at period 1 under constant discounting. The condition can be extended to nonlinear utility by specifying the relevant e levels: $\pi(\varphi, t, e_0, e_t) = \pi(\pi(\varphi, 1, e'_0 = e_{t-1}, e'_1 = e_t), t-1, e_0, e_{t-1})$. Because of the many e levels, the latter condition is not very transparent. Our formulations using tomorrow's value are more transparent. This case suggests that it is not always easy to find simple reformulations of preference conditions in terms of present values. We were also unable to find an easy condition in terms of present values for the representation $\sum_{t=0}^{T}\lambda_t U(x_t)$ (Equation (11)).

The preference conditions directly corresponding with our present value conditions are weaker (leading to stronger theorems) than the ones commonly used in the literature. First, our present value conditions relate to indifferences rather than preferences. Conditions for indifferences are logically weaker, making their implications logically stronger.[14] Second, our conditions only involve the simplest trade-offs possible, involving the change of the present outcome and one future outcome. This further enhances their generality. For example, commonly used stationarity conditions are more restrictive than are the conditions in our Statements 4(ii) and 5(ii). The derivations of the full force preference conditions used in the literature from our preference conditions are based on known techniques (including Gorman 1968).

An empirical advantage of our preference conditions is that they can be directly tested using statistical techniques such as analyses of variance and regressions. For example, if we take PV as the dependent variable, Equation (3) predicts that φ, t, e_0, and e_1 may be significant predictors, but the e_j's with $j \neq 0$, t are not. We can test this prediction using standard regression analyses. These allow us to use the sophisticated probabilistic error theories underlying econometric regressions, which are easier to use than the more recently developed error theories for preferences (Wilcox 2008). There is extensive data on the present values of future options in the financial market that can be used to test the various independence conditions proposed in this paper. For individual choice, we are not aware of tests of preference axioms using regression techniques. Such tests become possible through the theorems presented here.

In the main text, we confined our analysis to periods with upper bound T. Many papers have studied extensions of representations to infinitely many periods. Usually, in the first stage representation results are established for finitely many periods. Then in the next stage, the extension to infinitely many periods, continuity conditions are added to avoid diverging or undefined summations. Such two-stage techniques can readily be used to extend our results to infinitely many periods, where we can simply copy the second stage of previous analyses. An advanced general reference is Pivato (2014). Further references for nondiscounted utility include Alcantud and Dubey (2014), Basu and Mitra (2007), and Marinacci (1998); for constant discounted utility, see Harvey (1995), Bleichrodt et al. (2008), and Kopylov (2010); for time-dependent discounted utility, see Hübner and Suck (1993), Streufert (1995), and Wakker and Zank (1999).

We assumed that present values always exist, which implies that utility (in period 0) is unbounded from both sides. These restrictions can be dropped if we modify the preference conditions to hold only if all present values involved exist. In proofs of theorems, we first obtain the preference conditions in full force for every outcome sequence only in a neighborhood of that outcome sequence. This neighborhood is small enough to ensure that all present values required there exist. Next we combine these local representations into one global representation using the techniques of Chateauneuf and Wakker (1993). Their technique works for our most general model, time-dependent discounted utility and, hence, covers all cases considered in this paper.

The follow-up paper Keskin (2015) provides extensions of our results to some popular hyperbolic discount models, while still maintaining intertemporal separability. Extensions to more general intertemporal models are a topic for future research, as are further extensions to other optimization contexts.

9. Conclusion

We have introduced a new kind of preference condition for intertemporal choice that requires (quantitative) present values to be independent of particular other variables. The quantitative index should be directly observable so that the independence requirements are observable preference conditions that can be directly tested qualitatively and can be used in theoretical preference axiomatizations. Unlike usual preference conditions, our conditions and their independence requirements can also be directly tested quantitatively. Our conditions are more concise and transparent than conditions proposed before in the literature, and they are weaker, leading to stronger theorems. The technique of expressing preference conditions as independence conditions for directly observable quantitative indexes can be extended to other decision contexts such as decision under uncertainty to give new concise conditions that can easily be tested empirically.

Endnotes

1. See Attema (2012); Dolan and Kahneman (2008, p. 228); Epper et al. (2011); Frederick et al. (2002); Keller and Kirkwood (1999); Loewenstein and Prelec (1993); and Tsuchiya and Dolan (2005).
2. See Broome (1991); Gold et al. (1996; p. 100), Parfit (1984, Ch. 14); and Strotz (1956, p. 178).
3. See de Wit (1671), Fisher (1930), and Smith and McCardle (1999). Present values are used to compute a company's value when determining stock prices (LeRoy and Porter 1981) and to make investment decisions (Ingersoll and Ross 1992). In such financial decisions at firm or market levels, utility is usually assumed to be linear and discount rates follow market interest rates.
4. In individual choice experiments, indifferences between a future stream of outcomes and an immediate outcome (the present value) are usually obtained using choice lists. Ahlbrecht and Weber (1997) used both choice lists and direct matching to measure present values. Reviews are in Frederick et al. (2002) and Soman et al. (2005).
5. Predictors are often called "independent variables." We avoid this term so as to avoid confusion.
6. The notation here is short for $\pi = \pi(\varphi, t, e) = \pi(\varphi, t) = \tau(\varphi, t+1) = \tau(\varphi, t+1, e')$. The two endowments e and e' are immaterial and are allowed to be different. Because of the shift by one period, the condition is imposed only for all $t < T$. Existence of π and the equality in Statement 2(ii) imply that τ also exists.
7. Here, t refers to the period where φ is added and e_t specifies the endowment in that period. Note that whereas π depends on the level of e_t, it does not depend explicitly on period t, which we denote by suppressing t from the arguments of $\pi(\cdots)$.
8. The condition implies that the endowment levels e_j, $j \neq 0$, t of the PV endowment e, and similarly the endowment levels e'_j, $j \neq 1$, $t+1$ of the tomorrow-value endowment e', are immaterial. Existence of π and the equality imply that τ also exists. The condition holds for all $0 \leqslant t \leqslant T-1$, and whenever $e'_1 = e_0$ and $e'_{t+1} = e_t$. We use the convenient argument matching notation popular in programming languages (R: see R Core Team, http://www.R-project.org/; Python: see Python Software Foundation, http://www.python.org; Scala: see École Polytechnique Fédérale de Lausanne, http://www.scala-lang.org, among others) to express the latter two restrictions. Here the formal argument of a function (e'_1 or e'_{t+1}) is assigned a value (e_0 or e_t).
9. See Debreu (1960), Gorman (1968), Krantz et al. (1971), and Wakker (1989).
10. To wit, if the upper indifference is changed into a strict preference, then we find $\pi' < \pi$ to give indifference. The lower indifference follows with π' instead of π. By monotonicity, replacing π' by π leads to the lower strict preference.
11. See Debreu (1960); Gorman (1968); Krantz et al. (1971, Theorem 6.13); and Wakker (1989, Theorem III.6.6).
12. In the following equation we write $(e_t)_{t'}$ for $e'_{t'}$ to indicate that the t' level of e' is the same as e_t, the tth level of e.
13. This condition was suggested by a referee.
14. They follow from preference conditions by applying the latter twice, first with preferences one way and then with preferences reversed.

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