U

uncertainty

In most economic decisions where agents face uncertainties, no probabilities are available. This point was first emphasized by Keynes (1921) and Knight (1921). It was recently reiterated by Greenspan (2004, p. 38):

> ... how ... the economy might respond to a monetary policy initiative may need to be drawn from evidence about past behavior during a period only roughly comparable to the current situation. ... In pursuing a risk-management approach to policy, we must confront the fact that only a limited number of risks can be quantified with any confidence.

Indeed, we often have no clear statistics available. Knight went so far as to call probabilities unmeasurable in such cases. Soon after Knight's suggestion, Ramsey (1931), de Finetti (1931) and Savage (1954) showed that probabilities can be defined in the absence of statistics after all, by relating them to observable choice. For example, $P(E) = 0.5$ can be derived from an observed indifference between receiving a prize under event E and receiving it under not-E (the complement to E). Although widely understood today, the idea that something as intangible as a subjective degree of belief can be made observable through choice behaviour, and can even be quantified precisely, was a major intellectual advance.

Ramsey, de Finetti and Savage assumed that the agent, after having determined the probabilities subjectively (as required by some imposed rationality axioms), proceeds as under expected utility for given objective probabilities. The Allais (1953) paradox (explained later) revealed a descriptive difficulty: for known probabilities, people often do not satisfy expected utility. Hence, we need to generalize expected utility. Another, more fundamental, difficulty was revealed by the Ellsberg (1961) paradox (also explained later): for unknown probabilities, people behave in ways that cannot be reconciled with any assignment of subjective probabilities at all, so that further generalizations are needed. (The term 'subjective probability' always refers to additive probabilities in this article.)

Despite the importance and prevalence of unknown probabilities, understood since 1921, and the impossibility of modelling these through subjective probabilities, understood since Ellsberg (1961), decision theorists continued to confine their attention to decision under risk with given probabilities until the late 1980s. Wald's (1950) multiple priors model did account for unknown probabilities, but attracted little attention outside statistics.

As a result of an idea of David Schmeidler (1989, first version 1982), the situation changed in the 1980s. Schmeidler introduced the first theoretically sound decision model for unknown probabilities without subjective probabilities, called rank-dependent utility or Choquet expected utility. At the same time, Wald's multiple priors model was revived when Gilboa and Schmeidler (1989) established its decision-theoretic soundness; a similar result was obtained independently by Chateauneuf (1991, Theorem 2). These discoveries provided the basis for non-expected utility with unknown probabilities that had been sorely missing since 1921. Since the late 1980s, the table has turned in decision theory. Nowadays, most studies concern unknown probabilities. Gilboa (2004) contains recent papers and applications. This article concentrates on conceptual issues of individual decisions in the possible absence of known probabilities.

Theoretical studies of non-expected utility have usually assumed risk aversion for known probabilities (leading to concave utility and convex probability weighting), and ambiguity aversion for unknown probabilities (Camerer and Weber, 1992, section 2.3). These phenomena best fit with the existence of equilibria and can be handled using conventional tools of convex analysis (Mukerji and Tallon, 2001). Empirically, however, a more complex fourfold pattern has been found. For gains with moderate and high likelihoods, and for losses with low likelihoods, risk aversion is prevalent indeed, but for gains with low likelihoods and for losses with high likelihoods the opposite, risk seeking, is prevalent.

The fourfold pattern resolves the classical paradox of the coexistence of gambling and insurance, and leads, for instance, to new views on insurance. Whereas all classical studies of insurance explain insurance purchasing through concave utility, empirical measurements of utility have suggested that utility is not very concave for losses, often exhibiting more convexity than concavity (surveyed by Köbberling, Schwieren and Wakker, 2006). This finding is diametrically opposite to what has been assumed throughout the insurance literature. According to modern decision theories, insurance is primarily driven by consumers' overweighting of small probabilities rather than by marginal utility.

The fourfold pattern found for risk has similarly been found for unknown probabilities, and usually to a more pronounced degree. Central questions in uncertainty today concern how to analyse not only classical marginal utility but also new concepts such as probabilistic risk attitudes (how people process known probabilities), loss aversion and reference dependence (the framing of outcomes as gains and losses), and, further, states of belief and decision attitudes regarding unknown probabilities ('ambiguity attitudes').

We end this introduction with some notation and definitions. *Decision under uncertainty* concerns choices between *prospects* such as $(E_1{:}x_1, \ldots, E_n{:}x_n)$, yielding *outcome* x_j if *event* E_j obtains, $j = 1, \ldots, n$. Outcomes are monetary. The E_js are events of which an agent does not know for sure whether they will obtain, such as who of n candidates will win an election. The E_js are mutually exclusive and exhaustive. No probabilities of the events need to be given. Because the agent is uncertain about which event obtains, he is uncertain about which outcome will result from the prospect, and has to make decisions under uncertainty.

Decision under risk and non-expected utility through rank dependence

Because risk is a special and simple subcase of uncertainty (as explained later), this article on uncertainty begins with a discussion of *decision under risk*, where the probability $p_j = P(E_j)$ is given for each event E_j. We can then write a prospect as $(p_1{:}x_1, \ldots, p_n{:}x_n)$, yielding x_j with probability p_j, $j = 1, \ldots, n$. Empirical violations of expected-value maximization because of risk aversion (prospects being less preferred than their expected value) led Bernoulli (1738) to propose expected utility, $\sum_{j=1}^{n} p_j U(x_j)$, to evaluate prospects, where U is the utility function. Then risk aversion is equivalent to concavity of U.

Several authors have argued that it is intuitively unsatisfactory that risk attitude be modelled through the utility of money (Lopes, 1987, p. 283). It would be more satisfactory if risk attitude were also related to the way people feel about probabilities. Economists often react very negatively to such arguments, based as they are on introspection and having no clear link to revealed preference. Arguments against expected utility that are based on revealed preference were put forward by Allais (1953).

Figure 1 displays preferences commonly found, with K denoting $1,000:

(0.06:25K, 0.07:25K, *0.87:0*) $\prec$ (0.06:75K, 0.07:0, *0.87:0*) and
(0.06:25K, *0.87:25K*, 0.07:25K) $\succ$ (0.06:75K, *0.87:25K*, 0.07:25K).

Preference symbols $\succcurlyeq, \succ, \prec$ and $\preccurlyeq$ are as usual. We denote the outcomes in a rank-ordered manner, from best to worst. In Figure 1a, people usually prefer the 'risky' (r) prospect because the high payment of 75K is attractive. In Figure 1b, people usually prefer the 'safe' (s) certainty of 25K for sure. These preferences violate expected utility because, after dropping the common (italicized) term 0.87U(0) from the expected-utility inequality for Figure 1a and dropping the common term 0.87U(25K) from the expected-utility inequality for Figure 1b, the two inequalities become the same. Hence, under expected utility either both preferences should be for the safe prospect or both preferences should be for the risky one, and they cannot switch as in Figure 1. The special preference for safety in the second choice (the *certainty effect*) cannot be captured in terms of utility. Hence, alternative, non-expected utility models have been developed.

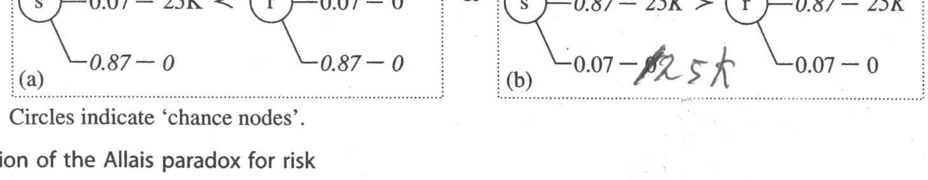
Based on the valuable intuition that risk attitude should have something to do with how people feel about probabilities, Quiggin (1982) introduced rank-dependent utility theory for risk. The same theory was discovered independently for the broader and more subtle context of uncertainty by Schmeidler (1989, first version 1982), a contribution that will be discussed later. A *probability weighting function* w: $[0,1] \to [0,1]$ satisfies $w(0) = 0$, $w(1) = 1$, and is strictly increasing and continuous. It reflects the (in)sensitivity of people towards probability. Assume that the outcomes of a prospect $(p_1{:}x_1, \ldots, p_n{:}x_n)$ are rank-ordered, $x_1 \geq \cdots \geq x_n$. Then its *rank-dependent utility (RDU)* is $\sum_{j=1}^{n} \pi_j U(x_j)$, where *utility* U is as before, and π_j, the *decision weight* of outcome x_j, is $w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1})$ (which is $w(p_1)$ for $j = 1$).

Tversky and Kahneman (1992) adapted their widely used original prospect theory (Kahneman and Tversky, 1979) by incorporating the rank dependence of Quiggin and Schmeidler. Prospect theory generalizes rank dependence by allowing a different treatment of gains from that of losses, which is desirable for empirical purposes. In this article on uncertainty, I focus on gains, in which case prospect theory in its modern version, sometimes called cumulative prospect theory, coincides with RDU.

With rank dependence, we can capture psychological (mis)perceptions of unfavourable outcomes being more likely to arise, in agreement with Lopes's (1987) intuition. We can also capture decision attitudes of deliberately paying more attention to bad outcomes. An extreme example of the latter pessimism concerns worst-case analysis, where all weight is given to the most unfavourable outcome. Rank dependence can explain the Allais paradox because the weight of the 0.07 branch in Figure 1b may exceed that in Figure 1a:

$$w(1) - w(0.93) \geq w(0.13) - w(0.06). \tag{1}$$

This inequality holds for w-functions that are steeper near 1 than in the middle region, a shape that is empirically prevailing indeed.



Circles indicate 'chance nodes'.

Figure 1 A version of the Allais paradox for risk

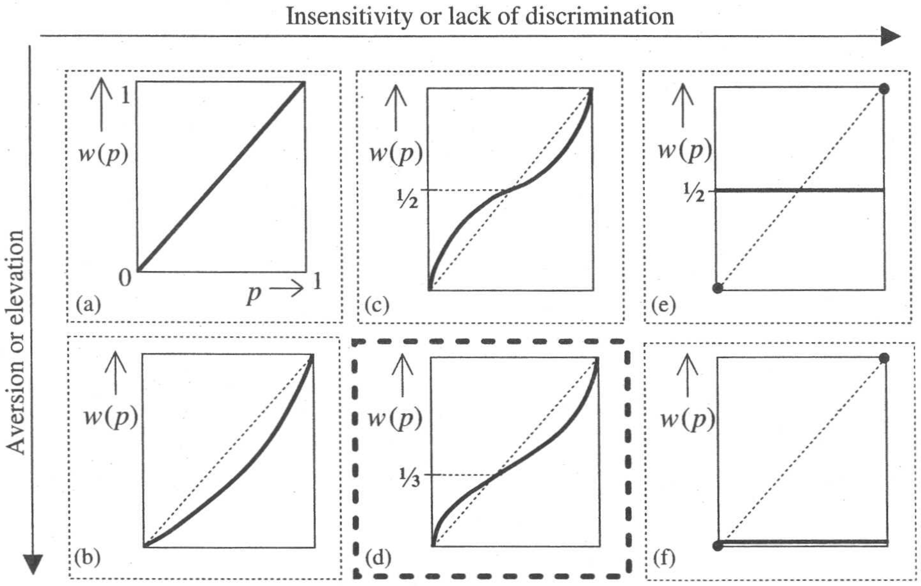


Figure 2 (a) Expected utility: linearity; (b) Aversion to risk: convexity; (c) Insensitivity: inverse-S (d) Prevailing empirical finding; (e) Extreme insensitivity; (f) Extreme aversion and insensitivity

Figure 2

~~The following figures~~ depict some probability weighting functions. Figure 2a concerns expected utility, and Figure 2b a *convex* w, which means that

$$w(p+r) - w(r) \tag{2}$$

is increasing in r for all $p \geq 0$. This is equivalent to w' being increasing, or w'' being positive. Equation (1) illustrates this property. Equation (2) gives the decision weight of an outcome occurring with probability p if there is an r probability of better outcomes. Under convexity, outcomes receive more weight as they are ranked worse (that is, r is larger), reflecting *pessimism*. It implies low evaluations of prospects relative to sure outcomes, enhancing risk aversion.

Empirical studies have found that usually $w(p) > p$ for small p, contrary to what convexity would imply, and that $w(p) < p$ only for moderate and high probabilities p (inverse-S; Abdellaoui, 2000; Bleichrodt and Pinto, 2000; Gonzalez and Wu, 1999; Tversky and Kahneman, 1992), as in Figures 2c and 2d. It leads to extremity-oriented behaviour with both best and worst outcomes overweighted. The curves in Figures 2c and 2d also satisfy eq. (1) and also accommodate the Allais paradox. They predict risk seeking for prospects that with a small probability generate a high gain, such as in public lotteries. The inverse-S shape suggests a cognitive insensitivity to probability, generating insufficient response to intermediate variations of probability and then, as a consequence, overreactions to changes from impossible to possible and from possible to certain. These phenomena arise prior to any 'motivational' (value-based) preference or dispreference for risk. Extreme cases of such behaviour are in Figures 2e and 2f (where we relaxed the continuity requirement for w).

Starmer (2000) surveyed non-expected utility for risk. The main alternatives to the rank-dependent models are the betweenness models (Chew, 1983; Dekel, 1986), with Gul's (1991) disappointment aversion theory as an appealing special case. Betweenness models are less popular today than the rank-dependent models. An important reason, besides their worse empirical performance (Starmer, 2000), is that models alternative to the rank-dependent ones did not provide concepts as intuitive as the sensitivity to probability/information modelled through the probability weighting w of the rank-dependent models. For example, consider a popular special case of betweenness, called weighted utility. The value of a prospect is

$$\frac{\sum_{i=1}^{n} p_i f(x_i) U(x_i)}{\sum_{j=1}^{n} p_j f(x_j)} \tag{3}$$

for a function $f: \mathbb{R} \to \mathbb{R}^+$. This new parameter f can, similar to rank dependence, capture pessimistic attitudes of overweighting bad outcomes by assigning high values to bad outcomes. It, however, applies to outcomes and not to probabilities. Therefore, it captures less extra variance of the data in the presence of utility than w, because utility also applies to outcomes. For example, for fixed outcomes, eq. (3) cannot capture the varying sensitivity to small, intermediate and high probabilities found empirically. Both pessimism and marginal utility are entirely specified by the range of outcomes considered without regard to the probabilities involved. It seems more interesting if new concepts, besides marginal utility,

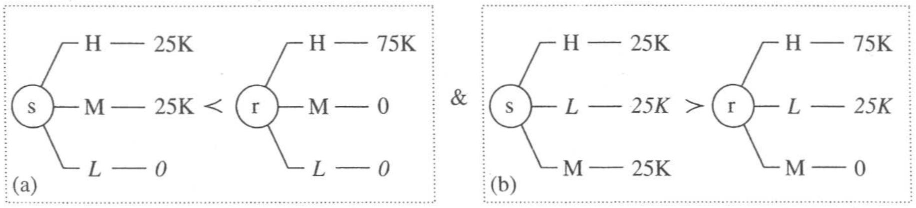


Figure 3 The certainty effect (Allais paradox) for uncertainty

concern the probabilities and the state of information of the decision maker rather than outcomes and their valuation. This may explain the success of rank-dependent theories and prospect theory.

Phenomena under uncertainty that naturally extend phenomena under risk

The first approach to deal with uncertainty was the Bayesian approach, based on de Finetti (1931), Ramsey (1931) and Savage (1954). It assumes that people assign, as well as possible, subjective probabilities $P(E_j)$ to uncertain events E_j. They then evaluate prospects (E_1: $x_1, \ldots, E_n$:x_n) through their (subjective) expected utility $\sum_{j=1}^{n} P(E_j)U(x_j)$. This model was the basis of Bayesian statistics and of much of the economics of uncertainty (Greenspan, 2004). The empirical measurement of subjective probabilities has been studied extensively (Fishburn, 1986; Manski, 2004; McClelland and Bolger, 1994). We confine our attention in what follows to models that have been introduced since the mid-1980s, models that deviate from Bayesianism. To Bayesians (including this author) such models are of interest for descriptive purposes.

Machina and Schmeidler (1992) characterized *probabilistic sophistication*, where a decision maker assigns subjective probabilities to events with unknown probabilities and then proceeds as for known probabilities. The decision maker may, however, deviate from expected utility for known probabilities, contrary to the Bayesian approach, and Allais-type behaviour can be accommodated.

The difference between objective, exogenous probabilities and subjective, endogenous probabilities is important. The former are stable, and readily available for analyses, empirical tests and communication in group decisions. The latter can be volatile and can change at any time by mere further thinking by the agent. For descriptive studies, subjective probabilities may become observable only after complex measurement procedures. Hence, I prefer not to lump objective and subjective probabilities together into one category, as has been done in several economic works (Ellsberg, 1961, p. 645; Epstein, 1999). In this article, the term risk refers exclusively to exogenous objective probabilities. Such probabilities can be considered a limiting case of subjective probabilities, in the same way as decision under risk can be considered a limiting case of decision under uncertainty (Greenspan, 2004, pp. 36–7). Under a differentiability assumption for state spaces, Machina (2004) formalized this interpretation. Risk, while not occurring very frequently, is especially suited for applications of decision theory.

The Allais paradox is as relevant to uncertainty as it is to risk (MacCrimmon and Larsson, 1979, pp. 364–5; Wu and Gonzalez, 1999). Figure 3 presents a demonstration by Tversky and Kahneman (1992, section 1.3). The analogy with Figure 1 should be apparent. The authors conducted the following within-subjects experiment. Let d denote the difference between the closing value of the Dow Jones on the day of the experiment and on the day after, where we consider the events H(igh): $d > 35$, M(iddle): $35 \geq d \geq 30$, L(ow): $30 > d$. The total Dow Jones value at the time of the experiment was about 3000. The right prospect in Figure 3b is (H:75K, L:25K, M:0), and the other prospects are denoted similarly. Of 156 money managers during a workshop, 77 per cent preferred the risky prospect r in Figure 3a, but 68 per cent preferred the safe prospect s in Figure 3b. The majority preferences violate expected utility, just as they do under risk: after dropping the common terms $P(\text{L})U(0)$ and $P(\text{L})U(25\text{K})$ (P denotes subjective probabilities), the same expected-utility inequality results for Figure 3a as for Figure 3b. Hence, either both preferences should be for the safe prospect, or both preferences should be for the risky one, and they cannot switch as in Figure 3. This reasoning holds irrespective of what the subjective probabilities $P(\text{H})$, $P(\text{M})$ and $P(\text{L})$ are. (The condition of expected utility that is falsified here, Savage's (1954) 'sure-thing principle', can be related to the separability preference condition of consumer theory.)

Schmeidler's (1989) *rank-dependent utility* (*RDU*) can accommodate the Allais paradox for uncertainty. We consider a *weighting function* (or non-additive probability or capacity) W that assigns value 0 to the vacuous (empty) event $\emptyset$, value 1 to the universal event, and satisfies monotonicity with respect to set-inclusion (if $A \supset B$ then $W(A) \geq W(B)$). Probabilities are special cases of weighting functions that satisfy *additivity*: $W(A \cup B) = W(A) + W(B)$ for disjoint events A and B. General weighting functions need not satisfy additivity. Assume that the outcomes of a prospect (E_1:$x_1, \ldots, E_n$: x_n) are rank-ordered, $x_1 \geq \cdots \geq x_n$. Then the prospect's *rank-dependent utility* (*RDU*) is $\sum_{j=1}^{n} \pi_j U(x_j)$ where *utility* U is as before, and π_j, the *decision weight* of outcome x_j, is $W(E_1 \cup \cdots \cup E_j) - W(E_1 \cup \cdots \cup E_{j-1})$ ($\pi_1 = W(E_1)$). The decision weight of x_j is the marginal W contribution of E_j to the event of receiving a better outcome.

Quiggin's RDU for risk is the special case with probabilities $p_j = P(E_j)$ given for all events, and $W(E_j) = w(P(E_j))$ with w the probability weighting function. Tversky and Kahneman (1992) improved their 1979 prospect theory not only by avoiding violations of stochastic dominance, but also, and more importantly, by extending their theory from risk to uncertainty, by incorporating Schmeidler's RDU.

Figure 3 can, just as in the case of risk, be explained by a larger decision weight for the M branches in Figure 3b than in Figure 3a:

$$W(\mathrm{M} \cup \mathrm{H} \cup \mathrm{L}) - W(\mathrm{H} \cup \mathrm{L}) \geq W(\mathrm{M} \cup \mathrm{H}) - W(\mathrm{H}). \quad (4)$$

This inequality occurs for W-functions that are more sensitive to changes of events near the certain universal event $\mathrm{M} \cup \mathrm{H} \cup \mathrm{L}$ than for events of moderate likelihood such as $\mathrm{M} \cup \mathrm{H}$. Although for uncertainty we cannot easily draw graphs of W functions, their properties are natural analogs of those depicted in Figures 2a–f. W is *convex* if the marginal W contribution of an event E to a disjoint event R is increasing in R, that is,

$$W(E \cup R) - W(R) \quad (5)$$

is increasing in R (with respect to set inclusion) for all E. This agrees with eq. (4), where increasing R from H to $\mathrm{H} \cup \mathrm{L}$ leads to a larger decision weight for $E = M$. Our definition of convexity is equivalent to other definitions in the literature such as $W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$. (Take $E = A - B$, and compare $R = A \cap B$ with the larger $R = A$.)

For probabilistic sophistication ($W(\cdot) = w(P(\cdot))$, convexity of W is equivalent to convexity of w under usual richness conditions, illustrating once more the close similarity between risk and uncertainty. Equation (5) gives the decision weight of an outcome occurring under event E if better outcomes occur under event R. Under convexity, outcomes receive more weight as they are ranked worse (that is, R is larger), reflecting *pessimism*. Theoretical economic studies usually assume convex W's, implying low evaluations of prospects relative to sure outcomes.

Empirical studies have suggested that weighting functions W for uncertainty exhibit patterns similar to Figure 2d, with unlikely events overweighted rather than, as convexity would have it, underweighted (Einhorn and Hogarth, 1986; Tversky and Fox, 1995; Wu and Gonzalez, 1999). As under risk, we get extremity orientedness, with best and worst outcomes overweighted and lack of sensitivity towards intermediate outcomes (Chateauneuf, Eichberger and Grant, 2005; Tversky and Wakker, 1995).

Phenomena for uncertainty that do not show up for risk: the Ellsberg paradox

Empirical studies have suggested that phenomena found for risk occur for uncertainty as well, and do so to a more pronounced degree (Fellner, 1961, p. 684; Hansen, Sargent and Tallarini, 1999; Kahn and Sarin, 1988, p. 270; Kahneman and Tversky, 1979, p. 281), in particular regarding the empirically prevailing inverse-S shape and its extension to uncertainty (Abdellaoui, Vossmann and Weber, 2005; Hogarth and Kunreuther, 1989; Kilka and Weber, 2001; Weber, 1994, pp. 237–8). It is plausible, for example, that the absence of known probabilities adds to the inability of people to sufficiently distinguish between various degrees of likelihood not very close to impossibility and certainty. In such cases, inverse-S shapes will be more pronounced for uncertainty than for risk. This observation entails a within-person comparison of attitudes for different sources of uncertainty, and such comparisons are the main topic of this section.

For Ellsberg's paradox, imagine an urn K with a known composition of 50 red balls and 50 black balls, and an ambiguous urn A with 100 red and black balls in unknown proportion. A ball is drawn at random from each urn, with R_k denoting the event of a red ball from the known urn, and the events B_k, R_a and B_a defined similarly. People prefer to bet on the known urn rather than on the ambiguous urn, and common preferences are:

$$(B_k{:}100,\ R_k{:}0) \succ (B_a{:}100,\ R_a{:}0) \text{ and}$$
$$(B_k{:}0,\ R_k{:}100) \succ (B_a{:}0,\ R_a{:}100).$$

Such preferences are also found if people can themselves choose the colour to bet on so that there is no reason for suspecting an unfavourable composition of the unknown urn. Under probabilistic sophistication with probability measure P, the two preferences would imply $P(B_k) > P(B_a)$ and $P(R_k) > P(R_a)$. However, $P(B_k) + P(R_k) = 1 = P(B_a) + P(R_a)$ yields a contradiction, because two big numbers cannot give the same sum as two small numbers. Ellsberg's paradox consequently violates probabilistic sophistication and, a fortiori, expected utility. Keynes (1921, p. 75) discussed the difference between the above two urns before Ellsberg did, but did not put forward the choice paradox and deviation from probabilistic sophistication as Ellsberg did. We now analyse the example assuming RDU.

In many studies of uncertainty, such as Schmeidler (1989), expected utility is assumed for risk, primarily for the sake of simplicity. Then, $W(B_k) = W(R_k) = 0.5$ in the above example, with these W values reflecting objective probabilities. Under RDU, the above preferences imply $W(B_a) = W(R_a) < 0.5$, in agreement with convex (or eventwise dominance, or inverse-S; for simplicity of presentation, I focus on convexity hereafter) weighting functions W. This finding led to the widespread misunderstanding that it is primarily the Ellsberg paradox that implies convex weighting functions for unknown probabilities, a condition that was sometimes called 'ambiguity aversion'. I have argued above that it is the Allais paradox, and not the Ellsberg paradox, that implies these conclusions, and I propose another interpretation of the Ellsberg paradox hereafter, following works by Amos Tversky in the early 1990s.

First, it is more realistic not to commit to expected utility under risk when studying uncertainty. Assume, therefore, that $W(B_k) = W(R_k) = w(P(B_k)) = w(P(R_k)) = w(0.5)$ for a nonlinear probability weighting function. It follows from the Ellsberg paradox that $W(B_a) = W(R_a) < w(0.5)$. This suggests:

Hypothesis. *In the Ellsberg paradox, the weighting function is more convex for the unknown urn than for the known urn.* □

Thus, the Ellsberg paradox itself does not speak to convexity in an absolute sense, and does not claim convexity for known or for unknown probabilities. It speaks to convexity in a relative (within-person) sense, suggesting *more* convexity for unknown probabilities than for known probabilities. It is, for instance, possible that the weighting function is concave, and not convex, for both known and unknown probabilities, but is less concave (and thus more convex) for the unknown probabilities (Wakker, 2001, section 6; cf. Epstein, 1999, pp. 589–90, or Ghirardato and Marinacci, 2002, example 25).

With information only about observed behaviour, and without additional information about the compositions of the urns or the agent's knowledge thereof, we cannot conclude which of the urns is ambiguous and which is not. It would then be conceivable that urn K were ambiguous and urn A were unambiguous, and that the agent satisfied expected utility for A and was optimistic or ambiguity seeking (concave weighting function, eq. (5) decreasing in *R*) for K, in full agreement with the Ellsberg preferences. Which of the urns is ambiguous and which is not is based on extraneous information, being our knowledge about the composition of the urns and about the agent's knowledge thereof. This point suggests that no endogenous definition of (un)ambiguity is possible.

The Ellsberg paradox entails a comparison of attitudes of one agent with respect to different sources of uncertainty. It constitutes a *within-agent* comparison. Whereas the Allais paradox concerns violations of expected utility in an absolute sense, the Ellsberg paradox concerns a relative aspect of such violations, finding more convexity (or eventwise dominance, or inverse-S) for the unknown urn than for the known urn. Such a phenomenon cannot show up if we study only risk, because risk is essentially only one source of uncertainty. Apart from some volatile psychological effects (Kirkpatrick and Epstein, 1992; Piaget and Inhelder, 1975), it seems plausible that people do not distinguish between different ways of generating objective known probabilities.

Uncertain events of particular kinds can be grouped together into sources of uncertainty. Formally, let *sources* be particular algebras of events, which means that sources are closed under complementation and union, and contain the vacuous and universal events. For example, source $\mathscr{A}$ may concern the performance of the Dow Jones stock index tomorrow, and source $\mathscr{B}$ the performance of the Nikkei stock index tomorrow. Chew and Sagi (2006) used the term small-world domain, and Ergin and Gul (2004) the term issue, for similar concepts. Assume that A from source $\mathscr{A}$ designates the event that the Dow Jones index goes up tomorrow, and B from source $\mathscr{B}$ the event that the Nikkei index goes up tomorrow. If we prefer (A:100, not-A,0) to (B:100, not-B:0), then this may be caused by a special source preference for $\mathscr{A}$ over $\mathscr{B}$, say, if $\mathscr{A}$ comprises less ambiguity for us than $\mathscr{B}$ does. However, it may also occur simply because we think that event A is more likely to occur than event B. To examine ambiguity attitudes we have to find a way to 'correct' for differences in perceived levels of likelihood.

One way to detect (*strong*) *source preference* for $\mathscr{A}$ over $\mathscr{B}$ is to find an $\mathscr{A}$-partition $(A_1, \ldots, A_n)$ and a $\mathscr{B}$-partition $(B_1, \ldots, B_n)$ of the universal event such that for each j, (A_j:100, not-A_j,0)$\succ$(B_j:100, not-B_j:0) (Nehring, 2001, definition 4; Tversky and Fox, 1995; Tversky and Wakker, 1995). Because both partitions span the whole universal event, we cannot have stronger belief in every A_j than B_j (under some plausible assumptions about beliefs), and hence there must be a preference for dealing with $\mathscr{A}$ events beyond belief. Formally, the condition requires that a similar preference of $\mathscr{B}$ over $\mathscr{A}$ is never detected. The Ellsberg paradox is a special case of this procedure.

Under the above approach to source preference, there is a special role for probabilistic sophistication. For a source $\mathscr{A}$ for which not some of its events are more ambiguous than others, it is plausible that $\mathscr{A}$ exhibits source indifference with respect to itself. This condition can be seen to amount to the additivity axiom of qualitative probability (if A_1 is as likely as A_3, and A_2 is as likely as A_4, then $A_1 \cup A_2$ is as likely as $A_3 \cup A_4$ whenever $A_1 \cap A_2 = A_3 \cap A_4 = \emptyset$), which, under sufficient richness, implies probabilistic sophistication for $\mathscr{A}$ under RDU, and does so in general (without RDU assumed) under an extra dominance condition (Fishburn, 1986; Sarin and Wakker, 2000). The condition also comprises source sensitivity (Tversky and Wakker, 1995). Probabilistic sophistication, then, entails a *uniform degree* of ambiguity of a source.

In theoretical economic studies it has usually been assumed that people are averse to ambiguity, corresponding with convex weighting functions. Empirical studies, mostly by psychologists, have suggested a more varied pattern, where different sources of ambiguity can arouse all kinds of emotions. For example, Tversky and Fox (1995) found that basketball fans exhibit source preference for ambiguous uncertain events related to basketball over events with known probabilities, which entails ambiguity seeking. This finding is not surprising in an empirical sense, but its conceptual implication is important: attitudes towards ambiguity depend on many ad hoc emotional aspects, such as a general aversion to deliberate secrecy about compositions of urns, or a general liking of basketball. Uncertainty is a large domain, and fewer regularities can be expected to hold universally for uncertainty than for risk, in the same way as fewer regularities will hold universally for the utility of non-monetary outcomes (hours of listening to music, amounts of milk to be drunk, life duration, and so on) than for the utility of monetary outcomes. It means that there is much yet to be discovered about uncertainty.

Models for uncertainty other than rank-dependence

Multiple priors

An interesting model of ambiguity by Jaffray (1989), with a separation of ambiguity beliefs and ambiguity attitudes, unfortunately has received little attention as yet. A surprising case of unknown probabilities can arise when the expected utility model perfectly well describes behaviour, but utility is state-dependent. The (im)possibility of defining probability in such cases has been widely discussed (Drèze, 1987; Grant and Karni, 2005; Nau, 2006).

The most popular alternative to Schmeidler's RDU is the multiple priors model introduced by Wald (1950). It assumes a set $\mathscr{P}$ of probability measures plus a utility function U, and evaluates each prospect through its minimal expected utility with respect to the probability distributions contained in $\mathscr{P}$. The model has an overlap with RDU: if W is convex, then RDU is the minimal expected utility over $\mathscr{P}$ where $\mathscr{P}$ is the CORE of W, that is, the set of probability measures that dominate W eventwise. Drèze (1961; 1987) independently developed a remarkable analog of the multiple priors model, where the maximal expected utility is taken over $\mathscr{P}$, and $\mathscr{P}$ reflects moral hazard instead of ambiguity. Drèze also provided a preference foundation. Similar functionals appear in studies of robustness against model misspecification in macroeconomics (Hansen and Sargent, 2001).

Variations of multiple priors, combining pessimism and optimism, employ convex combinations of the expected utility minimized over $\mathscr{P}$ and the expected utility maximized over $\mathscr{P}$ (Ghirardato, Maccheroni and Marinacci, 2004, proposition 19). Such models can account for extremity orientedness, as with inverse-S weighting functions and RDU. Arrow and Hurwicz (1972) proposed a similar model where a prospect is evaluated through a convex combination of the minimal and maximal utility of its outcomes (corresponding with $\mathscr{P}$ being the set of all probability measures). This includes maximin and maximax as special cases. Their approach entails a level of ambiguity so extreme that no levels of belief other than 'sure-to-happen', 'sure-not-to-happen' and 'don't know' play a role, similar to Figures 2e and 2f, and suggesting a three-valued logic. Other non-belief-based approaches, including minimax regret, are in Manski (2000) and Savage (1954), with a survey in Barberà, Bossert and Pattanaik (2004).

Other authors proposed models where for each single event a separate interval of probability values is specified (Budescu and Wallsten, 1987; Kyburg, 1983; Manski, 2004). Such interval-probability models are mathematically different from multiple priors because there is no unique relation between sets of probability measures over the whole event space and intervals of probabilities separately for each event (Škulj, 2006, pp. 192–193). The latter models are more tractable than multiple priors because probability intervals for some relevant event are easier to specify than probability measures over the whole space, but these models did not receive a preference foundation and never became popular in economics. Similar models of imprecise probabilities received attention in the statistics field (Walley, 1991).

Wald's multiple priors model did receive a preference axiomatization (Gilboa and Schmeidler, 1989), and consequently became the most popular alternative to RDU for unknown probabilities. The evaluating formula is easier to understand at first than RDU. The flexibility of not having to specify precisely what 'the' probability measure is, while usually perceived as an advantage at first acquaintance, can turn into a disadvantage when applying the model. We then have to specify exactly what 'the' set of probability distributions is, which is more complex than exactly specifying only one probability measure (cf. Lindley, 1996; Tversky and Koehler, 1994, p. 563).

The simple distinction between probability measures that are either possible (contained in $\mathscr{P}$) or impossible (not contained in $\mathscr{P}$), on the one hand adds to the tractability of the model, but on the other hand cannot capture cognitive states where different probability measures are plausible to different degrees. To the best of my knowledge, the multiple priors model cannot yet be used in quantitative empirical measurements today, and there are no empirical assessments of sets of priors available in the literature to date. Multiple priors are, however, well suited for general theoretical analyses where only general properties of the model are needed. Such analyses are considered in many theoretical economic studies, where the multiple priors model is very useful.

The multiple priors model does not allow deviations from expected utility under risk, and a desirable extension would obviously be to combine the model with non-expected utility for risk. Promising directions for resolving the difficulties of the multiple priors model are being explored today (Maccheroni, Marinacci and Rustichini, 2006).

Model-free approaches to ambiguity

Dekel, Lipman and Rustichini (2001) considered models where outcomes of prospects are observed but the state space has not been completely specified, as relevant to incomplete contracts. Similar approaches with ambiguity about the underlying states and events appeared in psychology in repeated-choice experiments by Hertwig et al. (2003), and in support theory (Tversky and Koehler, 1994). This section discusses two advanced attempts to define ambiguity in a model-free way that have received much attention in the economic literature.

In a deep paper, Epstein (1999) initiated one such approach, continued in Epstein and Zhang (2001). Epstein sought to avoid any use of known probabilities and tried to endogenize (un)ambiguity and the use of probabilities. (He often used the term uncertainty as equivalent to ambiguity.) For example, he did not define risk neutrality with respect to known probabilities, as we did above, but with respect to subjective probabilities derived from preferences as in probabilistic sophistication (Epstein, 1999, eq. (2)). He qualified probabilistic sophistication as ambiguity neutrality (not uniformity as done above). Ghirardato and Marinacci (2002) used another approach that is similar to Epstein's. They identified absence of ambiguity not with probabilistic sophistication, as did Epstein, but, more restrictively, with expected utility.

The above authors defined an agent as ambiguity averse if *there exists* another, hypothetical, agent who behaves the same way for unambiguous events, but who is ambiguity neutral for ambiguous events, and such that the real agent has a stronger preference than the hypothetical agent for sure outcomes (or unambiguous prospects, but these can be replaced by their certainty equivalents) over ambiguous prospects. This definition concerns traditional between-agent within-source comparisons as in Yaari (1969). The stronger preferences for certainty are, under rank-dependent models, equivalent to eventwise dominance of weighting functions, leading to non-emptiness of the CORE (Epstein, 1999, lemma 3.4; Ghirardato and Marinacci, 2002, corollary 13). These definitions of ambiguity aversion are not very tractable because of the 'there exists' clause. It is difficult to establish which ambiguity neutral agent to take for the comparisons. To mitigate this problem, Epstein (1999, section 4) proposed eventwise derivatives as models of local probabilistic sophistication. Such derivatives exist only for continua of events with a linear structure, and are difficult to elicit. They serve their purpose only under restrictive circumstances (ambiguity aversion throughout plus constancy of the local derivative, called coherence; see Epstein's Theorem 4.3).

In both above approaches, ambiguity and ambiguity aversion are inextricably linked, making it hard to model attitudes towards ambiguity other than aversion or seeking (such other attitudes include insensitivity), or to distinguish between ambiguity-neutrality or -absence (Epstein, 1999, p. 584, 1st para; Epstein and Zhang, 2001, p. 283; Ghirardato and Marinacci, 2002, p. 256, 2nd para). Both approaches have difficulties distinguishing between the two Ellsberg urns. Each urn in isolation can be taken as probabilistically sophisticated with, in our interpretation, a uniform degree of ambiguity, and Epstein's definition cannot distinguish which of these is ambiguity neutral (cf. Ghirardato and Marinacci, 2002, middle of p. 281). Ghirardato and Marinacci's definition does so, but only because it selects expected utility (and the urn generating such preferences) as the only ambiguity-neutral version of probabilistic sophistication. Any other form of probabilistic sophistication, that is, any non-expected utility behaviour under risk, is then either mismodelled as ambiguity attitude (Ghirardato and Marinacci, 2002, pp. 256–7), or must be assumed not to exist.

We next discuss in more detail a definition of (un)ambiguity by Epstein and Zhang (2001), whose aim was to make (un)ambiguity endogenously observable by expressing it directly in terms of a preference condition. They called an event E *unambiguous* if

$$
\begin{aligned}
&(E{:}c,\ E_2{:}\gamma, E_3{:}\beta, E_4{:}x_4, \ldots, E_n{:}x_n) \succcurlyeq \\
&(E{:}c,\ e(E{:}c,\ E_2{:}\beta, E_3{:}\gamma, E_4{:}x_4, \ldots, E_n{:}x_n) \text{ implies} \\
&(E{:}c',\ E_2{:}\gamma, E_3{:}\beta, E_4{:}x_4, \ldots, E_n{:}x_n) \succcurlyeq \\
&(E{:}c',\ E_2{:}\beta, E_3{:}\gamma, E_4{:}x_4, \ldots, E_n{:}x_n)
\end{aligned} \tag{6}
$$

for all partitions $E_2, \ldots, E_n$ of not-E, and all outcomes $c, c', x_4, \ldots, x_n, \gamma \succcurlyeq \beta$, with a similar condition imposed on not-E. In words, changing a common outcome c into another common outcome c' under E does not affect preference, but this is imposed only if the preference concerns nothing other than to which event (E_2 or E_3) a good outcome γ is to be allocated instead of a worse outcome β. Together with some other axioms, eq. (6) implies that probabilistic sophistication holds on the set of events satisfying this condition, which in the interpretation of the authors designates absence of ambiguity (rather than uniformity). As we will see next, it is not clear why eq. 6 would capture the absence of ambiguity.

Example. Assume that events are subsets of [0,1), $E = [0, 0.5)$, not-$E = [0.5, 1)$, and E has unknown probability π. Every subset A of E has probability $2\pi\lambda(A)$ (λ is the usual Lebesgue measure, that is, the uniform distribution over [0,1)) and every subset B of not-E has probability $2(1-\pi)\lambda(B)$. Then it seems plausible that event E and its complement not-E are ambiguous, but conditional on these events ('within them') we have probabilistic sophistication with respect to the conditional Lebesgue measure and without any ambiguity. In Schmeidler (1989), the ambiguous events E and not-E are called horse events, and the unambiguous events conditional on them are called roulette events. Yet, according to eq. (6), events E and not-E themselves are unambiguous, both preferences in eq. (6) being determined by whether $\lambda(E_2)$ exceeds $\lambda(E_3)$. □

In the example, the definition in eq. (6) erroneously ascribes the unambiguity that holds for events conditional on E, so 'within E', to E as a whole. Similar examples can be devised where E and not-E themselves are unambiguous, there is 'non-uniform' ambiguity conditional on E, this ambiguity is influenced by outcomes conditional on not-E through non-separable interactions typical of non-expected utility, and eq. (6) erroneously ascribes the ambiguity that holds within E to E as a whole (see the author's homepage).

A further difficulty with eq. (6) is that it is not violated in the Ellsberg example with urns A and K as above (nor if the uncertainty regarding each urn is extended to a 'uniform' continuum as in Example 5.8ii of Abdellaoui and Wakker, 2005), and cannot detect which of the urns is ambiguous. The probabilistic sophistication that is obtained in Epstein and Zhang (2001, Theorem 5.2) for events satisfying eq. (6), and that rules out the two-urn Ellsberg paradox and its continuous extension of Abdellaoui and Wakker (2005), is mostly driven by their Axioms 4 and 6 (the latter is not satisfied by all rank-dependent utility maximizers contrary to the authors' claim at the end of their Section 4; their footnote 18 is incorrect) and the necessity to consider also intersections of different-urn events (see their Appendix E). This imposes, in my terminology, a uniformity of ambiguity over the events satisfying eq. (6) that, rather than eq. (6) itself, rules out the above counterexamples.

Multi-stage approaches to ambiguity

Several authors have considered two-stage approaches with intersections of first-stage events $A_i, i = 1, \ldots, \ell$ and second-stage events $K_j, j = 1, \ldots, k$, so that $n = \ell k$ events A_iK_j result, and prospects $(A_iK_j{:}x_{ij})_{i=1}^{\ell}\,{}_{j=1}^{k}$ are considered. It can be imagined that in a first stage it is determined which event A_i obtains, and then in a second stage, conditional on A_i, which event K_j obtains. Many authors considered such two-stage models with probabilities given for the events in both stages, the probabilities of the first stage interpreted as ambiguity about the probabilities of the second stage, and non-Bayesian evaluations used (Levi, 1980; Segal, 1990; Yates and Zukowski, 1976).

Other authors considered representations

$$\sum_{i=1}^{\ell} Q(A_i)\varphi\left(\sum_{j=1}^{k} P(K_j)U(x_{ij})\right) \qquad (7)$$

for probability measures P and Q, a utility function U, and an increasing transformation φ. For φ the identity or, equivalently, φ linear, traditional expected utility with backwards induction results. Nonlinear φ's give new models. Kreps and Porteus (1979) considered eq. (7) for intertemporal choice, interpreting nonlinear φ's as non-neutrality towards the timing of the resolution of uncertainty. Ergin and Gul (2004), Nau (2006) and Neilson (1993) reinterpreted the formula, where now the second-stage events are from a source of different ambiguity than the first-stage events. A concave φ, for instance, suggests stronger preference for certainty, and more ambiguity aversion, for the first-stage uncertainty than for the second.

Klibanoff, Marinacci and Mukerji (2005) considered cases where the decomposition into A- and K-events is endogenous rather than exogenous. This approach greatly enlarges the scope of application, but their second-order acts, that is, prospects with outcomes contingent on aspects of preferences, are hard to implement or observe if those aspects cannot be related to exogenous observables.

Equation (7) has a drawback similar to eq. (3). All extra mileage is to come from the outcomes, to which also utility applies, so that there will not be a great improvement in descriptive performance or in new concepts to be developed.

Conclusion

The Allais paradox reveals violations of expected utility in an absolute sense, leading to convex or inverse-S weighting functions for risk and, more generally, for uncertainty. The Ellsberg paradox reveals deviations from expected utility in a relative sense, showing that an agent can deviate more from expected utility for one source of uncertainty (say one with unknown probabilities) than for another (say, one with known probabilities). It demonstrates the importance of within-subject between-source comparisons.

The most popular models for analysing uncertainty today are based on rank dependence, with multiple priors a popular alternative in theoretical studies. The most frequently studied phenomenon is ambiguity aversion. Uncertainty is, however, a rich empirical domain with a wide variety of phenomena, where ambiguity aversion and ambiguity insensitivity (inverse-S) are prevailing but are not universal patterns. The possibility of relating the properties of weighting functions for uncertainty to cognitive interpretations such as insensitivity to likelihood-information makes RDU and prospect theory well suited for links with other fields such as psychology, artificial intelligence (Shafer, 1976) and neuroeconomics (Camerer, Loewenstein and Prelec, 2004).

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See also; **Allais, Maurice; Allais paradox; ambiguity and ambiguity aversion; Bernoulli, Daniel; certainty equivalence; de Finetti, Bruno; decision theory in econometrics; expected utility hypothesis; Kahneman, Daniel; non-expected utility theory; rational behaviour; rational expectations; revealed preference theory; risk; risk aversion; Savage, Leonard J.; Savage's subjective expected utility model; separability; statistical decision theory; stochastic dominance; Tversky, Amos; utility.**

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Bibliography

Abdellaoui, M. 2000. Parameter-free elicitation of utilities and probability weighting functions. *Management Science* 46, 1497–512.

Abdellaoui, M., Vossmann, F. and Weber, M. 2005. Choice-based elicitation and decomposition of decision weights for gains and losses under uncertainty. *Management Science* 51, 1384–99.

Abdellaoui, M. and Wakker, P.P. 2005. The likelihood method for decision under uncertainty. *Theory and Decision* 58, 3–76.

Allais, M. 1953. Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine. *Econometrica* 21, 503–46.

Arrow, K. and Hurwicz, L. 1972. An optimality criterion for decision making under ignorance. In *Uncertainty and Expectations in Economics: Essays in Honour of G.L.S. Shackle*, ed. C. Carter and J. Ford. Oxford: Basil Blackwell.

Barberà, S., Bossert, W. and Pattanaik, P. 2004. Ranking sets of objects. In *Handbook of Utility Theory, Vol. 2, Extensions*, ed. S. Barberà, P. Hammond and C. Seidl. Dordrecht: Kluwer Academic Publishers.

Bernoulli, D. 1738. Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5, 175–92.

Bleichrodt, H. and Pinto, J. 2000. A parameter-free elicitation of the probability weighting function in medical decision analysis. *Management Science* 46, 1485–96.

Budescu, D. and Wallsten, T. 1987. Subjective estimation of precise and vague uncertainties. In *Judgmental Forecasting*, ed. G. Wright and P. Ayton. New York: Wiley.

Camerer, C., Loewenstein, G. and Prelec, D. 2004. Neuroeconomics: why economics needs brains. *Scandinavian Journal of Economics* 106, 555–79.

Camerer, C. and Weber, M. 1992. Recent developments in modelling preferences: uncertainty and ambiguity. *Journal of Risk and Uncertainty* 5, 325–70.

Chateauneuf, A. 1991. On the use of capacities in modeling uncertainty aversion and risk aversion. *Journal of Mathematical Economics* 20, 343–69.

Chateauneuf, A., Eichberger, J. and Grant, S. 2005. Choice under uncertainty with the best and worst in mind: NEO-additive capacities. CERMSEM, CEM, University of Paris I.

Chew, S.H. 1983. A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox. *Econometrica* 51, 1065–92.

Chew, S.H. and Sagi, J. 2006. Small worlds: modeling attitudes towards sources of uncertainty. Haas School of Business, University of California, Berkeley.

de Finetti, B. 1931. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae* 17, 298–329.

Dekel, E. 1986. An axiomatic characterization of preferences under uncertainty: weakening the independence axiom. *Journal of Economic Theory* 40, 304–18.

Dekel, E., Lipman, B. and Rustichini, A. 2001. Representing preferences with a subjective state space. *Econometrica* 69, 891–934.

Drèze, J. 1961. Les fondements logiques de l'utilité cardinale et de la probabilité subjective. *La Décision, Colloques Internationaux du CNRS*, 73–83.

Drèze, J. 1987. *Essays on Economic Decision under Uncertainty.* Cambridge: Cambridge University Press.

Einhorn, H. and Hogarth, R. 1986. Decision making under ambiguity. *Journal of Business* 59, S225–S250.

Ellsberg, D. 1961. Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics* 75, 643–69.

Epstein, L. 1999. A definition of uncertainty aversion. *Review of Economic Studies* 66, 579–608.

Epstein, L. and Zhang, J. 2001. Subjective probabilities on subjectively unambiguous events. *Econometrica* 69, 265–306.

Ergin, H. and Gul, F. 2004. A subjective theory of compound lotteries. Working paper, MIT.

Fellner, W. 1961. Distortion of subjective probabilities as a reaction to uncertainty. *Quarterly Journal of Economics* 75, 670–89.

Fishburn, P. 1986. The axioms of subjective probability. *Statistical Science* 1, 335–58.

Ghirardato, P., Maccheroni, F. and Marinacci, M. 2004. Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory* 118, 133–73.

Ghirardato, P. and Marinacci, M. 2002. Ambiguity made precise: a comparative foundation. *Journal of Economic Theory* 102, 251–89.

Gilboa, I., ed. 2004. *Uncertainty in Economic Theory: Essays in Honor of David Schmeidler's 65th Birthday.* London: Routledge.

Gilboa, I. and Schmeidler, D. 1989. Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics* 18, 141–53.

Gonzalez, R. and Wu, G. 1999. On the shape of the probability weighting function. *Cognitive Psychology* 38, 129–66.

Grant, S. and Karni, E. 2005. Why does it matter that beliefs and valuations be correctly represented? *International Economic Review* 46, 917–34.

Greenspan, A. 2004. Innovations and issues in monetary policy: the last fifteen years. *American Economic Review, Papers and Proceedings* 94, 33–40.

Gul, F. 1991. A theory of disappointment aversion. *Econometrica* 59, 667–86.

Hansen, L. and Sargent, T. 2001. Robust control and model uncertainty. *American Economic Review, Papers and Proceedings* 91, 60–6.

Hansen, L.P., Sargent, T.J. and Tallarini, T.D. 1999. Robust permanent income and pricing. *Review of Economic Studies* 66, 873–908.

Hertwig, R., Barron, G., Weber, E. and Erev, I. 2003. Decisions from experience and the effect of rare events in risky choice. *Psychological Science* 15, 534–9.

Hogarth, R.M. and Kunreuther, H.C. 1989. Risk, ambiguity, and insurance. *Journal of Risk and Uncertainty* 2, 5–35.

Jaffray, J.-Y. 1989. Linear utility theory for belief functions. *Operations Research Letters* 8, 107–12.

Kahn, B. and Sarin, R. 1988. Modeling ambiguity in decisions under uncertainty. *Journal of Consumer Research* 15, 265–72.

Kahneman, D. and Tversky, A. 1979. Prospect theory: an analysis of decision under risk. *Econometrica* 47, 263–91.

Keynes, J.M. 1921. *A Treatise on Probability.* London: Macmillan.

Kilka, M. and Weber, M. 2001. What determines the shape of the probability weighting function under uncertainty. *Management Science* 47, 1712–26.

Kirkpatrick, L. and Epstein, S. 1992. Cognitive-experiential self-theory and subjective probability: further evidence for two conceptual systems. *Journal of Personality and Social Psychology* 63, 534–44.

Klibanoff, P., Marinacci, M. and Mukerji, S. 2005. A smooth model of decision making under ambiguity. *Econometrica* 73, 1849–92.

Knight, F. 1921. *Risk, Uncertainty, and Profit.* New York: Houghton Mifflin.

Köbberling, V., Schwieren, C. and Wakker, P.P. 2006. Prospect-theory's diminishing sensitivity versus economics' intrinsic utility of money: how the introduction of the euro can be used to disentangle the two empirically. Mimeo, Department of Economics, University of Maastricht, The Netherlands.

Kreps, D. and Porteus, E. 1979. Dynamic choice theory and dynamic programming. *Econometrica* 47, 91–100.

Kyburg, H., Jr. 1983. *Epistemology and Inference.* Minneapolis: University of Minnesota Press.

Levi, I. 1980. *The Enterprise of Knowledge.* Cambridge, MA: MIT Press.

Lindley, D. 1996. Discussion of Walley (1996). *Journal of the Royal Statistical Society B* 58, 47–8.

Lopes, L. 1987. Between hope and fear: the psychology of risk. *Advances in Experimental Psychology* 20, 255–95.

MacCrimmon, K. and Larsson, S. 1979. Utility theory: axioms versus 'paradoxes'. In *Expected Utility Hypotheses and the Allais Paradox*, ed. M. Allais and O. Hagen. Dordrecht: Reidel.

Machina, M. 2004. Almost-objective uncertainty. *Economic Theory* 24, 1–54.

Machina, M. and Schmeidler, D. 1992. A more robust definition of subjective probability. *Econometrica* 60, 745–80.

Maccheroni, F., Marinacci, M. and Rustichini, A. 2006. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica* 74, 1447–98.

Manski, C. 2000. Identification problems and decisions under ambiguity: empirical analysis of treatment response and normative analysis of treatment choice. *Journal of Econometrics* 95, 415–42.

Manski, C. 2004. Measuring expectations. *Econometrica* 72, 1329–76.

McClelland, A. and Bolger, F. 1994. The calibration of subjective probabilities: theories and models 1980–1994. In *Subjective Probability*, ed. G. Wright and P. Ayton. New York: Wiley.

Mukerji, S. and Tallon, J.-M. 2001. Ambiguity aversion and incompleteness of financial markets. *Review of Economic Studies* 68, 883–904.

Nau, R. 2006. Uncertainty aversion with second-order utilities and probabilities. *Management Science* 52, 136–45.

Nehring, K. 2001. Ambiguity in the context of probabilistic beliefs. Mimeo, UC Davis.

Neilson, W.S. 1993. Ambiguity aversion: an axiomatic approach using second order probabilities. Mimeo, Dept. of Economics, University of Tennessee, Knoxville.

Piaget, J. and Inhelder, B. 1975. *The Origin of the Idea of Chance in Children.* New York: Norton.

Quiggin, J. 1982. A theory of anticipated utility. *Journal of Economic Behaviour and Organization* 3, 323–43.

Ramsey, F. 1931. Truth and probability. In *The Foundations of Mathematics and other Logical Essays.* London: Routledge and Kegan Paul.

Sarin, R. and Wakker, P.P. 2000. Cumulative dominance and probabilistic sophistication. *Mathematical Social Sciences* 40, 191–6.

Savage, L.J. 1954. *The Foundations of Statistics.* New York: Wiley.

Schmeidler, D. 1982. Subjective probability without additivity. Working paper, Foerder Institute of Economic Research, Tel Aviv University.

Schmeidler, D. 1989. Subjective probability and expected utility without additivity. *Econometrica* 57, 571–87.

Segal, U. 1990. Two-stage lotteries without the reduction axiom. *Econometrica* 58, 349–77.

Shafer, G. 1976. *A Mathematical Theory of Evidence.* Princeton: Princeton University Press.

Škulj, D. 2006. Jeffrey's conditioning rule in neighbourhood models. *International Journal of Approximate Reasoning* 42, 192–211.
Starmer, C. 2000. Developments in non-expected utility theory: the hunt for a descriptive theory of choice under risk. *Journal of Economic Literature* 38, 332–82.
Tversky, A. and Fox, C. 1995. Weighing risk and uncertainty. *Psychological Review* 102, 269–83.
Tversky, A. and Kahneman, D. 1992. Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5, 297–323.
Tversky, A. and Koehler, D. 1994. Support theory: a nonextensional representation of subjective probability. *Psychological Review* 101, 547–67.
Tversky, A. and Wakker, P.P. 1995. Risk attitudes and decision weights. *Econometrica* 63, 1255–80.
Wakker, P.P. 2001. Testing and characterizing properties of nonadditive measures through violations of the sure-thing principle. *Econometrica* 69, 1039–59.
Wald, A. 1950. *Statistical Decision Functions*. New York: Wiley.
Walley, P. 1991. *Statistical Reasoning with Imprecise Probabilities*. London: Chapman and Hall.
Weber, E. 1994. From subjective probabilities to decision weights: the effects of asymmetric loss functions on the evaluation of uncertain outcomes and events. *Psychological Bulletin* 115, 228–42.
Wu, G. and Gonzalez, R. 1999. Nonlinear decision weights in choice under uncertainty. *Management Science* 45, 74–85.
Yaari, M. 1969. Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory* 1, 315–29.
Yates, J. and Zukowski, L. 1976. Characterization of ambiguity in decision making. *Behavioral Science* 21, 19–25.

uncertainty and general equilibrium

One of the notable intellectual achievements of economic theory during the second half of the 20th century has been the rigorous elaboration of the Walras–Pareto theory of value; that is, the theory of the existence and optimality of competitive equilibrium. Although many economists and mathematicians contributed to this development, the resulting edifice owes so much to the pioneering and influential work of Arrow and Debreu that in this paper we shall refer to it as the 'Arrow–Debreu theory'. (For comprehensive treatments, together with references to previous work, see Debreu, 1959; Arrow and Hahn, 1971.)

The Arrow–Debreu theory was not originally put forward for the case of uncertainty, but an ingenious device introduced by Arrow (1953), and further elaborated by Debreu (1953), enabled the theory to be reinterpreted to cover the case of uncertainty about the availability of resources and about consumption and production possibilities. (See Debreu, 1959, ch. 7, for a unified treatment of time and uncertainty.)

Subsequent research has extended the Arrow–Debreu theory to take account of (*a*) differences in information available to different economic agents, and the 'production' of information, (*b*) the incompleteness of markets, and (*c*) the sequential nature of markets. The consideration of these complications has stimulated the developments of new concepts of equilibrium, two of which will be elaborated in this article under the headings: (*a*) equilibrium of plans, prices, and price expectations (EPPPE) and (*b*) rational expectations equilibrium (REE). The exploration of these features of real-world markets has also made possible a general-equilibrium analysis of money and securities markets, institutions about which the original Arrow–Debreu theory could provide only limited insights. It has also led to a better understanding of the limits to the ability of the 'invisible hand' in attaining a Pareto optimal allocation of resources.

Review of the Arrow–Debreu model of a complete market for present and future contingent delivery

In this section, we review the approach of Arrow (1953) and Debreu (1959) to incorporating uncertainty about the environment into a Walrasian model of competitive equilibrium. The basic idea is that commodities are to be distinguished, not only by their physical characteristics and by the location and dates of their availability and/or use, but *also by the environmental event in which they are made available and/or used*. For example, ice cream made available (at a particular location on a particular date) if the weather is hot may be considered to be a different commodity from the same kind of ice cream made available (at the same location and date) if the weather is cold. We are thus led to consider a list of 'commodities' that is greatly expanded by comparison with the corresponding case of certainty about the environment. The standard arguments of the theory of competitive equilibrium, applied to an economy with this expanded list of commodities, then require that we envisage a 'price' for each commodity, the resulting set of price ratios specifying the market rate of exchange between each pair of commodities.

Just what institutions could, or do, effect such exchanges is a matter of interpretation that is, strictly speaking, outside the model. We shall present one straightforward interpretation, and then comment briefly on an alternative interpretation.

First, however, it will be useful to give a more precise account of concepts of environment and event that we shall be employing. The description of the 'physical world' is decomposed into three sets of variables: (*a*) decision variables, which are controlled (chosen) by