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# Relative concave utility for risk and ambiguity

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ABSTRACT

This paper presents a general technique for comparing the concavity of different utility functions when probabilities need not be known. It generalizes: (a) Yaari's comparisons of risk aversion by not requiring identical beliefs; (b) Kreps and Porteus' information-timing preference by not requiring known probabilities; (c) Klibanoff, Marinacci, and Mukerji's smooth ambiguity aversion by not using subjective probabilities (which are not directly observable) and by not committing to (violations of) dynamic decision principles; (d) comparative smooth ambiguity aversion by not requiring identical second-order subjective probabilities. Our technique completely isolates the empirical meaning of utility. It thus sheds new light on the descriptive appropriateness of utility to model risk and ambiguity attitudes.



## 1. Introduction

This paper presents a discrete version of marginal rates of substitution for analyzing utility under risk and ambiguity. Our version is based on preference midpoints and a standard sequence technique of Krantz et al. (1971). We show that an agent (called Ringo) has more concave utility than another agent (called George) if and only if Ringo's preference midpoints are always below those of George. Although our result is elementary in a mathematical sense, it has not been known before and it generalizes several classical results in the literature. Thus the first contribution of this paper is positive in that it offers new results that facilitate utility analyses. Further, for readers who equate utility with risk attitude as in expected utility (von Neumann and Morgenstern, 1944; Savage, 1954), or with ambiguity attitude as in recursive expected utility (Klibanoff, Marinacci and Mukerji, 2005; KMM henceforth), our results simply facilitate the study of such attitudes, which is again a positive contribution.

For other readers, however, our results may raise doubts about the properness of utility to model risk or ambiguity attitudes descriptively. Because we can isolate the empirical meaning of utility more clearly than done before, we see that the above theories entail an implausible recommendation: to completely capture risk and ambiguity attitudes for a set of events, we should inspect outcomes throughout their domain in every detail, while almost completely ignoring the relevant probabilities and events. Such a recommendation is not plausible. Risk and uncertainty should concern probabilities and events. Ambiguity is, indeed, mostly taken to concern interactions between events, violating separability. Drawing inferences about such interactions requires an inspection of various events.

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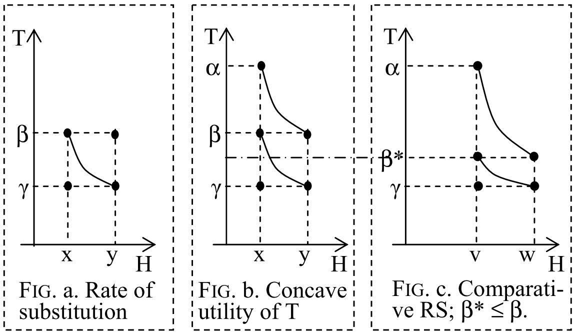


**Fig. 2.1.** Rates of substitution.

Our paper considers risk and ambiguity from a purely descriptive perspective. None of our claims are made for normative purposes. Most models in the theoretical literature on ambiguity, such as KMM's model, have been introduced for normative purposes. Hence, our paper is not a criticism of existing models, but rather we point to problems for descriptive applications.

For risk, our argument contributes to a view widely held a generation ago, and one of the main motivations for the introduction of nonexpected utility in the 1980s: Utility of outcomes cannot completely capture risk attitudes, and there must be more to risk attitudes (Schoemaker, 1982). This view was also widely held in the psychological literature. For example, Lopes (1987) wrote: "Risk attitude is more than the psychophysics of money" (p. 283). Rabin (2000) derived a calibration paradox from expected utility's modeling of risk attitude through utility, adding support to the view of the 1980s. By our way of isolating utility in a pure form, we add yet further support to this view for decisions under risk. Moreover, and this is our main purpose, we extend this view to decisions under ambiguity. Our analysis thus supports theories that model ambiguity through functions on events, which automatically entails a deviation from expected utility.[1]

## 2. Rates of substitution: from consumer theory to decision under uncertainty

In consumer theory, marginal rates of substitution provide a useful tool for analyzing utility. For example, assume that at the starting point $(x, \gamma)$ in Fig. 2.1(a), an increase of $\beta - \gamma$ of one commodity, say tea ($T$), exactly offsets an increase of $y - x$ of hats ($H$). The rate of substitution of $H$ for $T$ (RS) is $\frac{\beta-\gamma}{y-x}$, where marginal rates of substitution concern the case where these quantities are taken infinitesimal. A common empirical finding is that RSs diminish: The more tea you have, the more extra tea you need to offset one unit of hats. If we take the starting point $(x, \beta)$ in Fig. 2.1(b), with more tea than at $(x, \gamma)$, then $\alpha - \beta \geqslant \beta - \gamma$. Because it will be convenient to work with midpoints, we rewrite:

$$\beta \leqslant \frac{\alpha + \gamma}{2}. \tag{2.1}$$

We call $\beta$ a *preference midpoint* between $\alpha$ and $\gamma$ whenever we have a configuration with indifferences as in Fig. 2.1(b); i.e., whenever there exist $x$, $y$ such that[2]

$$(x, \alpha) \sim (y, \beta) \quad \text{and} \quad (x, \beta) \sim (y, \gamma). \tag{2.2}$$

Similar concepts can obviously be defined with the coordinates interchanged. Lemma 2.1 will show that $\beta$ is independent of the particular choices of $x$ and $y$ under expected utility (with strictly increasing utility). Eq. (2.1) can be phrased as: *preference midpoints are lower than physical midpoints*.

Fig. 2.1(c) presents a comparative result, comparing the situation at $v$ (another initial hat-wealth position) to that at $x$. The preference midpoint at $v$, $\beta^*$ is lower than $\beta$, the preference midpoint at $x$. This suggests that RSs (or marginal utility if we have cardinal models) is more strongly diminishing at $v$ than at $x$.

As pointed out by Arrow (1953) and Debreu (1959), techniques used to analyze preferences over commodity bundles can also be used to analyze preferences under uncertainty. Suppose that a coin is flipped, with either heads (event $H$) or tails (event $T$) coming up. The point $(x, \gamma)$ now designates an event-contingent payment, yielding $x$ if $H$ and $\gamma$ if $T$. Such event-contingent payments are called *acts* (Savage, 1954). *Expected utility* (*EU*) holds if acts are evaluated by

$$(x, \gamma) \longrightarrow pU(x) + (1 - p)U(\gamma), \tag{2.3}$$

where: (a) $0 < p < 1$ is the probability of $H$ ($H$ is *nondegenerate*); (b) $U$ denotes utility; (c) $U$ is continuous and strictly increasing. We assume that the outcome set is a nonpoint subinterval of the reals. The following lemma shows that preference midpoints are utility midpoints.

[1] Such theories include Gilboa (1987), Gilboa and Schmeidler (1989), Schmeidler (1989), and many other theories.

[2] Alternative definitions of preference midpoints are in Abdellaoui et al. (2007), Fishburn and Edwards (1997, Axiom 8), Ghirardato et al. (2003, Definition 4), Harvey (1986 above Eq. (4)), Köbberling and Wakker (2003, p. 408), and Vind (2003, §IV.2, above Theorem IV.2.1).

**Lemma 2.1.** *If $\beta$ is a preference midpoint between $\alpha$ and $\gamma$, then under EU*

$$U(\beta) = \frac{U(\alpha) + U(\gamma)}{2}. \tag{2.4}$$

*Risk aversion* means that every act is less preferred than its expected value. In the following theorem, the condition in (iii), as elementary as it is, is new.

**Theorem 2.2.** *Under EU, the following three statements are equivalent*:

(i) *$U$ is concave*;
(ii) *risk aversion holds*;
(iii) *preference midpoints are below physical midpoints* (*Eq.* (2.1)).

Condition (iii) relates risk aversion to diminishing RSs from consumer theory. Condition (ii) is not very useful for decision under uncertainty, where probabilities are subjective. Then they are not directly observable, and they must be derived from preferences. Preference conditions using them, such as condition (ii) that needs probabilities to determine the expected value needed in the definition of risk aversion, are then not directly observable. They are not directly observable in the same way as preference conditions using utilities as inputs are not directly observable.[3] Condition (iii), to the contrary, is directly observable also if probabilities are subjective. This observation shows that our technique, based on RSs, better isolates utility than traditional methods have done so far. Harvey (1986, Theorem 3) and Wakker (1986, Theorem 5.2) characterized concave utility in terms of derived tradeoffs, which similarly are directly observable.

An appealing alternative condition to characterize concave utility that does not use probability mixtures (subjective or objective) either is quasiconcavity with respect to outcome mixing (Chateauneuf and Tallon, 2002, Theorem 1 and Proposition 1; Debreu and Koopmans, 1982, p. 4; Strzalecki, 2011, p. 62). We will not study this condition because we are not aware of comparative versions, the topic to which we now turn.

For two preference relations $\succcurlyeq^*$ and $\succcurlyeq$, the *preference midpoints for $\succcurlyeq^*$ are below those of $\succcurlyeq$* if, for all outcomes $\alpha$ and $\gamma$ with $\succcurlyeq$-midpoint $\beta$ and $\succcurlyeq^*$-midpoint $\beta^*$, we have

$$\beta^* \leqslant \beta. \tag{2.5}$$

This condition adapts the comparisons of diminishing RS of Fig. 2.1(c) to uncertainty. It is allowed in this definition that the events $H$ and $T$ in Fig. 2.1(c) are different from those in Fig. 2.1(b). $U^*$ is *more concave than $U$* if $U^*(\cdot) = \varphi(U(\cdot))$ for a concave transformation $\varphi$. Given that concavity of utility is commonly taken to reflect risk aversion, more concavity is usually interpreted as bigger risk aversion.

**Theorem 2.3.** *Assume EU for two preference relations $\succcurlyeq^*$ and $\succcurlyeq$, with the same outcome sets and with different utilities $U^*$ and $U$. The nondegenerate outcome relevant events $H$ (and $T$) are allowed to be different for $\succcurlyeq$ than for $\succcurlyeq^*$. The following two statements are equivalent*:

(i) *$U^*$ is more concave than $U$*;
(ii) *the preference midpoints of $\succcurlyeq^*$ are below those of $\succcurlyeq$* (*Fig.* 2.1(c)).

We have stated the above result for decisions depending only on one nondegenerate event. The result can be applied to any general event space with expected utility, simply by fixing one nondegenerate event $H$ and considering binary acts whose outcomes depend only on this event.

**Observation 2.4.** The results of Theorems 2.2 and 2.3 hold for general event spaces with expected utility. In Theorem 2.3, the event spaces may be different for the two decision makers. Both event spaces should contain a nondegenerate event.

## 3. Applications of our technique for comparing concavity

### 3.1. Yaari's interpersonal comparisons of risk aversion

Yaari (1969) showed that Ringo's utility function is more concave than George's if Ringo's certainty equivalents are below those of George. (Yaari used an equivalent formulation in terms of acceptance sets.) Yaari's condition is directly applicable to uncertainty with unknown probabilities because it does not use the latter as inputs. It requires that the two decision makers

[3] Preference conditions concern fundamental measurement in the terminology of Hempel (1952) and Krantz et al. (1971, §10.9.2). That is, only directly observable primitives such as preferences are used as inputs. If theoretical constructs such as subjective probabilities (Budescu and Wallsten, 1987, p. 68) or utilities are used as inputs, then the measurement is called derived.

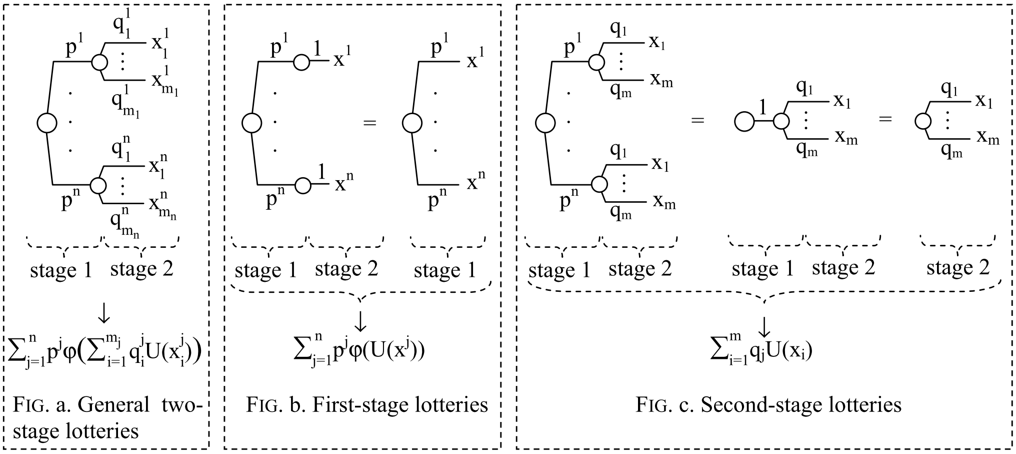


**Fig. 3.1.** Multi-stage lotteries and recursive utility.

face the same set of acts and the same uncertain events. It then has another implication, besides comparing concave utility: the two decision makers must have the same subjective probabilities. Thus only decision makers with the same beliefs can be compared, and only for them is Yaari's condition necessary and sufficient for relative concavity.

**Observation 3.1.** Theorem 2.3 generalizes Yaari's (1969, §I) characterization of more concave utility functions to decision makers with possibly different subjective probabilities and even with possibly different event spaces.

Unlike Yaari's (1969) result, Observation 3.1 allows comparisons of utility (= risk attitude under EU) separated from beliefs, where beliefs are allowed to be different. We can handle the case where George and Ringo are agents in two separate markets. We cannot directly compare what they trade, but we can compare their rates of substitution and hence, under EU, their risk attitudes.

### *3.2. Kreps and Porteus' recursive expected utility for preference for the timing of the resolution of uncertainty*

This subsection considers decision under risk, where *lotteries* designate probability distributions over outcomes. We consider two-stage lotteries, depicted in Fig. 3.1(a), with the method of evaluation (*recursive expected utility*) indicated in the figure. The model entails a commitment to backward induction. In the first stage, the probability distribution is not over outcomes, but over second-stage lotteries with, as always, the probabilities $p^j$ nonnegative and summing to 1. Throughout, superscripts designate indexes and not powers. In the second stage, a lottery over outcomes results that with probability $q_i^j$ yields outcome $x_i^j$, where $j$ depends on the resolution of the first-stage uncertainty and where for each $j$ the probabilities $q_i^j$ are nonnegative and sum to 1.

Kreps and Porteus (1978, 1979) considered the two-stage setup assuming that stage 2 comes after stage 1 in a temporal sense. They gave preference conditions for $\varphi$ being concave or convex ($\varphi$ is defined in Fig. 3.1). We will focus on concave $\varphi$'s, the convex case being similar. The preference condition of Kreps and Porteus amounts to a weak aversion to mean-preserving spreads in terms of second-stage probabilities. It is equivalent to a preference for a second-stage lottery on the right-hand side of Fig. 3.1(c) over a first-stage lottery on the right-hand side of Fig. 3.1(b) whenever these happen to concern the same lottery ($m = n$, $p^j = q_j$, and $x^j = x_j$ for all $j$). That is, the decision maker prefers the resolution of uncertainty in stage 2 to the resolution of the same uncertainty in stage 1. This entails a violation of the reduction of compound lotteries. Kreps and Porteus's condition can be restated as lower certainty equivalents for stage 1 than for stage 2, and in this sense can be taken to be equivalent to Yaari's (1969) condition for concavity of $\varphi$.

We can characterize concavity of $\varphi$ simply by comparing the preference midpoints in stage 1 with those in stage 2 (Theorem 2.3 and Observation 2.4). In this way, we do not need to consider any two-stage lottery.

**Observation 3.2.** Theorem 2.3 generalizes Kreps and Porteus (1978, Theorem 3), characterizing more concave utility functions, by

(i) not committing to particular (violations of) dynamic decision principles;
(ii) allowing for subjective (unknown) instead of objective probabilities, which furthermore may be different at different stages, as may be the underlying events;
(iii) requiring only one nontrivial probability, rather than all probabilities, to be available at each stage.

Commitments to dynamic decision principles such as backward induction and (violations of) reduction of compound lotteries, as in Kreps and Porteus (1978), are controversial in nonexpected utility (Machina, 1989). Kreps and Porteus could similarly have avoided the use of multi-stage lotteries by using Yaari's certainty equivalent condition as described above. However, then they would still have the restrictions of Yaari's technique discussed in Section 3.1. von Gaudecker et al. (2011) tested the Kreps and Porteus' (1978) model empirically while avoiding the use of two-stage lotteries. They did not directly test our preference condition or compare certainty equivalents, but they parametrically fitted concave utility to data. They found no difference in utility. Klibanoff and Ozdenoren (2007) provided a subjective generalization of Kreps and Porteus' model. Hence our results can be used in their model.

### *3.3. KMM's recursive expected utility for smooth attitudes towards ambiguity*

KMM introduced an influential recursive expected utility model to analyze ambiguity. We discuss their concepts in some detail. At the outset, the uncertainty in the second stage in their model concerns a Savage event space $S$. The uncertainty in the first stage concerns what is the appropriate subjective second-stage probability measure on $S$. *First-stage acts* (called second-order acts in their model; they are the analogs of our first-stage lotteries in Section 3.2[4]) assign outcomes contingent on what the appropriate second-stage subjective probability measure on $S$ is. For example, an act yielding outcome 10 if the appropriate subjective probability measure on $S$ is $P$, and outcome 5 if it is $Q$, is a first-stage act.

KMM assume that the two-stage decomposition is endogenous. Thus there is no need for an extraneous physical mechanism to determine what the appropriate subjective probability measure on $S$ is (such as compositions of urns in the paradoxes of Ellsberg, 1961), upon which we could easily condition. Instead, the second-stage events are based exclusively on the subjective perception of the decision maker. KMM's model is, therefore, not sequential in a traditional sense. For a sequential extension of their model, see Klibanoff et al. (2009). By assuming preferences between first-stage acts to be available despite the absence of an observable physical mechanism, KMM make it possible to still analyze this endogenous two-stage decomposition. Thus their model becomes considerably more broadly applicable while at the same time coming close to our psychological perceptions of ambiguity. As pointed out by KMM (p. 1856), a drawback of their first-stage acts is that these may not be observable.

By assuming that the second-stage probabilities over $S$ are completely specified by the first-stage events, KMM achieve that subjective probabilities over the event space $S$ can be treated as known probabilities in the second stage (KMM, Lemma 1, Definition 2, and Assumption 3). This leads to the paradoxical but useful result that the Savage second-stage uncertainty can be treated as objective risk. All aspects of ambiguity are captured in the first stage. We thus reach the configuration of Fig. 3.1 with the first-stage probabilities $p^j$ subjective. KMM have one fixed $n$-tuple $(p_1, \ldots, p_n)$ in Fig. 3.1 whereas Kreps and Porteus consider all $n$-tuples of probabilities (for all $n$) there. The second-stage probabilities $q_i^j$ in KMM are known and objective.

An aspect of KMM's model that complicates the mathematical analysis is that, for one two-stage lottery, the second-stage lotteries at different first-stage branches must all have been generated by one and the same Savage act. They can only be different because of different "true" probabilities assumed on the Savage space. Thus, not all combinations of second-stage lotteries can be considered. KMM add some richness assumptions to ensure sufficient combinations to apply known mathematical techniques for preference conditions:

KMM assume an Anscombe and Aumann (1963) structure within the Savage space, implying that all lotteries can appear as second-stage lotteries on the right end of Fig. 3.1(c). (3.1)

KMM assume that all first-stage acts are available; i.e. all $n$-tuples of outcomes $x^1, \ldots, x^n$ can appear on the right end of Fig. 3.1(b). (3.2)

KMM assume that some second-stage probability distributions over the Savage space are available that have disjoint supports.[5] (3.3)

Assumption 3.2 involves acts that are not generated by Savage acts, and it entails the exception mentioned before. The beliefs with different support underlying Assumption 3.3 are maximally contradictory. By one belief, an event is certainly true, but by the other belief, that same event is certainly false.

**Observation 3.3.** Theorem 2.2 generalizes KMM (2005, Proposition 1; characterizing smooth ambiguity aversion) by

(i) not committing to particular (violations of) dynamic decision principles;
(ii) not using subjective probabilities as input in characterizing conditions;
(iii) not using Richness Assumption 3.3.

[4] The different terminology in this paper regarding what the first and the second stage are, relative to KMM, is caused by different conventions for second-order probability theory than for dynamic decision theory. Unfortunately, this inconvenience cannot be avoided.
[5] In a mathematical sense this implies, roughly, that a full two-dimensional subspace of combinations of second-order lotteries is available.

Regarding point (ii), to characterize concavity of $\varphi$, KMM adopt the preference condition of weak aversion to mean-preserving spreads (preferring any two-stage lottery less than the second-stage lottery that results from probability multiplication), interpreted as smooth ambiguity aversion. Mean-preserving spreads require first-stage probabilities as input. Given their subjective nature in KMM's model, this preference condition is not directly observable (Strzalecki, 2011, p. 62).

A useful property of KMM's model is the possibility to define comparative ambiguity aversion in a manner similar to Yaari's definition of comparative risk aversion, using well-known utility analyses. If Ringo and George share the same first-stage beliefs, then Ringo is *more smooth ambiguity averse* than George if Ringo preferring an act to a (second-stage) lottery implies that George also prefers the act to the lottery. KMM showed that this is equivalent to Ringo and George having the same second-stage utility ($U$) but Ringo's $\varphi$ being more concave than George's. For this result they assumed, as did Yaari (1969), that Ringo and George share the same first-stage beliefs.

**Theorem 3.4.** *Assume recursive EU as in Fig.* 3.1 *for two different agents facing the same outcome set but possibly different first- and second-stage events and probabilities. Then the following two statements are equivalent*:

(i) *The second agent has the same "risky" utility $U$ as the first agent does and the first agent has a more concave transformation $\varphi$ than the first agent does* (*more smooth ambiguity aversion*)*.*
(ii) *The second agent has the same preference midpoints for second-stage lotteries, and lower preference midpoints for first-stage acts, than the first agent does.*

**Observation 3.5.** Theorem 3.4 generalizes KMM (2005, Theorem 2), characterizing more smooth ambiguity aversion, by

(i) not committing to particular (violations of) dynamic decision principles;
(ii) not using Richness Assumption 3.3;
(iii) not requiring identical first-stage beliefs or events.

### *3.4. An intuitive objection*

An objection can be raised when our preference condition in terms of utility midpoints is not just used to analyze utility, but is also interpreted as a condition for risk or ambiguity aversion. Our midpoint condition does not speak to the empirical nature of risk, timing (as in Kreps and Porteus' model), or ambiguity, unlike the conditions that other authors have used. However (and this is our message) if a theory such as EU or recursive EU implies that our condition is still equivalent to the others, then this implication of the theory cannot be empirically appropriate, which raises doubts about the theory itself.

## 4. Other utility-based models of ambiguity and discussion

Neilson (1993, published 2010) presented a recursive expected utility model similar as in KMM, but with the two-stage structure being exogenous. He also assumed backward induction and aversion to mean-preserving spreads with respect to subjective probabilities. Thus our results apply to his model as they do to KMM. Nau (2006) added event dependence of utility. Ergin and Gul (2009) used this model with subjective probabilities and ambiguity attitudes too, but extended the preference conditions of Kreps and Porteus to a recursive version of Machina and Schmeidler's (1992) probabilistic sophistication rather than EU. That is, they moved away from EU, a move supported by our paper. Dobbs (1991) used recursive expected utility with second-stage probabilities being updates after observed outcomes. Seo (2009) added an extra stage of exogenous objective probabilities. Halevy and Ozdenoren (2008) considered a combination of Seo (2009) and Ergin and Gul (2009), but did not assume expected utility within each stage, a move supported by our paper. Empirical support for their model is in Halevy (2007).

Epstein (2010) started by criticizing the problematic empirical status of the endogenous two-stage decomposition of KMM. His first example shows that KMM is not able to model ambiguity *within* a stage, which is related to our criticism of KMM's use of expected utility within each stage. Epstein's second example shows that KMM is not able to model different degrees of ambiguity within a stage, which naturally follows from his first example. His Section 3 criticizes KMM for deviating from multiple priors.

Chew et al. (2008) provided a static alternative, with no stages, to recursive utility. It distinguished between different sources of events (subgroups of events), assuming expected utility within each source, and taking utility source dependent. The more ambiguity aversion there is for a source, the more concave the utility function for that source is. Our theorems can be applied to all models mentioned, including Chew et al.'s, because they need no stages.

When axiomatizing concave utility, researchers have always used conditions that involve probabilities, as in risk aversion (condition (ii) in Theorem 2.2) or in aversion to mean-preserving spreads. These conditions suggest that utility speaks to probabilities. Our conditions better isolate utility, showing that utility is virtually unrelated to probabilities or ambiguous events. This leads to our criticism of the utility-based modeling of ambiguity in KMM's model and other related models. The utility-based approach implies, for example, that ambiguity attitudes are determined by the subdomain of outcomes faced. If they come from a subdomain where utility is very concave, then there is much ambiguity aversion, but if they come from another subdomain where utility is almost linear, then there is little ambiguity aversion. This holds alike for all events related to the utility function considered. We think that ambiguity should be related to (nonseparable) events rather than to outcomes only. Utility-based models are not well suited to describe ambiguity attitudes in the same way as expected utility is not well suited to describe risk attitudes.

## 5. Conclusion

We have introduced a new tool for analyzing utility under risk and ambiguity. Our tool shows more clearly than done before that utility can be analyzed by focusing on outcomes while almost entirely ignoring the probabilities and ambiguous events. For risk, this reinforces arguments by Allais (1953), Kahneman and Tversky (1979), Rabin (2000), and Schoemaker (1982), that utility in expected utility cannot fully capture risk attitudes descriptively. We extend these arguments to ambiguity. Thus our analysis supports the use of nonexpected utility (nonseparable events) to analyze ambiguity, as propagated by Gilboa (1987), Gilboa and Schmeidler (1989), Schmeidler (1989), and others.

We hope that readers who disagree with our intuitive interpretations will nevertheless appreciate the positive results of this paper. We have introduced a simple tool to analyze concave utility under expected utility that can also be used if probabilities are subjective. We have provided generalizations of several classical results (Yaari, 1969; Kreps and Porteus 1978, 1979; Klibanoff et al., 2005). Our tool for analyzing utility can easily be extended to nonexpected utility models by adding appropriate comonotonicity restrictions.

## Appendix A. Proofs

We begin with a lemma that will be useful for the elaboration of our main results.

**Lemma A.1.** *Let $f : I \to \mathbb{R}$ be continuous, with $I \subset \mathbb{R}$ an interval. Then $f$ is concave if and only if, for every $\alpha, \gamma \in I$, there exists a $p_{\alpha,\gamma}$ with $0 < p_{\alpha,\gamma} < 1$ and $f(p_{\alpha,\gamma}\alpha + (1 - p_{\alpha,\gamma})\gamma) \geqslant p_{\alpha,\gamma} f(\alpha) + (1 - p_{\alpha,\gamma}) f(\gamma)$.*

**Proof.** This result follows from Hardy et al. (1934, Observation 88). We will only use the case where $p_{\alpha,\gamma} = p$ is independent of $\alpha, \gamma$. □

**Proof of Lemma 2.1.** The indifferences in Eq. (2.2) imply $p(U(x) - U(y)) = (1 - p)(U(\beta) - U(\alpha)) = (1 - p)(U(\gamma) - U(\beta))$. Because $1 - p > 0$, $U(\beta) - U(\alpha) = U(\gamma) - U(\beta)$, implying Eq. (2.4). □

**Proof of Theorem 2.2.** (i) ⇒ (ii) is well known. (ii) ⇒ (i) follows from Lemma A.1, with $p_{\alpha,\gamma} = p$ (the probability in the EU model) and $0 < p < 1$. By Lemma 2.1, concavity of $U$ implies (iii).

We finally assume (iii), and derive (i). Assume some $\theta$ in the interior of the outcome set. By continuity, there exists an $\varepsilon > 0$ such that: (a) the interval $(\theta - \varepsilon, \theta + \varepsilon)$ is contained in the outcome set; (b) $U(\theta + \varepsilon) - U(\theta - \varepsilon) > 0$ is so small that for all $\alpha > \beta$ in the interval there exist outcomes $y > x$ such that

$$p\big(U(y) - U(x)\big) = (1 - p)\big(U(\alpha) - U(\beta)\big).$$

Take any $\alpha > \beta > \gamma$ in the interval such that $\beta$ is the utility midpoint of $\alpha$ and $\gamma$, and take $y > x$ as just described. Eq. (2.2) results, so that $\beta$ is the preference midpoint between $\alpha$ and $\gamma$. It implies Eq. (2.1) by condition (iii). Because $U$ is strictly increasing, it implies, for $p_{\alpha,\gamma} = \frac{1}{2}$: $U(p_{\alpha,\gamma}\alpha + (1 - p_{\alpha,\gamma})\gamma) \geqslant p_{\alpha,\gamma} U(\alpha) + (1 - p_{\alpha,\gamma})U(\gamma)$. By Lemma A.1, $U$ is concave on $(\theta - \varepsilon, \theta + \varepsilon)$; i.e. $U$ is locally concave on the interior of the outcome domain. Then it is concave on the whole interior domain and, by continuity, on the whole outcome set. □

**Proof of Theorem 2.3.** Because of strict increasingness and continuity, we can write $U^*(\cdot) = \varphi(U(\cdot))$ with $\varphi$ continuous and strictly increasing. If we replace all outcomes by their $U$ values, so that $U$ values take the role of physical outcomes, then $\varphi$ carries physical outcomes into $U^*$ values. Thus concavity of $\varphi$ implies that its midpoints are below the corresponding physical midpoints, so that $U^*$ midpoints are always below the corresponding $U$ midpoints and, by Lemma 2.1, $\succcurlyeq^*$ preference midpoints are always below $\succcurlyeq$ preference midpoints. This establishes the implication (i) ⇒ (ii).

The reversed implication can at first be established only locally (to follow globally next). The reason is, roughly, that configurations in Eq. (2.2) sometimes can be constructed only locally. To that we now turn. We assume (ii) henceforth, and derive (i).

Take an outcome $\theta$ in the interior of the outcome domain. By continuity, there exists an $\varepsilon > 0$ similar as in the proof of Theorem 2.2; i.e.: (a) the interval $(\theta - \varepsilon, \theta + \varepsilon)$ is contained in the outcome set; (b) $U(\theta + \varepsilon) - U(\theta - \varepsilon) > 0$ is so small that for all $\alpha > \beta$ in the interval there exist outcomes $y > x$ such that

$$p\big(U(y) - U(x)\big) = (1 - p)\big(U(\alpha) - U(\beta)\big);$$

(c) $U^*(\theta + \varepsilon) - U^*(\theta - \varepsilon) > 0$ is so small that for all $\alpha > \beta$ in the interval there exist outcomes $y^* > x^*$ such that

$$p^*\big(U^*\big(y^*\big) - U^*\big(x^*\big)\big) = \big(1 - p^*\big)\big(U^*(\alpha) - U^*(\beta)\big).$$

As in the proof of Theorem 2.2, on the interval $(\theta-\varepsilon, \theta+\varepsilon)$ every $U$ midpoint is a $\succcurlyeq$ preference midpoint and every $U^*$ midpoint is a $\succcurlyeq^*$ preference midpoint.

On the interval $(U(\theta-\varepsilon), U(\theta+\varepsilon))$ within the domain of $\varphi$, every $\varphi$ midpoint $U(\beta^*)$ of $U(\alpha)$ and $U(\gamma)$ in the sense that $\varphi(U(\beta^*)) = (\varphi(U(\gamma)) + \varphi(U(\alpha)))/2$, satisfies $U^*(\beta^*) = (U^*(\gamma) + U^*(\alpha))/2$ and, furthermore, has $\beta^*$ as a $\succcurlyeq^*$ preference midpoint of $\alpha$ and $\gamma$. That is, we can arrange the configuration of Fig. 2.1(c). We can also obtain the $\succcurlyeq$ preference midpoint $\beta$ of $\alpha$ and $\gamma$, and by statement (ii) we have $\beta \geqslant \beta^*$. This implies that $U(\beta^*) \leqslant U(\beta) = (U(\gamma) + U(\alpha))/2$. Thus, on $(U(\theta-\varepsilon), U(\theta+\varepsilon))$, every $\varphi$ midpoint $U(\beta^*)$ is below the physical midpoint $U(\beta)$, implying that $\varphi$ is concave on the interval $(U(\theta-\varepsilon), U(\theta+\varepsilon))$. $\varphi$ is locally concave on the interior of its domain. This implies that $\varphi$ is concave on the interior of its domain and, by continuity, is so on its entire domain. $\square$

**Proof of Theorem 3.4.** The same preference midpoints implies, by two-fold application of Theorem 2.3, that one function is both more concave and more convex than the other, so that it is a linear transform. Under expected utility this means that the utility functions can be taken identical. This proves the claims of the theorem regarding risky utility $U$. We now turn to stage 1 and ambiguity, for $\succcurlyeq$, $\varphi$, and $U$ versus $\succcurlyeq^*$, $\varphi^*$, and $U^*$ with $U^* = U$. Lower $\succcurlyeq^*$ midpoints is equivalent, by Theorem 2.3, to $\varphi^*(U(\cdot)) = \psi(\varphi(U(\cdot)))$ for a concave $\psi$, which is equivalent to $\varphi^*$ being more concave than $\varphi$. $\square$

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