OPERATIONS RESEARCH
Vol. 62, No. 1, January–February 2014, pp. 207–218
ISSN 0030-364X (print) | ISSN 1526-5463 (online)
http://dx.doi.org/10.1287/opre.2013.1230

Average Utility Maximization: A Preference Foundation

Amit Kothiyal
Max Planck Institute for Human Development, 14195 Berlin, Germany, kothiyal@mpib-berlin.mpg.de

Vitalie Spinu
Anderson School of Management, University of California, Los Angeles, Los Angeles, California 90095, spinuvit@gmail.com

Peter P. Wakker
Erasmus School of Economics, Erasmus University, Rotterdam 3000 DR, The Netherlands, wakker@ese.eur.nl

This paper provides necessary and sufficient preference conditions for average utility maximization over sequences of variable length. We obtain full generality by using a new algebraic technique that exploits the richness structure naturally provided by the variable length of the sequences. Thus we generalize many preceding results in the literature. For example, continuity in outcomes, a condition needed in other approaches, now is an option rather than a requirement. Applications to expected utility, decisions under ambiguity, welfare evaluations for variable population size, discounted utility, and quasilinear means in functional analysis are presented.

Subject classifications: utility; multiattribute; mathematics functions; decision analysis; multiple criteria.
Area of review: Decision Analysis.
History: Received May 3, 2013; revision received September 6, 2013; accepted October 3, 2013. Published online in *Articles in Advance* January 29, 2014.

1. Introduction

This paper considers preferences over sequences $(x_1, \ldots, x_n)$ of variable length n. We provide a preference foundation of average utility representations $(\sum_{j=1}^{n} U(x_j)/n)$. Average utility describes, for instance, (a) a representative agent in utilitarian welfare evaluations with variable population size (Blackorby et al. 2005, Harsanyi 1955); (b) the price index when different countries have different basic commodities (Balk 1995); (c) the quality of life of different medical treatments tested on different samples (Weinstein et al. 1980); (d) the average happiness in different countries (van Praag and Ferrer-i-Carbonell 2004); (e) decisions under complete ignorance (Gravel et al. 2012); (f) the subjectively perceived nuisance of waiting times (Carmon et al. 1995); and so on. Averages of transformed observations serve as summary indices in statistics (Norris 1976) and in many other contexts.

Because an exponential transformation of a representing function does not affect preference, our analysis also covers geometric means. Thus we can, for instance, cover geometric averages of discount rates rather than the arithmetic average of discount factors when determining present values. For simplicity, we will consider arithmetic averages in what follows.

In our preference foundations, we take optimal choices between sequences, represented by a (binary) preference relation over the sequences, as empirical primitive. We then give necessary and sufficient conditions in terms of those preferences for representability by average utility. Preference foundations serve to justify or falsify the appropriateness of particular kinds of representing functions (Keeney and Raiffa 1976). Previous preference foundations first used results for sequences of fixed length n, where continuity in outcomes is needed to scale utility U, and then extended their representation to variable length.[1] We show that the variable length, available anyhow, in fact simplifies the analysis. It provides all the richness needed to scale utility. Using the variable length, we can define a concatenation operation that pastes sequences together and then use Hölder's (1901) simple but powerful lemma to scale utility. Continuity in outcomes is simply redundant.

It is well known that without any cardinal-type operation (like addition, or taking differences or averages), no cardinal representation can be obtained. This led the committee of Ferguson et al. (1940), assuming that no cardinal-type operations are available in the social sciences, to conclude that cardinal measurement is impossible in social sciences. Thousands of studies have since refuted this conclusion. Our concatenation operation will serve as an addition-type operation. As an example of concatenation in a welfare context, we may consider the average happiness in separate European countries but also in their concatenations (unions), such as in the whole European union. These concatenations are natural and readily available.

The preference conditions for axiomatizing average utility provided in §2 are necessary and sufficient in full generality and generalize all previous results in the literature, comprising dozens of papers. Section 3 shows that although we can do without continuity, we can add it if desirable. That is, in our approach, continuity is optional rather than required. The topological results with continuity comprised in §3 apply to every U with an interval as image, which generalizes Euclidean continuity.

Section 4 presents two applications of our theorems in some detail. The first concerns decision under risk. Our result here in fact shows how Hölder's addition-based lemma can be used to simplify and generalize the commonly used, mixture-based, von Neumann-Morgenstern (1944) derivations of expected utility. The second application concerns the mathematical theory of generalized (or quasilinear) means, showing that the simplicity and generality of our technique has not been known before in the mathematical literature. We generalize results by Aczél (1966), Hardy et al. (1934), Kolmogorov (1930), and many more recent papers. We show again that continuity, commonly assumed here too, need not be imposed as an extra assumption because it is implied by the other assumptions. Section 5 concludes.

Appendix A gives details on how exactly we generalize preceding results in the literature on decision under risk and functional analysis. Appendix B gives details on the literature in other domains of application, in particular welfare models with variable population size. Proofs are in Appendices C through E.

2. General Preference Foundation for Average Utility

X is a nonempty *outcome space*. It can be finite or infinite. Greek letters $\alpha, \beta, \ldots$ and indexed Roman letters x_i denote outcomes. *Prospects* are sequences of the form $x = (x_1, \ldots, x_n) \in X^n$, with $n \in \mathbb{N}$. $\#x = n$ is the length of the prospect. The x_js are *coordinates* of x. $\mathscr{X} = \bigcup_{n \in \mathbb{N}} X^n$ is the set of all prospects. That is, n is variable and we consider finite sequences of any length. The *concatenation* (x, y) denotes $(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Further, $1x = x$, $2x = (x, x)$, and $kx = (x, (k-1)x)$ for all $k > 1$.

A preference relation $\succcurlyeq$ is given on $\mathscr{X}$. The notation $\succ$, $\preccurlyeq$, $\prec$, $\sim$ is as usual. A function M *represents* $\succcurlyeq$ if $M: \mathscr{X} \to \mathbb{R}$ and $x \succcurlyeq y \Leftrightarrow M(x) \geqslant M(y)$. Other terms used in the literature are that $\succcurlyeq$ maximizes the objective function M or that prospects are evaluated by M. If a representing function exists, then $\succcurlyeq$ is a *weak order*; i.e., it is *complete* ($x \succcurlyeq y$ or $y \succcurlyeq x$ for all x, y) and transitive. The axiomatizations provided here concern preference foundations.

We identify outcomes with the corresponding sequences of length 1. The restriction of $\succcurlyeq$ to X resulting this way is also denoted $\succcurlyeq$. A function U *represents* $\succcurlyeq$ *on* X if $U: X \to \mathbb{R}$ and $\alpha \succcurlyeq \beta \Leftrightarrow U(\alpha) \geqslant U(\beta)$.

Definition 1. *Average utility* (*AU*) holds if there exists $U: X \to \mathbb{R}$ such that

$$x \mapsto \frac{\sum_{i=1}^{\#x} U(x_i)}{\#x} \tag{1}$$

represents $\succcurlyeq$. U is the *utility function*.

AU implies that U represents $\succcurlyeq$ on X. It also implies symmetry:

Definition 2. *Symmetry* holds if

$$(x_1, \ldots, x_n) \sim (x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all permutations π.

Symmetry is often called anonymity in welfare evaluations. The following implication is characteristic of both additive ($\sum_{i=1}^{\#x} U(x_i)$) and average utility.

Definition 3. *Joint independence* holds if

$$\begin{aligned} &(c_1, x_2, \ldots, x_n) \succcurlyeq (c_1, y_2, \ldots, y_n) \\ \Rightarrow &(d_1, x_2, \ldots, x_n) \succcurlyeq (d_1, y_2, \ldots, y_n). \end{aligned} \tag{2}$$

The condition implies that preferences between two n-tuples are independent of a common first coordinate. Because of symmetry, preference then is independent of any common coordinate. By repeated application, preference is independent of any number of common coordinates. This condition is often called separability. We do not use this term to avoid confusion with a topological condition of the same name defined later. Joint independence is called the sure-thing principle in decision under uncertainty (Savage 1954) and is central in nontransitive models of uncertainty (Bell 1982, Loomes and Sugden 1982).

We often use the following generalization of joint independence:

Definition 4. *Expansion independence* holds if

$$x \succcurlyeq y \;\Leftrightarrow\; (\alpha, x) \succcurlyeq (\alpha, y) \quad \text{whenever } \#x = \#y. \tag{3}$$

In the presence of symmetry, the condition implies that inserting an extra common coordinate at any place does not affect optimality. By repeated application, inserting any number of common coordinates does not affect optimality. In general, this condition is somewhat stronger than joint independence (Lemma 18) because it links preferences between prospects of different length. In the presence of other conditions that we will use (mainly replication equivalence, which will link preferences between prospects of different lengths), the two conditions become equivalent (Lemma 19).[2] In welfare evaluations, expansion independence means that a choice between welfare allocations over a fixed population need not be altered if a new member, unaffected by the choice, joins. In a mathematical sense, expansion independence amounts to compatibility of preference and sequence concatenation. Our application of Hölder's lemma will be based on this compatibility.

Because real numbers satisfy the Archimedean axiom, such an axiom is necessary for any real-valued representation. The following axiom will be used for average utility.

Definition 5. *The Archimedean axiom* holds if for all $x, y \in \mathscr{X}$ with $\#x = \#y$ and $x \succ y$, and all $v, w \in \mathscr{X}$ with $\#v = \#w$:

$$(nx, v) \succcurlyeq (ny, w)$$

for some $n \in \mathbb{N}$.

In words, a sufficient number n of advantages $x \succ y$ can offset any disadvantage $v \prec w$.

The conditions defined so far are satisfied both by additive and by average utility representations. The following condition is natural for average utility and distinguishes it from additive utility:[3]

Definition 6. *Replication equivalence* holds if $x \sim mx$ for all x, m.

The following theorem is our main result. All the proofs are delegated to appendices.

Theorem 7. *The following two statements are equivalent for $\succcurlyeq$ on $\mathscr{X} = \bigcup_{n=1}^{\infty} X^n$.*

(i) *There exists a utility function U: $X \to \mathbb{R}$ such that average utility* (*Equation* (1)) *represents $\succcurlyeq$.*

(ii) *$\succcurlyeq$ satisfies the following five conditions*:

1. *weak ordering*
2. *joint independence*
3. *symmetry*
4. *replication equivalence, and*
5. *the Archimedean axiom.*

Further, the utility function U in statement (i) *is unique up to level and unit.*

The first four conditions in statement (ii) (the Archimedean axiom not included) will often be called the *intuitive conditions*. The rest of this section informally explains why our theorem can do without additional richness and why this is desirable.

Several authors have pointed out the problematic nature of continuity in preference axiomatizations,[4] adding to the desirability of having it optional. The only preference foundations of average utility that did not use any version of continuity are Fishburn (1972) and Gravel et al. (2012). They both considered finite subsets of a set rather than sequences as we do. Fishburn's preference conditions use a technique by Scott (1964), based on methods for solving linear inequalities. As pointed out by Gravel et al. (2012, end of §3), this leads to complex axioms. The latter paper used a richness Axiom 4 that is preferable to continuity but still has similar observability problems (Wakker 1988b).

Our main theorem combines the advantages of Fishburn's (1972) technique and the papers that used continuity or richness. We achieve complete generality as does Fishburn, leaving continuity optional. At the same time, our intuitive axioms are the simplest and weakest ones used in the literature on continuous or rich representations. As suggested above, we will not first establish fixed-finite dimensional results, but we immediately turn to general dimensions n.

We next illustrate how we can scale cardinal utility in our general setup. Assume outcomes $\gamma \succ \beta \succ \alpha$ and $n \in \mathbb{N}$ ($\succ$ denotes strict preference). We can find k such that the following preferences hold between three n-tuples, where $k\gamma$ denotes k-fold concatenation and not multiplication:

$$\big(k\gamma, \alpha, (n-k-1)\alpha\big) \preccurlyeq n\beta \preccurlyeq \big(k\gamma, \gamma, (n-k-1)\alpha\big). \quad (4)$$

These preferences reveal

$$\frac{k}{n} \leqslant \frac{U(\beta) - U(\alpha)}{U(\gamma) - U(\alpha)} \leqslant \frac{k+1}{n}. \quad (5)$$

In plain words, we count how many advantages $\gamma \succ \beta$ it takes to offset a number of disadvantages $\alpha \prec \beta$. Because n can be taken arbitrarily large, we can thus identify cardinal utility as accurately as we want. This is why we can do without the richness assumption of a continuum domain.

3. Preference Foundation for Continuous Average Utility

This section considers the special case where X is a topological space. It is *connected* if the only sets that are both open and closed are $\varnothing$ and X. It is *separable* if there exists a countable dense subset. A subset is *dense* if every nonempty open set contains an element of the subset. To obtain continuous representations, we use a continuity condition introduced by Gravel et al. (2011). It is the weakest condition used in the literature (see below), leading to the strongest theorems. It is a remarkable weakening of the simple continuity (continuity with respect to finite-dimensional product topologies; see Appendix B.2) assumed in the other papers in the literature.

Definition 8. $\succcurlyeq$ on $\mathscr{X}$ is *continuous* if the sets $\{\alpha \in X: \alpha \succ x\}$ and $\{\alpha \in X: \alpha \prec x\}$ are open for every $x \in \mathscr{X}$.

Theorem 9. *The following two statements are equivalent for $\succcurlyeq$ on $\mathscr{X} = \bigcup_{n=1}^{\infty} X^n$.*

(i) *There exists a utility function U: $X \to \mathbb{R}$ such that average utility* (*Equation* (1)) *represents $\succcurlyeq$, with $U(X)$ an interval.*

(ii) *$\succcurlyeq$ satisfies the following five conditions*:

1. *weak ordering*
2. *joint independence*
3. *symmetry*
4. *replication equivalence, and*
5. *continuity with respect to some connected and separable topology on X.*

Further, the utility function U in statement (i) *is continuous* (*with respect to the topology*[5] *in point* (5) *in statement* (ii)), *and it is unique up to level and unit.*

Theorem 9 differs from Theorem 7 in the following ways. In statement (i) we added that $U(X)$ is an interval.[6] In statement (ii) we replaced the Archimedean axiom with the stronger requirement of continuity of $\succcurlyeq$.

The topological requirements in the theorem are satisfied, for instance, if X is a convex subset of a Euclidean space. Then, given the other conditions, $\succcurlyeq$ is continuous if and only if U is, and the topology can be taken to be the usual Euclidean one. This is the most common case in applications.

Continuity of $\succcurlyeq$ when restricted to the outcome set X is a relatively weak condition. It is, for instance, satisfied whenever $X = \mathbb{R}$ and $\succcurlyeq$ is monotonic, in which case the restriction of $\succcurlyeq$ to the outcome set is simply the natural order on $\mathbb{R}$. Continuity of $\succcurlyeq$ on the domain of all prospects, $\mathscr{X}$, is stronger. It implies the existence of constant equivalents for all prospects. We call $\alpha \in X$ a *constant equivalent* (*CE*) of $x \in \mathscr{X}$ if $\alpha \sim x$. In general, a prospect x can have no CE, or many CEs, which then are all equivalent. The *CE condition* holds if every prospect has a CE.

Lemma 10. *Assume the intuitive conditions* (1–4 *in Theorem* 7). *Continuity of $\succcurlyeq$ on the outcome set X with respect to a connected topology and the CE condition are equivalent to continuity of $\succcurlyeq$ on the set of prospects $\mathscr{X}$ with respect to a connected topology on X.*

4. Applications

This section briefly applies the preceding theorems to two fields, decision under risk and functional analysis. We provide our results in the main text but defer detailed comparisons of the literature to Appendix A.

4.1. Expected Utility for Decision Under Risk

Lotteries designate probability distributions over X. We interpret n-tuples as $1/n$ probability-lotteries. Average utility then is expected utility. Our domain contains every simple rational-probability lottery (Blackorby et al. 1977, p. 354; Grabisch et al. 2011a, §2.3). For example, the lottery $(2/5{:}8, 3/5{:}0)$, where notation is as usual, corresponds with the equally probable five-tuple $(8, 8, 0, 0, 0)$. Thus our theorems axiomatize expected utility for all simple rational-probability lotteries.

In decision under risk, replication equivalence and symmetry are satisfied by definition because they are different ways of writing the same lottery.[7] We obtain the following axiomatization.

Corollary 11. *Expected utility holds on the set of simple rational-probability lotteries if and only if $\succcurlyeq$ is an Archimedean weak order satisfying joint independence.*

The corollary immediately follows from Theorem 7. It generalizes Shepherdson (1980, Theorem 2.3 and Corollary 5.3; see our Appendix A), who considered rational probabilities as we do. Continuity of utility can be characterized using Theorem 9. Corollary 11 can be extended to all lotteries, including those with nonrational probabilities, by reinforcing the Archimedean axiom. We do not elaborate on this point.

The axiomatizations of expected utility provided in the literature so far used a stronger condition than joint independence—NM independence. When all weighted n-tuples (simple lotteries) $(p_1{:}x_1, \ldots, p_n{:}x_n)$, denoted C, P, Q, and so on (with the p_j's possibly irrational) are incorporated in the domain, then *NM-independence* requires

$$P \succcurlyeq Q \Leftrightarrow \lambda C + (1-\lambda)P \succcurlyeq \lambda C + (1-\lambda)Q \quad \text{for all } 0 < \lambda < 1. \qquad (6)$$

Here the probabilistic mixture operation is defined the usual way.[8]

Lemma 12. *NM-independence implies joint independence on the subset of simple rational-probability lotteries.*

Thus traditional derivations of expected utility can be obtained as corollaries from our analysis. We can similarly axiomatize Savage's (1954) subjective expected utility for uncertainty if we relate n-tuples to Savage's uniform partitions. We again do not elaborate on this point.

To illustrate the intuitive nature of our concatenation-based approach, we restate expansion-independence for risk. It requires that a preference between random selections from two different n-tuples of outcomes remain unaffected if the same outcome is added to both n-tuples. This condition is less restrictive, and easier to understand, than is the mixture condition in Equation (6).

4.2. Functional Analysis

Foundations of average utility representations can be obtained if preference theory is linked to the mathematical theory of generalized means.[9] This theory is a subfield of functional analysis (Aczél 1966) and makes the following assumption:

Assumption 13 (Structural Assumption). *$X \subset \mathbb{R}$ is an interval, $\mathscr{X} = \bigcup_{n\in\mathbb{N}} X^n$, and M: $\mathscr{X} \to \mathbb{R}$ is* monotonic (*strictly increasing in each coordinate*).

Taking a function, rather than the preference relation, as primitive, is common in many fields. In production theory (Nicholson 2005), for example, the x_j's are the inputs of a production process, and M is the output quantity. Preference foundations give insights into relationships between properties of production processes and production outputs. For price indices (Balk 1995), the x_j's concern prices of particular commodities, and it is discussed whether life has become more or less expensive. There have been many debates about what proper price indices are, and preference foundations provide arguments for those discussions. The influential Artzner et al. (1999) considered sequences of uncertain outcomes x_j conditional on uncertain events, with M reflecting the degree of riskiness. Bonferroni (1924) introduced the following concept.

DEFINITION 14. M is a *generalized mean* if there exists a strictly increasing function U such that

$$M(x_1, \ldots, x_n) = U^{-1}\Big(\sum U(x_j)/n\Big) \tag{7}$$

(with U^{-1} assumed well defined).[10]

For our purposes it is convenient not to require continuity of U from the outset, although it will follow from the other conditions (Theorem 15). U is the *utility function*.

AU preference theorems can be related to axiomatizations of generalized means by defining M as the CE function of the preference relation $\succcurlyeq$. This way we can derive the theorems presented next from the results of preceding sections, as the proofs in the appendix will demonstrate. We first give direct reformulations of our AU preference conditions for the function M. M satisfies

- *reflexivity* if $M(\alpha) = \alpha$ for all outcomes α;
- *symmetry* if M is invariant under permutations of coordinates;
- *joint independence* if $M(c_1, x_2, \ldots, x_n) \geqslant M(c_1, y_2, \ldots, y_n) \Rightarrow M(d_1, x_2, \ldots, x_n) \geqslant M(d_1, y_2, \ldots, y_n)$;
- *replication equivalence* if $M(x) = M(nx)$ for all n, x.

We need not reformulate the Archimedean axiom because our setup will imply continuity (and hence the Archimedean axiom) as in Theorem 9.

THEOREM 15 (CHARACTERIZING GENERALIZED MEANS). *Assume Structural Assumption* 13. *Then the following two statements are equivalent*:

(i) *M is a generalized mean* (*Equation* (7)).

(ii) *M satisfies reflexivity, symmetry, joint independence, and replication equivalence.*

U in (i) *is continuous and it is unique up to level and unit.*

That U must be continuous can be seen, for instance, because for every $U(\alpha)$ and $U(\gamma)$ in its range, the midpoint $U(\beta)$ for $\beta = M(\alpha, \gamma)$ must also be contained in its range. Thus U, increasing on an interval, cannot have "jumps."

5. Conclusion

We have studied average utility representations on sequences of variable length. Our main contribution is to demonstrate how the richness generated by the variable length can be exploited to simplify measurements (Equation (5)) and axiomatizations. This richness allows for a concatenation operation on sequences to which Hölder's (1901) Lemma can be applied. We thus obtain necessary and sufficient conditions for average utility in full generality, and we generalize all results in the literature for various fields (detailed in Appendix B).

In particular, we show that continuity in outcomes, commonly assumed, is redundant, as it is in von Neumann-Morgenstern expected utility for risk. This point had been overlooked in the literature on sequences of variable length. Given the problematic empirical status of this continuity for preference foundations, our relaxation is desirable.

An obvious topic for future research concerns the use of our technique, exploiting the richness of variable length, to analyze functionals other than AU, including many other welfare criteria for variable population size (Blackorby et al. 2005; Grabisch et al. 2011a, b).

Appendix A. Preceding Literature on Risk Theory and Functional Analysis

This appendix gives details on how the results of §4 generalize preceding results in the literature.

A.1. Risk Theory

The relation between our results and classical derivations of expected utility was explained in the main text. An alternative derivation of expected utility for rational probabilities is in Shepherdson (1980, Theorem 2.3 and Corollary 5.3). His main axiom is NM-independence (Equation (6)) restricted to $\lambda = 0.5$ which, together with a continuity-in-probability that is stronger than the Archimedean axiom, implies NM-independence for all rational probabilities and, thus, joint independence. The domain of simple rational-probability lotteries was used for instance by Peleg and Peters (2009).

Blackorby et al. (1977) considered equally likely lotteries as we do. They assumed $X = \mathbb{R}^n_+$. Their Assumption 3 is simple continuity, implying our continuity. Their Assumption 4 is joint independence.[11] Symmetry is their Definition 1. Replication equivalence is an implication of their implicit assumption (in their Lemma 2) that a prospect is sufficiently described by the subjective probability distribution that it generates over outcomes, a condition sometimes called probabilistic sophistication (Machina and Schmeidler 1992). Thus their main result, Theorem 1, follows from our Theorem 9.

A.2. Functional Analysis

Theorem 15 generalizes results by Kolmogorov (1930) and Nagumo (1930). These authors used an *associativity* condition:

$$M(x_1, \ldots, x_k) = \alpha \Rightarrow M(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = M(k\alpha, x_{k+1}, \ldots, x_n). \tag{A1}$$

Associativity can be equated with the substitution principle of decision under risk, which is a weak version of the NM-independence condition. It allows replacement of a conditional lottery by an equivalent other conditional lottery (such as its CE) without affecting preference. The Blackorby et al. (2005, p. 125) population substitution principle, when imposed on their representative agent, is equivalent. Kolmogorov (1930) and Nagumo (1930) also used *idempotence*, a reinforcement of reflexivity: $M(n\alpha) = \alpha$.[12] It can be seen, under symmetry, that associativity is equivalent to joint independence and replication equivalence and that it implies idempotence. We only prove the implications needed for our analysis. Reversed implications can be derived from Lemma 21, but we will not elaborate on this.

LEMMA 16. *Assume Structural Assumption* 13 *and symmetry. Then associativity implies idempotence, reflexivity, replication equivalence, and joint independence.*

Thus the theorems of Nagumo and Kolmogorov follow as corollaries of Theorem 15. Remarkably, Nagumo and Kolmogorov assumed continuity of M, but this assumption can be dropped. It is implied by the other assumptions, as Theorem 15 showed. This also implies that our continuity condition in §3 entails no restriction for the study of generalized means because it is always implied.

Several authors studied generalized means for weighted prospects $(p_1{:}x_1, \ldots, p_n{:}x_n)$ (Chew 1983; de Finetti 1931; Hardy et al. 1934, §6.20 and Theorem 215; Muliere and Parmigiani 1993). Then the *generalized mean* is defined as $U^{-1}(\sum_{j=1}^{n} p_j U(x_j))$. An obvious interpretation is that the prospects are lotteries and the generalized mean is the CE under expected utility, as in §4.1. Axiomatizations similar to those of Nagumo (1930) and Kolmogorov (1930) were given, using a modified associativity condition:

$$M(P) = M(Q) \Leftrightarrow M\big(\lambda C + (1-\lambda)P\big) = M\big(\lambda C + (1-\lambda)Q\big) \quad \text{for all } 0 < \lambda < 1. \quad \text{(A2)}$$

The condition implies associativity because we can apply it to the case where Q is the CE of P, and then it implies that a conditional subpart of a prospect can be replaced by its CE. The condition is a weakened version of the independence condition (Equation (6)) because it only considers indifferences/equalities. Our Theorem 15 can be used here similarly as in §4.1, with our domain containing all prospects with rational weights. In the context of weighted n-tuples, Hardy et al. (1934, Theorem 215) noticed that continuity of the function M need not be imposed because it is implied by the other conditions, similarly as in our results.

Appendix B. Comparing and Generalizing Preference Foundations Published in Other Fields

This appendix discusses related preference foundations in the literature. We give details on the way in which our preference foundations generalize preference foundations in the literature that were not discussed in Appendix A.

B.1. Additive and Average Utility

The two most common ways to subjectively evaluate sequences are by sums or by averages of their utilities. These two ways generate the same preferences between n-tuples if n is fixed, but part ways if n is variable. Then sequences of different length have to be compared with each other.

Additive representations have been extensively studied, and numerous preference foundations have been provided, often based on bisymmetry conditions. We will not review the large literature on additive representations in detail[13] but instead focus this appendix on average representations for variable length.

B.2. Alternative Preference Foundations of Average Utility

We discussed Fishburn's (1972) foundation of average utility in §2. We now discuss some other contributions. All references discussed assume weak ordering and we also assume it throughout this section.

Many papers assume the following continuity condition, which is stronger (more restrictive) than the Gravel et al. (2011) continuity condition as used in our paper. *Simple continuity* holds if, for each n, the restriction of $\succcurlyeq$ to X^n is continuous with respect to the product topology. In the presence of the requirement $\alpha \sim k\alpha$ as implied by replication equivalence, this condition implies our continuity (Definition 8).

Instead of our replication equivalence,[14] Gravel et al. (2011) used an *averaging* preference condition: $x \succcurlyeq y \Rightarrow x \succcurlyeq (x, y) \succcurlyeq y$ that is stronger:

Lemma 17. *Under weak ordering, averaging implies replication equivalence.*

The *same number enlargement* and *same number existence consistency* of Gravel et al. (2011) are equivalent to expansion independence, which in turn implies our joint independence (Lemma 18). Their *existence of critical levels* is, in the presence of some other conditions such as averaging, equivalent to our CE condition, which implies our continuity (Lemma 10). Their main result, Theorem 2, shows that an AU representation exists for $X = \mathbb{R}^k$ if weak ordering, symmetry, averaging, expansion independence, continuity, and minimal increasingness (a monotonicity condition on $\mathbb{R}^k$) hold. Because $\mathbb{R}^k$ is a connected separable topological space, their conditions imply the conditions of our Theorem 9 and their result follows from our theorem.

Blackorby et al. (2005, Theorem 6.15 and the text following the theorem) assume $X = \mathbb{R}$. Outcomes are interpreted as individual utilities, assumed available as observed inputs. The authors assume simple continuity, Pareto weak preference and minimal increasingness (two monotonicity conditions), replication equivalence, symmetry (see their p. 198 line 2, "anonymity"), and same-people independence, which is our joint independence (see their p. 198), to obtain AU. Again, all conditions in our Theorem 9 are satisfied and their result follows from our result.

Gravel et al. (2012) considered AU representations over sets instead of sequences. We focus on sequences, which are an essentially different domain. Hence there is no logical relation between their result and ours. Their main result does not use continuity in outcomes but a solvability condition (their richness Axiom 4). This condition is similarly problematic in preference axiomatizations as utility (see our discussion following Theorem 7, and Wakker 1988b for further discussion).

Marinacci (1998) considered limits of average utility for infinite sequences and provided axiomatizations of these (Corollaries 6 and 16) and of nonadditive generalizations. He assumed an Anscombe–Aumann (1963) model, which mathematically amounts to linear utility on a convex (mixture) outcome space.

B.3. Different Domains and More General Functionals

There exist many preference foundations of models that generalize average utility by bringing in other subjective factors, such as nonequal subjective weightings of coordinates that may be additive[15] or nonadditive.[16] In many of these, our finite n-tuples of outcomes are replaced with continuous streams of outcomes. In all these cases, our equally weighted average utility is present as a special case in a substructure. However, there is no easy way to obtain preference foundations of unweighted average utility from those more general models. They derive weaker conclusions from weaker axioms and essentially use the extra structure in their derivations. Hence these results are logically independent of the results of this paper.

Appendix C. Proofs Without Continuity

LEMMA 18. *Under weak ordering, expansion independence implies joint independence.*

PROOF. Assume $(x_2, \ldots, x_n)\, R\, (y_2, \ldots, y_n)$, with either $R = \succ$, or $R = \sim$, or $R = \prec$. By expansion independence, both

$$(c_1, x_2, \ldots, x_n)\, R\, (c_1, y_2, \ldots, y_n) \quad \text{and} \quad (d_1, x_2, \ldots, x_n)\, R\, (d_1, y_2, \ldots, y_n).$$

Hence the latter two relationships are always the same, implying joint independence. □

The reversed implication need not hold, for instance under average utility with utility depending on length n. It does hold in the presence of the other intuitive conditions.

LEMMA 19. *The intuitive conditions imply expansion independence.*

PROOF. Assume an outcome α and $\#x = \#y = n$. Take some c with $\#c = n^2$ and assume the relationship $(x, c)\, R\, (y, c)$ where either $R = \succ$, or $R = \sim$, or $R = \prec$. We will show below that both $x\, R\, y$ and $(x, \alpha)\, R\, (y, \alpha)$. That is, the latter two relationships are the same whatever they are. This implies expansion independence.

By joint independence (applied n^2 times) we have $(x, d)\, R\, (y, d)$ for all d of length n^2. In words, replacing an n-tuple x by an n-tuple y in any $n(n+1)$ tuple generates an R relation.

We, hence, have $((j+1)x, (n-j)y)\, R\, (jx, (n+1-j)y)$ for all j, which by transitivity implies $(n+1)x\, R\, (n+1)y$. By replication equivalence, it gives xRy.

We also have $((j+1)x, (n-j-1)y, n\alpha)\, R\, (jx, (n-j)y, n\alpha)$ for all j, which by transitivity implies $(nx, n\alpha)\, R\, (ny, n\alpha)$. By replication equivalence, $(x, \alpha)\, R\, (y, \alpha)$. □

Monotonicity holds if weakly improving an outcome in a prospect gives a weakly preferred prospect and strictly improving an outcome in a prospect gives a strictly preferred prospect. Repeated application of the condition implies the same if several outcomes are replaced.

LEMMA 20. *The intuitive axioms imply monotonicity.*

PROOF. Let $R = \succcurlyeq$ or $R = \succ$. By expansion independence applied $n-1$ times (for $x_2, \ldots, x_n$), $\beta\, R\, \alpha$ implies $(\beta, x_2, \ldots, x_n) \cdot (\alpha, x_2, \ldots, x_n)$. □

Strong associativity holds if

$$(x_1, \ldots, x_k) \succcurlyeq (y_1, \ldots, y_k) \Leftrightarrow (x_1, \ldots, x_k, c_{k+1}, \ldots, c_n) \succcurlyeq (y_1, \ldots, y_k, c_{k+1}, \ldots, c_n). \tag{C1}$$

In words, improving a subprospect improves the whole prospect. The condition reinforces monotonicity by considering subprospects of length exceeding 1.

LEMMA 21. *Expansion independence implies strong associativity.*

PROOF. Apply expansion independence $n-k$ times (for $c_{k+1}, \ldots, c_n$). □

The following result concerns the Archimedean axiom.

LEMMA 22. *Assume the intuitive axioms, $\#x = \#y$, $x \succ y$, and $(nx, v) \succcurlyeq (ny, w)$. Then $(mx, v) \succcurlyeq (my, w)$ for all $m \geqslant n$.*

PROOF. By applying strong associativity twice, we get

$$((n+1)x, v) \succcurlyeq (y, nx, v) \succcurlyeq (y, ny, w) = ((n+1)y, w). \quad \square$$

PROOF OF THEOREM 7. Necessity of the five conditions in statement (ii) follows from substitution. We, hence, assume the five conditions of statement (ii) and derive the AU representation (and after that establish the uniqueness result). We will use Theorem 3.2.1.1 of Krantz et al. (1971; Hölder's 1901 lemma without the requirement that every element have an inverse) and will verify its conditions. Our concatenation operation of prospects corresponds with the concatenation operation denoted $\circ$ by Krantz et al. We define a binary relation $\succcurlyeq^*$ on prospects that will turn out to be the additive counterpart of the "averaging" binary relation $\succcurlyeq$. To this effect, we take an arbitrary outcome $\theta \in X$. We will let θ play the role of a neutral element with respect to the concatenation ("addition") and $\succcurlyeq^*$, later setting $U(\theta) = 0$.

DEFINITION 23. $x \succcurlyeq^* y$ if there exists $m \geqslant \max\{\#x, \#y\}$ such that $(x, (m-\#x)\theta) \succcurlyeq (y, (m-\#y)\theta)$.

The preference $\succcurlyeq$ in Definition 23 refers only to prospects of the same length, in which case additive and average utility give the same result. If there exists an $m \geqslant \max\{\#x, \#y\}$ such that the preference in Definition 23 holds, then by expansion independence (Lemma 19), the preference holds for all $m \geqslant \max\{\#x, \#y\}$. Roughly, $\succcurlyeq^*$ is derived from $\succcurlyeq$ by starting from equal-length comparisons and then adding or deleting θs as is desirable. The symmetric part $\sim^*$ and the asymmetric part $\succ^*$ are defined as usual. We verify that $\succcurlyeq^*$ satisfies all four conditions of Theorem 3.2.1.1 of Krantz et al. (1971). For conditions 2 through 4, we first give their definition, and then their derivation.

1. (Weak ordering): Completeness of $\succcurlyeq^*$ follows immediately from completeness of $\succcurlyeq$. For transitivity, assume that $x \succcurlyeq^* y$ and $y \succcurlyeq^* z$. Then $(x, (m-\#x)\theta) \succcurlyeq (y, (m-\#y)\theta) \succcurlyeq (z, (m-\#z)\theta)$ for all $m \geqslant \max\{\#x, \#y, \#z\}$, implying, by transitivity of $\succcurlyeq$, $(x, (m-\#x)\theta) \succcurlyeq (z, (m-\#z)\theta)$ for all such m. This implies $x \succcurlyeq^* z$, and transitivity follows.

2. (Weak associativity): $(x, (y, z)) \sim^* ((x, y), z)$. This follows from idempotence because we even have equality here. (Our concatenation satisfies what is sometimes called associativity for operations.)

3. (Krantz et al. monotonicity): $x \succcurlyeq^* y \Leftrightarrow (c, x) \succcurlyeq^* (c, y) \Leftrightarrow (x, c) \succcurlyeq^* (y, c)$. This term of Krantz et al. deviates from our term monotonicity. It is in fact expansion independence extended to prospects x, y of different length. $(x, (m-\#x)\theta) \succcurlyeq (y, (m-\#y)\theta)$ is, by expansion independence, equivalent to $(z, x, (m-\#x)\theta) \succcurlyeq (z, y, (m-\#y)\theta)$, which gives the first logical equivalence in monotonicity. The second logical equivalence follows from symmetry.

4. (The Archimedean axiom): If $x \succ^* y$, then for all $v, w \in \mathscr{X}$ there exists an n such that $(nx, v) \succ^* (ny, w)$. Assume $x \succ^* y$; that is, $(x, (m-\#x)\theta) \succcurlyeq (y, (m-\#y)\theta)$. By the Archimedean axiom of $\succcurlyeq$, $(n(x, (m-\#x)\theta), v) \succcurlyeq (n(y, (m-\#y)\theta), w)$ for some n. All the θs cancel, and hence we get $(nx, v) \succcurlyeq^* (ny, w)$, as required.

All conditions in Krantz et al. (1971, Theorem 3.2.1.1) are satisfied. Hence there exists a real valued function $U(\cdot)$, unique up to a positive scale factor, that is additive with respect to the concatenation operation ($U(x, y) = U(x) + U(y)$).[17] We get $U(\theta) = 0$. The idea underlying their proof is to, first, define U through Equations (4) and (5) and taking limits for $n \to \infty$.

In these equations we only compare sequences of the same length, implying that the preferences $\succcurlyeq$ agree with $\succcurlyeq^*$. We obtain U correctly irrespective of whether the representation is additive, as with $\succcurlyeq^*$, or average, as with $\succcurlyeq$. The preference conditions imply that the revelations of utility do not generate inconsistencies. The conditions of Krantz et al. (1971, Theorem 3.2.1.1), similar to Hölder (1901), imply that sums of U values represent $\succcurlyeq^*$. Mostly their monotonicity condition for $\succcurlyeq^*$ implies that its representation is additive and not average.

For $\succcurlyeq$ we do not have monotonicity in the Krantz et al. sense, but replication equivalence. We finally show that this implies that averages of U represent $\succcurlyeq$. Because of replication equivalence, we have $x \succcurlyeq y \Leftrightarrow (\#y)x \succcurlyeq (\#x)y$. Because the latter preference concerns two prospects of the same length, it is equivalent to $(\#y)x \succcurlyeq^* (\#x)y$, or $(\#y)\sum_{i=1}^{\#x} U(x_i) \geqslant (\#x)\sum_{j=1}^{\#y} U(y_j)$. This is equivalent to $(\sum_{i=1}^{\#x} U(x_i))/\#x \geqslant (\sum_{i=1}^{\#y} U(y_i))/\#y$. The AU representation of $\succcurlyeq$ has been derived.

We finally consider the uniqueness result. Substitution immediately shows that we are free to choose level and unit of U. To see that there is no other liberty, consider $U(\theta)$ for some θ and define $\succcurlyeq^*$ as above with respect to θ. The above proof has demonstrated, using Krantz et al. (1971, Theorem 3.2.1.1), that only the unit of U then can be changed. ☐

An alternative way in which we could have derived results similar to ours (primarily Theorem 7) from existing expected utility theorems is as follows. We can, first, relate n-tuples to rational-probability lotteries. Then, by properly weakening the NM axioms, including the NM type mixture independence, to such lotteries, the commonly used restrictive continuity conditions on outcomes can be replaced by the nonrestrictive (necessary) NM type Archimedean axiom, similarly as we did. We would then have used an independence condition based on an underlying mixture operation instead of our Hölder-type joint independence. We preferred our route based on the latter independence because concatenation is more basic than mixing.[18] Hence our conditions are simpler and give more general theorems. In short, the Hölder-concatenation approach is more efficient than the NM-mixture approach.

Appendix D. Proofs with Continuity (§3)

Definition 24. For the preference relation $\succcurlyeq$ on X, the *order topology* is the smallest topology that makes $\succcurlyeq$ *continuous*, i.e., that contains all sets $\{\alpha \in X\colon \alpha \succ \beta\}$ and $\{\alpha \in X\colon \alpha \prec \beta\}$.

Because the order topology is coarser than any other topology with respect to which $\succcurlyeq$ is continuous, we have

Lemma 25. *$\succcurlyeq$ on X is continuous with respect to a connected topology if and only if the order topology is connected. $\succcurlyeq$ on X is continuous with respect to a separable topology if and only if the order topology is separable.*

Proof of Lemma 10. Assume continuity of $\succcurlyeq$. Then continuity of $\succcurlyeq$ on X holds trivially. For the CE condition, consider any $x = (x_1, \ldots, x_n)$. We find a CE of x. Assume, without loss of generality, that $x_1 \succcurlyeq \cdots \succcurlyeq x_n$. By replication equivalence and monotonicity (Lemma 20), $x_1 \sim (x_1, \ldots, x_1) \succcurlyeq x \succcurlyeq (x_n, \ldots, x_n) \sim x_n$. Hence the closed sets $\{\alpha \in X\colon \alpha \succcurlyeq x\}$ and $\{\alpha \in X\colon \alpha \preccurlyeq x\}$ are nonempty. Their union is X and, by connectedness, they must intersect (in fact, at their infimum/supremum). This intersection is a CE of x. The CE condition holds.

Next assume the CE condition and continuity of $\succcurlyeq$ on X. Take an arbitrary x, with its constant equivalent denoted CE. Then $\{\alpha \in X\colon \alpha \succ x\} = \{\alpha \in X\colon \alpha \succ \text{CE}\}$ and $\{\alpha \in X\colon \alpha \prec x\} = \{\alpha \in X\colon \alpha \prec \text{CE}\}$ are open because $\succcurlyeq$ on X is continuous. Hence $\succcurlyeq$ (on $\mathscr{X}$) is continuous. ☐

Our continuity condition (taken from Gravel et al. 2011) is the closed sections condition of Fuhrken and Richter (1991), but restricted to one dimension, which underscores its generality. The above results show that it is essentially stronger than continuity of $\succcurlyeq$ on X because the latter does not imply the CE condition, such as with lexicographic preferences with respect to rank-ordered n-tuples from $\mathbb{R}^n$. The following result is a corollary of Lemmas 10 and 25.

Lemma 26. *Assume the intuitive conditions. $\succcurlyeq$ (on $\mathscr{X}$) is continuous with respect to a connected topology on X if and only if it is continuous with respect to the order topology of $\succcurlyeq$ on X and the latter is connected.*

Lemmas 25 and 26 show, effectively, that the topology taken on X is a refinement of the order topology and that it is immaterial which refinement it is.

It has sometimes been thought, erroneously, that a function representing a binary relation is always continuous with respect to the order topology. However, any strictly increasing discontinuous function from $\mathbb{R}$ to $\mathbb{R}$ provides a counterexample. The following well-known result (a corollary of Beardon and Mehta 1994, Proposition 1; see also Steen and Seebach 1970, pp. 67–68) gives details.

Lemma 27. *Assume that $\succcurlyeq$ on X is continuous with respect to a connected topology, and that U represents $\succcurlyeq$ on X. Then the following statements are equivalent*:

1. *U is continuous.*
2. *$U(X)$ is an interval.*
3. *$U(X)$ neither has a gap nonempty (maximal noncontained interval) of the form $(\sigma, \tau]$ nor of the form $[\sigma, \tau)$.*
4. *$U(X)$ is a dense subset of its convex hull.*
5. *For each pair of outcomes α, γ, there exists a utility midpoint β in the sense that $U(\beta) = (U(\alpha) + U(\gamma))/2$.*

The following lemma provides the main step in the derivation of Theorem 9 from Theorem 7.

Lemma 28. *The intuitive conditions and continuity imply the Archimedean axiom.*

Proof. For contradiction, assume that the Archimedean axiom is violated. That is, assume $x \succ y$ and $(nx, v) \prec (ny, w)$ for all n, with $\#x = \#y = k$ and $\#v = \#w = m$. Let $\delta = \text{CE}(v)$, $\gamma = \text{CE}(w)$ (this counteralphabetic notation will be most convenient). $v \sim \delta \sim m\delta$ and $w \sim \gamma \sim m\gamma$ and strong associativity imply $(nx, v) \sim (nx, m\delta)$ and $(ny, w) \sim (ny, m\gamma)$. Hence we have $(nx, m\delta) \prec (ny, m\gamma)$ for all n, which implies $(nmx, m\delta) \prec (nmy, m\gamma)$ for all n, and then

$$(nx, \delta) \prec (ny, \gamma) \quad \text{for all } n. \tag{D1}$$

Let $x \sim \alpha$, $y \sim \beta$. Then $x \sim k\alpha$, $y \sim k\beta$, $nx \sim nk\alpha$, and $ny \sim nk\beta$ imply, by strong associativity, $(nx, \delta) \sim (nk\alpha, \delta)$ and $(ny, \gamma) \sim (nk\beta, \gamma)$ for all n. Substituting this in Equation (D1) gives $(nk\alpha, \delta) \prec (nk\beta, \gamma)$ for all n. By Lemma 22 we cannot have $(n\alpha, \delta) \succcurlyeq (n\beta, \gamma)$, implying

$$(n\alpha, \delta) \prec (n\beta, \gamma) \quad \text{for all } n. \tag{D2}$$

We have $\alpha \succ \beta$. By monotonicity, $\gamma \succ \delta$. Equation (D2) suggests that the utility difference between α and β is infinitesimal relative to that between γ and δ.

We first define a set C, being an arc between γ and δ (i.e., C is isomorphic to $[0, 1]$). For each rational number q between 0 and 1, we choose some m and n with $q = m/n$. We then define $q\gamma + (1-q)\delta$, or $q\gamma$ for short, as a CE of $(m\gamma, (n-m)\delta)$. By replication equivalence, it does not depend on the particular m and n chosen. Thus we have an ordered set isomorphic to the rational numbers between 0 and 1. For each irrational number $0 < r < 1$, the two sets $\bigcup_{q\in\mathbb{Q}:\, q<r}\{\rho \in X: \rho \prec q\gamma\}$ and $\bigcup_{q\in\mathbb{Q}:\, q>r}\{\rho \in X: \rho \succ q\gamma\}$ are open and nonempty. By connectedness, there exists at least one outcome contained in neither set. It can be seen that there usually are many and that they are not all equivalent. At any rate, we take one and define it as $r\gamma$. Define the function that assigns $r\gamma$ to each $0 \leqslant r \leqslant 1$. (Formally, this can be done using the choice axiom from mathematical logic.) This ordered set is isomorphic to $[0, 1]$ (it is a continuum), denoted C.

For each $\sigma \in C$ we define a constant equivalent σ_α of (α, σ) and σ_β of (β, σ). It generates two sets $\{\sigma_\alpha: \sigma \in C\}$ and $\{\sigma_\beta: \sigma \in C\}$.

Lemma 29. *For each $\tau \succ \sigma \in C$ we have $\tau_\beta \succ \sigma_\alpha$.*

Proof. This proof will be ended by *QED*. Because the rational numbers are dense in $[0, 1]$, there are rational numbers m/n and $(m-k)/n$ (with $k > 0$) such that $\tau \succ (m/n)\gamma$ and $((m-k)/n)\gamma \succ \sigma$. Then, by monotonicity, also $\tau_\beta \succ ((m/n)\gamma)_\beta$ and $(((m-k)/n)\gamma)_\alpha \succ \sigma_\alpha$. It suffices to prove $((m/n)\gamma)_\beta \succ (((m-k)/n)\gamma)_\alpha$, or $((m/n)\gamma, \beta) \succ (((m-k)/n)\gamma, \alpha)$, or

$$\big(n((m/n)\gamma), n\beta\big) \succ \big(n(((m-k)/n)\gamma), n\alpha\big). \tag{D3}$$

Because $(m\gamma, (n-m)\delta) \sim (m/n)\gamma \sim n((m/n)\gamma)$, we have, by strong associativity, $(n((m/n)\gamma), n\beta) \sim (m\gamma, (n-m)\delta, n\beta)$. Similarly, $(n(((m-k)/n)\gamma), n\alpha) \sim ((m-k)\gamma, (n-m+k)\delta, n\alpha)$. We substitute these indifferences in Equation (D3) and get, as sufficient to prove,

$$(m\gamma, (n-m)\delta, n\beta) \succ ((m-k)\gamma, (n-m+k)\delta, n\alpha).$$

By expansion independence, dropping $n-k$ common coordinates, this is equivalent to $(k\gamma, n\beta) \succ (k\delta, n\alpha)$. We finally prove this. By Equation (D2), $(\gamma, n\beta) \succ (\delta, n\alpha)$. For induction, $(j\gamma, n\beta) \succ (j\delta, n\alpha)$ implies, by expansion independence and then monotonicity, $(\gamma, j\gamma, n\beta) \succ (\gamma, j\delta, n\alpha) \succ (\delta, j\delta, n\alpha)$, or $((j+1)\gamma, n\beta) \succ ((j+1)\delta, n\alpha)$. It follows by induction that $(k\gamma, n\beta) \succ (k\delta, n\alpha)$. QED

Every preference interval $\{\rho \in X: \sigma_\beta \prec \rho \prec \sigma_\alpha\}$ is nonempty because it contains $\mathrm{CE}(\sigma_\beta, \sigma_\alpha)(\sigma_\beta \sim (\sigma_\beta, \sigma_\beta) \prec (\sigma_\beta, \sigma_\alpha) \prec (\sigma_\alpha, \sigma_\alpha)$, mostly by monotonicity). These preference intervals are all disjoint by Lemma 29. Hence these are uncountably many disjoint nonempty open sets, which cannot be because of topological separability. Contradiction has resulted and Lemma 28 has been proved.

The beginning of this proof has in fact shown that under the CE condition the Archimedean axiom can be restricted to the case where x, y, v, and w are outcomes. ☐

Lemma 30. *Assume that average utility holds with utility U and that continuity holds with respect to a connected topology on X. Then U is continuous.*

Proof. For all values $U(\alpha)$ and $U(\gamma)$ in the image of U, the midpoint $(U(\alpha)+U\gamma))/2$ is also contained in the image of U because it is the utility of the CE of (α, γ), which exists by Lemma 10. By Lemma 27, U is continuous. ☐

Proof of Theorem 9. To derive (i) $\Rightarrow$ (ii), we assume statement (i). The first four conditions in (ii) follow from substitution. Condition 5 follows by taking the topology on X generated by U, i.e., by $\succcurlyeq$ on X. $U(X)$ being an interval implies connectedness and separability of this topology.

We finally assume statement (ii), and derive statement (i), continuity of U, and the uniqueness result. By Lemma 28, the Archimedean axiom is satisfied. By Theorem 7, we obtain an average-utility representation. By Lemma 30, U is continuous. By Lemma 27, $U(X)$ is an interval. The uniqueness follows from Theorem 7. ☐

If we replace continuity by simple continuity in Theorem 9, then the nontrivial implication, (ii) $\Rightarrow$ (i), can easily be derived from Debreu (1960), along the lines of Blackorby et al. (1977). First, on each X^n with $n \geqslant 3$ we then get an additive representation $\sum_{j=1}^n V_{j,n}(x_j)$, mainly because joint independence is what is often called (preferential) separability. The function $V_{j,n}$ at this stage can depend on n. For each n, the functions $V_{1,n}, \ldots, V_{n,n}$'s are identical, and we obtain a representation $\sum_{j=1}^n U_n(x_j)$ (Blackorby et al. 1977, Lemma 1; Blackorby et al. 2005, Theorem 4.7). Replication equivalence (relating dimensions k and m to each other through dimension km) implies that U_n is independent of n (as in Blackorby et al. 2005, Theorems 4.19, 6.1, and 6.2) and (dropping the n) that average utility $\sum_{j=1}^n U(x_j)/n$ represents preference on the whole domain $\mathscr{X}$. A special advantage of our more general analysis is that we obtain a uniform generalization of results on generalized means.

Appendix E. Further Proofs

Proof of Lemma 12. Both for $r_1 = c_1$ and $r_1 = d_1$,

$$(r_1, x_2, \ldots, x_n) = \frac{1}{n}r_1 + \frac{n-1}{n}(x_2, \ldots, x_n) \quad \text{and}$$
$$(r_1, y_2, \ldots, y_n) = \frac{1}{n}r_1 + \frac{n-1}{n}(y_2, \ldots, y_n).$$

By NM-independence, the preference between $(r_1, x_2, \ldots, x_n)$ and $(r_1, y_2, \ldots, y_n)$ must agree with that between $(x_2, \ldots, x_n)$ and $(y_2, \ldots, y_n)$ for both r_1, implying joint independence. We did not use weak ordering in this proof, and we only used NM-independence for rational probabilities. ☐

Proof of Theorem 15. The implication (i) $\Rightarrow$ (ii) follows from substitution. We, hence, assume (ii) and derive (i) and continuity and uniqueness. $\succcurlyeq$, the preference relation represented by M, satisfies the first four conditions in statement (ii) in Theorem 9. Continuity of $\succcurlyeq$ on X follows because it is the natural order on $\mathbb{R}$. The topology, the Euclidean one, is connected and separable. The CE condition follows from reflexivity. By Theorem 9, there exists a continuous AU representation. By monotonicity, U is strictly increasing. By idempotence, M is the CE function. (i) has been proved. The uniqueness follows from Theorem 9. ☐

Proof of Lemma 16. For idempotence, assume $M(n\alpha) = \beta$. By associativity, with $k = n$, we have $M(n\alpha) = M(n\beta)$. By monotonicity, $\alpha = \beta$.

For replication equivalence (also derived by Blackorby et al. 2005, Theorem 4.20), take the constant equivalent $M(x) = \alpha$. By k-fold application of associativity, $M(kx) = M(k(\#x\alpha))$. By idempotence, $M((k\#x)\alpha) = \alpha$. We conclude that $M(x) = M(kx)$.

For joint independence, assume $(x_1, \ldots, x_n) \succcurlyeq (y_1, \ldots, y_n)$. The CE condition is satisfied with M the CE function, by idempotence. Take constant equivalents $M(x) = \beta$ and $M(y) = \gamma$, respectively. Then $\beta \geqslant \gamma$. By monotonicity, $(\alpha, \beta, \ldots, \beta) \succcurlyeq (\alpha, \gamma, \ldots, \gamma)$ (both $n + 1$-tuples). By associativity, the former $(n + 1)$-tuple is indifferent to $(\alpha, x_1, \ldots, x_n)$ and the latter to $(\alpha, y_1, \ldots, y_n)$. Hence $(\alpha, x_1, \ldots, x_n) \succcurlyeq (\alpha, y_1, \ldots, y_n)$. Expansion independence has been shown. By Lemma 18, this implies joint independence of $\succcurlyeq$ and, hence, of M. □

Proof of Lemma 17. For induction, assume $kx \sim x$. It implies $kx \succcurlyeq x$ and $kx \preccurlyeq x$. By averaging, $kx \succcurlyeq (k+1)x \succcurlyeq x$ and $kx \preccurlyeq (k+1)x \preccurlyeq x$. $(k+1)x \sim x$ follows. Induction implies replication equivalence. □

Endnotes

1. See Blackorby et al. (1977, Lemma 2 and Assumption 5); Blackorby et al. (2005, Chapters 4–6); Gravel et al. (2011, following Theorem 1).
2. This equivalence is elementary in the sense that it does not use technical axioms such as the Archimedean axiom or continuity in its derivation. All logical relations between intuitive conditions claimed in this paper will be elementary in this sense.
3. Conversely, additive representations can be separated from average representations by a reinforced expansion independence where Equation (3) is also imposed for x, y of different length (Pivato 2013, Wakker 1986).
4. See Pfanzagl (1968, §6.6, pp. 107–108, Remark on p. 111, §9.5); Krantz et al. (1971, §9.1); Luce et al. (1990, Theorem 21.21); and Fuhrken and Richter (1991, p. 94). Bossert et al. (2007) argued for relaxing continuity and used it to obtain extensions to infinite sequences.
5. Lemmas 25 and 26 will show that this topology has to be a refinement of the order topology generated by $\succcurlyeq$ on X. These lemmas also show that it can be any such refinement. Lemmas 10 and 27 further illustrate that the choice of the refined topology is immaterial. This also holds for the other results in this section.
6. Equivalently, there exists a connected topology with respect to which U is continuous. This topology can always be taken to be separable. Note that topological separability need not be stated as an assumption, generalizing results by Gravel et al. (2011) and others.
7. To formalize the application of our results to decision under risk, we equate lotteries with the equivalence classes of n-tuples that generate them. We can then verify that our conditions for n-tuples naturally extend to lotteries under replication equivalence and symmetry. For brevity we do not elaborate on this formalization.
8. NM-independence is often defined with an implication in only one direction in Equation (6), even sometimes with indifference instead of preference and only for mixture 0.5. Then a stronger continuity-like version of the Archimedean axiom is used from which the two-sided implication follows for all mixture weights.
9. This was indicated by Bullen (2003); Chew (1983); Goovaerts et al. (2010); Grabisch et al. (2011a, p. 42, b); Muliere and Parmigiani (1993); Ozaki (2009); and Wakker (1988a).
10. U can also be taken strictly decreasing, but then we can replace it by $-U$. Studies on nonsymmetric quasilinear means include Aczél (1966), Marichal (2000), and Wakker (1988a).
11. Their Lemma 1, with U^s dependent on the length s of the prospects, suggests that their Assumption 4 is not meant to capture expansion independence.
12. Some papers interchange the terms idempotence and reflexivity.
13. References include Fuhrken and Richter (1991), Krantz et al. (1971), and Wakker (1986). Pivato (2013) provided advanced results for sequences of variable length, with extensions to infinitesimal representations without the Archimedean axiom (see also Fuhrken and Richter 1991, p. 84). Barberà and Pattanaik (1984) and Kannai and Peleg (1984) give axioms to separate additive and average representations.
14. Their term replication equivalence is similar to what we call idempotence.
15. Such models include subjective expected utility (Ahn 2008, Jeffrey 1965, Savage 1954); discounted utility (Koopmans 1972, Kopylov 2010); and utilitarian welfare (Blackorby et al. 2005, Harsanyi 1955, Kolm 2002).
16. Such models include rank-dependent utility (Gilboa 1987; Grabisch et al. 2011a, b; Quiggin 1982; Schmeidler 1986) and betweenness models (Chew 1983, Ozaki 2009).
17. This result for $\succcurlyeq^*$ generalizes Blackorby et al. (2005, Theorems 4.21 and 4.22) by dropping continuity of the representation and weakening the continuity preference condition to the Archimedean axiom.
18. It can be seen that our independence condition is close to Savage's sure-thing principle, which is more basic than NM-independence, again, because it does not use a mixture operation.

Acknowledgments

Thierry Marchant and three referees made useful comments.

References

Aczél J (1966) *Lectures on Functional Equations and Their Applications* (Academic Press, New York).

Ahn DS (2008) Ambiguity without a state space. *Rev. Econom. Stud.* 75:3–28.

Anscombe FJ, Aumann RJ (1963) A definition of subjective probability. *Ann. Math. Stat.* 34:199–205.

Artzner P, Delbaen F, Eber J-M, Heath D (1999) Coherent measures of risk. *Math. Finance* 9:203–228.

Balk BM (1995) Axiomatic price index theory: A survey. *Internat. Statist. Rev.* 63:69–93.

Barberà S, Pattanaik PK (1984) Extending an order on a set to the power set: Some remarks on Kannai and Peleg's approach. *J. Econom. Theory* 32:185–191.

Beardon AF, Mehta GB (1994) The utility theorems of Wold, Debreu, and Arrow-Hahn. *Econometrica* 62:181–186.

Bell DE (1982) Regret in decision making under uncertainty. *Oper. Res.* 30(5):961–981.

Blackorby C, Bossert W, Donaldson D (2005) *Population Issues in Social Choice Theory, Welfare Economics and Ethics* (Cambridge University Press, Cambridge, UK).

Blackorby C, Davidson R, Donaldson D (1977) A homiletic exposition of the expected utility hypothesis. *Economica* 44:351–358.

Bonferroni CE (1924) La media esponenziale in matematica finanziaria. *Annuario del Regio Istituto Superiore di Scienze Economiche e Commerciali di Bari AA* 23–24:1–14.

Bossert W, Sprumont Y, Suzumura K (2007) Ordering infinite utility streams. *J. Econom. Theory* 135:579–589.

Bullen PS (2003) *Handbook of Means and Their Inequalities*, Mathematics and Its Applications, Vol. 560 (Kluwer Academic Publishers, Dordrecht, The Netherlands).

Carmon Z, Shanthikumar JG, Carmon TF (1995) A psychological perspective on service segmentation models: The significance of accounting for consumers' perceptions of waiting and service. *Management Sci.* 41(11):1806–1815.

Chew SH (1983) A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox. *Econometrica* 51:1065–1092.

Debreu G (1960) Topological methods in cardinal utility theory. Arrow KJ, Karlin S, Suppes P, eds. *Mathematical Methods in the Social Sciences* (Stanford University Press, Stanford, CA), 16–26.

de Finetti B (1931) Sul concetto di media. *Giornale dell'Istituto Italiano degli Attuari* II(3):369–396.

Ferguson A, Meyers CS, Bartlett RJ, Banister H, Bartlett FC, Brown W, Campbell NR, et al. (1940) Quantitative estimates of sensory events. The advancement of science. *Rep. British Assoc. Advancement Sci.* 2:331–349.

Fishburn PC (1972) Even-chance lotteries in social choice theory. *Theory Decision* 3:18–40.

Fuhrken G, Richter MK (1991) Additive utility. *Econom. Theory* 1: 83–105.

Gilboa I (1987) Expected utility with purely subjective non-additive probabilities. *J. Math. Econom.* 16:65–88.

Goovaerts MJ, Kaas R, Laeven RJA (2010) Decision principles derived from risk measures. *Insurance: Math. Econom.* 47:294–302.

Grabisch M, Marichal J-L, Mesiar R, Pap E (2011a) Aggregation functions: Construction methods, conjunctive, disjunctive and mixed classes. *Inform. Sci.* 181:23–43.

Grabisch M, Marichal J-L, Mesiar R, Pap E (2011b) Aggregation functions: Means. *Inform. Sci.* 181:1–22.

Gravel N, Marchant T, Sen A (2011) Comparing societies with different numbers of individuals on the basis of their average advantage. Fleurbaey M, Salles M, Weymark JA, eds. *Social Ethics and Normative Economics* (Springer, Berlin), 261–277.

Gravel N, Marchant T, Sen A (2012) Uniform expected utility criteria for decision making under ignorance or objective ambiguity. *J. Math. Psych.* 56:297–315.

Hardy GH, Littlewood JE, Pólya G (1934) *Inequalities* (Cambridge University Press, Cambridge, UK).

Harsanyi JC (1955) Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *J. Political Econom.* 63:309–321.

Hölder O (1901) Die Axiome der Quantität und die Lehre vom Mass. *Berichte Verhand. König. Sächs. Gesell. Wiss. (*Leipzig*), Math. Phys., Classe* 53:1–64. [Part I is translated into English by Michell J, Ernst C (1996) The axioms of quantity and the theory of measurement. *J. Math. Psych.* 40:235–252. Part II is translated into English by Michell J, Ernst C (1997) The axioms of quantity and the theory of measurement. *J. Math. Psych.* 41:345–356.]

Jeffrey RC (1965) *The Logic of Decision* (McGraw-Hill, New York).

Kannai Y, Peleg B (1984) A note on the extension of an order on a set to the power set. *J. Econom. Theory* 32:172–175.

Keeney RL, Raiffa H (1976) *Decisions with Multiple Objectives* (John Wiley & Sons, New York).

Kolm S-C (2002) *Modern Theories of Justice* (MIT Press, Cambridge, MA).

Kolmogorov AN (1930) Sur la notion de moyenne. *Rendiconti della Academia Nazionale dei Lincei* 12:388–391.

Koopmans TC (1972) Representations of preference orderings with independent components of consumption; Representations of preference orderings over time. McGuire CB, Radner R, eds. *Decision and Organization* (North-Holland Publishing Company, Amsterdam), 57–100.

Kopylov I (2010) Unbounded probabilistic sophistication. *Math. Soc. Sci.* 60:113–118.

Krantz DH, Luce RD, Suppes P, Tversky A (1971) *Foundations of Measurement* (*Additive and Polynomial Representations*), Vol. I (Academic Press, New York).

Loomes G, Sugden R (1982) Regret theory: An alternative theory of rational choice under uncertainty. *Econom. J.* 92:805–824.

Luce RD, Krantz DH, Suppes P, Tversky A (1990) *Foundations of Measurement* (*Representation, Axiomatization, and Invariance*), Vol. III (Academic Press, New York).

Machina MJ, Schmeidler D (1992) A more robust definition of subjective probability. *Econometrica* 60:745–780.

Marichal J-L (2000) An axiomatic approach of the discrete Choquet integral as a tool to aggregate interaction criteria. *IEEE Trans. Fuzzy Systems* 8:800–807.

Marinacci M (1998) An axiomatic approach to complete patience and time invariance. *J. Econom. Theory* 83:105–144.

Muliere P, Parmigiani G (1993) Utility and means in the 1930s. *Statist. Sci.* 8:421–432.

Nagumo M (1930) Über eine Klasse der Mittelwerte. *Japanese J. Math.* 7:71–79.

Nicholson W (2005) *Microeconomic Theory; Basic Principles and Extensions*, 9th ed. (South-Western, Thomson Learning, London).

Norris N (1976) General means and statistical theory. *Amer. Statistician* 30:8–13.

Ozaki H (2009) Conditional implicit mean and the law of iterated integrals. *J. Math. Econom.* 45:1–15.

Peleg B, Peters HJM (2009) Nash consistent representation of effectivity functions through lottery models. *Games Econom. Behav.* 65: 503–515.

Pfanzagl J (1968) *Theory of Measurement* (Physica-Verlag, Vienna).

Pivato M (2013) Additive representation of separable preferences over infinite products. *Theory and Decision*, ePub ahead of print September 20.

Quiggin J (1982) A theory of anticipated utility. *J. Econom. Behav. Organ.* 3:323–343.

Savage LJ (1954) *The Foundations of Statistics* (Wiley, New York).

Schmeidler D (1986) Integral representation without additivity. *Proc. Amer. Math. Soc.* 97:255–261.

Scott D (1964) Measurement structures and linear inequalities. *J. Math. Psych.* 1:233–247.

Shepherdson JC (1980) Utility theory based on rational probabilities. *J. Math. Econom.* 7:91–113.

Steen LA, Seebach JA Jr (1970) *Counterexamples in Topology* (Holt, Rinehart and Winston, New York).

van Praag BMS, Ferrer-i-Carbonell A (2004) *Happiness Quantified* (Oxford University Press, Oxford, UK).

von Neumann J, Morgenstern O (1944) *Theory of Games and Economic Behavior* (Princeton University Press, Princeton, NJ).

Wakker PP (1986) The repetitions approach to characterize cardinal utility. *Theory Decision* 20:33–40.

Wakker PP (1988a) Characterizations of quasilinear representing functions, and specified forms of these. Eichhorn W, ed. *Measurement in Economics: Theory and Applications of Economic Indices* (Physica-Verlag, Heidelberg, Germany), 311–326.

Wakker PP (1988b) The algebraic versus the topological approach to additive representations. *J. Math. Psych.* 32:421–435.

Weinstein MC, Fineberg HV, Elstein AS, Frazier HS, Neuhauser D, Neutra RR, McNeil BJ (1980) *Clinical Decision Analysis* (Saunders, Philadelphia).

Amit Kothiyal is a postdoctoral fellow at the Max Planck Institute for Human Development in Berlin. His research interests include preference axiomatization of decision models, computational statistics and data analysis, and their applications in economics and finance.

Vitalie Spinu is a postdoctoral fellow at the Anderson School of Management, University of California, Los Angeles. His research interests include computational statistics and data analysis with a primary focus on Bayesian hierarchical modeling, foundations of statistics, extensions of Bayesian inference, behavioral models under risk and uncertainty, behavioral finance, and programming paradigms.

Peter P. Wakker is full professor at the Erasmus School of Economics, Erasmus University of Rotterdam. His research interests include preference axiomatizations of decision models and their empirical tests. His main model of interest is prospect theory for ambiguity (unknown probabilities that are empirically handled in non-Bayesian manners).