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EXPLAINING THE CHARACTERISTICS OF THE POWER (CRRA) UTILITY FAMILY

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SUMMARY

The power family, also known as the family of constant relative risk aversion (CRRA), is the most widely used parametric family for fitting utility functions to data. Its characteristics have, however, been little understood, and have led to numerous misunderstandings. This paper explains these characteristics in a manner accessible to a wide audience. Copyright © 2008 John Wiley & Sons, Ltd.

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INTRODUCTION

This paper provides a toolkit for empirical researchers who use the power family to fit data. We will mainly consider data fitting for interval scales, the most common scale type of utility functions. Although the power family is the most widely used parametric family for this purpose, the intricacies of the family have been little understood. This paper provides an accessible account of these intricacies.

The power family contains functions of the form $U(x) = x^r$, and is also known in the economic literature as the family of constant relative risk aversion (CRRA). It has been widely used for modeling risk aversion, not only in the economic domain (Holt and Laury, 2002; Palacios-Huerta and Serrano, 2006) but also in psychology (Luce and Krumhansi, 1988) and in the health domain (Bleichrodt *et al.*, 1999). It often gives a better fit than alternative families (Abellán *et al.*, 2006; Camerer and Ho, 1994, footnote 22). The input x can designate any quantity such as money, life duration, or a quality-of-life index to be converted into decision utility. Unfortunately, there have been many misunderstandings about the characteristics of this family, where the choice of origin $x = 0$ is crucial, the functions exhibit extreme behavior near that origin, and the negative powers have often been overlooked.

For interval scales (or 'cardinal' scales), the level (intercept) and unit (slope) have no empirical meaning, and for all purposes the function $U(x) = b + ax^r$ is equivalent to $U(x) = x^r$ for any $a > 0$ and $b \in \mathbb{R}$. U is, for instance, an interval scale if it designates utility to be maximized in expected utility. An example where U is not an interval scale occurs in production theory, with L designating labor, x designating capital, and output being the product $L \times x^r$. Whereas doubling a utility function has no empirical relevance in expected utility because it leaves the ordering of all prospects unaffected, doubling a production function obviously has much empirical relevance. The analysis of this paper, therefore, does not apply to production functions.

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Unlike the intercept and the slope, the curvature of U (with a formal index of curvature provided later) does have empirical meaning for interval scales. The status of utility as an interval scale implies, for instance—as will be demonstrated later—that the limit of $U(x) = x^r$ for r tending to 0 is not the constant function $U(x) = 1$, which would be the case if x^r were an absolute scale. The limit now rather is the logarithmic function $U(x) = \ln(x)$, to which x^r converges in terms of curvature. The subtle implications of U being an interval scale rather than an absolute scale have aggravated the misunderstandings about the power family.

So as to ensure accessibility to readers without a technical background, this paper will avoid the use of mathematical arguments, sometimes using graphs to illustrate certain points. The next section gives the definition of the power family. This definition will at first sight seem to be *ad hoc*. An example will illustrate a typical misunderstanding about the power family. The subsequent section shows that the definition of the power family is natural, so that it is not *ad hoc* after all. The section further considers the scaling of outputs ('values') $U(x)$. It does so for positive inputs ('arguments') $x>0$. The two subsequent sections discuss some numerical problems in the scalings of inputs and outputs of power utility and suggest remedies to the extent possible. These two sections are the most complex ones in this paper. The first one describes theoretical and empirical problems caused by the extreme behavior of outputs $U(x)$ near $x = 0$. The subsequent section considers scalings and rescalings of the inputs x. Invariance with respect to any change in the unit of input x characterizes the power family, which means that this invariance is the critical condition for the appropriateness of this family. The level of the inputs, i.e. the definition of the origin $x = 0$ of the input, is empirically relevant and should, therefore, be specified in empirical applications.

For an interval scale, being unique up to the unit and level of its outputs, all relevant aspects are captured by the Arrow–Pratt index of concavity, and an analysis of this index will confirm the observations made previously. Appendix A illustrates the theoretical observations of this paper through empirical implications for preferences, regarding indifference curves and their marginal rates of substitution. Appendix B contains some calculations for Appendix A.

DEFINITION AND EXAMPLE ILLUSTRATING A COMMON MISUNDERSTANDING

This section and the following section only consider inputs $x>0$. The subsequent section will consider extended domains. The power family of utility U, with parameter r, is defined as follows:

$$\text{If } r>0 \text{ then } U(x) = x^r \tag{1}$$

$$\text{if } r=0 \text{ then } U(x) = \ln(x) \tag{2}$$

$$\text{if } r<0 \text{ then } U(x) = -x^r \tag{3}$$

All functions are strictly increasing, which explains the minus sign for $r<0$. At first sight, the transition at $r = 0$ may seem to be *ad hoc*. In the sections to follow we will see that the functions in Equations (1)–(3) nevertheless belong together in a natural way. The following example illustrates a common misunderstanding about the power family.

Example 2.1 (Erroneously overlooking negative powers)
Assume that we use certainty equivalents to assess the utility of life duration for an individual. We call y the *certainty equivalent* of $(p : x, z)$ if living y years for sure is equally preferable to the *prospect* of living

Table I. Numerical results from parametric fittings of data

		Considered by researcher						Overlooked by researcher			
		U : exp	$U = x^r$					$U = \ln(x)$	$U = -x^r$		
	Data	$e^{-0.064x}$	$r = 2$	$r = 1$	$r = 0.1$	$r = 0.01$	$r = 0.001$	$r = 0$	$r = -0.1$	$r = -0.52$	$r = -1$
CE(0.75:50,10)	29	28.40	43.59	40	34.24	33.52	33.45	33.44	32.61	29.01	25.00
CE(0.50:50,10)	19	19.67	36.06	30	23.10	22.43	22.37	22.36	21.65	18.98	16.67
CE(0.25:50,10)	13.5	14.10	26.46	20	15.33	14.99	14.96	14.95	14.60	13.42	12.50
Distance	0	1.16	671	284	48	34	33.22	33.09	21.30	0.01	22.45

The entries in the U columns represent certainty equivalents (abbreviated CE) of prospects in corresponding rows under expected utility with the corresponding U. For example, under expected utility with $U(x) = x^2$, CE(0.50:50,10), the certainty equivalent of (0.50:50,10), is 36.06, deviating considerably from the observed certainty equivalent 19.

x years with probability p and living z years with probability $1-p$, denoted $y \sim (p : x, z)$. We assume expected utility, implying $U(y) = pU(x) + (1-p)U(z)$. Assume that our data set contains three observed certainty equivalents for the individual, exhibiting considerable risk aversion:

$$29 \sim (0.75 : 50, 10)$$

$$19 \sim (0.50 : 50, 10)$$

$$13.50 \sim (0.25 : 50, 10)$$

A researcher searches for a utility function U that best fits the data in the sense of minimizing the squared distances between the certainty equivalents observed and those theoretically predicted by expected utility with U.[1] He first considers the power family $U(x) = x^r$, but erroneously restricts attention to positive powers $r > 0$. Table I displays some numerical results. The middle block of columns under '$U(x) = x^r$' displays some of the cases now considered by the researcher.

The results are unsatisfactory, with a degenerate solution resulting: the optimal fit is for the smallest r considered, say $r = 0.001$. The theoretically predicted certainty equivalents for positive powers will never be better than 33.44, 22.36, and 14.95, respectively, which can be seen to be their limits as r tends to 0.[2] They deviate considerably from the certainty equivalents observed in the data and never get close. The sum of squared distances is always worse than $(33.44 - 29)^2 + (22.36 - 19)^2 + (14.95 - 13.50)^2 = 33.09$ (its limit if r tends to 0). The power family seemingly cannot fit the data well, and no good descriptions or predictions result. The functions considered fail to accommodate strong degrees of concavity and risk aversion.

The researcher next turns to the exponential family of utility with concave, risk averse, functions $U(x) = 1 - \exp(-\alpha x)$ for $\alpha > 0$. This family gives a considerably better fit. With $\alpha = 0.064$ (also known through the 'risk tolerance' $\frac{1}{0.064} \approx \$15$), the predicted certainty equivalents (see the third column in the table) fit the data considerably better, with sum of squared distances 1.16. The researcher concludes that the exponential utility fits the data better than the power family, but the fit never is very good.

The researcher's error in the preceding analysis was to overlook the negative powers and, thus, to leave out the most concave part of the power family. With negative powers incorporated, the optimal fit results for $r = -0.52$ and $U(x) = -x^{-0.52}$, with certainty equivalents fitting the data almost perfectly

[1] For prospect $(p : x, z)$, the theoretically predicted certainty equivalent is the sure amount with utility equal to the expected utility of the prospect, i.e. it is $U^{-1}\ (pU(x) + (1-p)U(z))$.

[2] The limiting data are as would result from $U(x) = \ln(x)$. The limit of x^r for r tending to 0 is $\ln(x)$ indeed, and not the constant function $U(x) = 1$, as we will see later.

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well; see the table. The squared distance is $(29.01 - 29)^2 + (18.98 - 19)^2 + (13.42 - 13.50)^2 = 0.01$. With negative powers properly incorporated, the power family clearly outperforms the exponential family and gives an almost perfect fit. □

THE NATURAL POWER FAMILY AND SCALING OUTPUTS $U(x)$ FOR POSITIVE INPUTS $x > 0$

This section demonstrates that the functions in Equations (1)–(3) naturally belong together under the following assumption made throughout this paper. As in the previous section, we consider only positive inputs x in this section.

Assumption 3.1

U is *unique up to unit and level*, i.e. it can be multiplied by any positive factor and any constant can be added without affecting any relevant empirical aspect (U is an *interval scale*). □

The assumption is satisfied in many applications, not only in expected utility but also under most nonexpected utility theories. It is also satisfied if health states are ordered according to an additive aggregation of health components, in separable orderings of commodity bundles or multi-criteria choice options, in additive intertemporal aggregations such as the quality-adjusted life years model (for a recent test see Brazier *et al.*, 2006) and discounted utility, and in utilitarian welfare evaluations of allocations of goods and services to people. Assumption 3.1 implies that the power family can be rewritten as

$$\text{If } r > 0 \text{ then } U(x) = ax^r + b \text{ for some } b \in \mathbb{R} \text{ and } a > 0 \tag{4}$$

$$\text{if } r = 0 \text{ then } U(x) = a \ln(x) + b \text{ for some } b \in \mathbb{R} \text{ and } a > 0 \tag{5}$$

$$\text{if } r < 0 \text{ then } U(x) = b - ax^r \text{ for some } b \in \mathbb{R} \text{ and } a > 0 \tag{6}$$

The choice of a and b is immaterial here. Sometimes Equations (4) and (6) are compactly combined into $(c/r)x^r + b$ for $c > 0$ and b, where $a = c/r$ for Equation (4) and $a = -c/r$ for Equation (6). In economics, $1 - r$, which is $-xU''(x)/U'(x)$, is often taken as an index of risk aversion, called the index of relative risk aversion, which is constant for the power family (*constant relative risk aversion, CRRA*).[3] In decision analysis, the reciprocal of the CRRA index, $1/(1 - r)$, called *risk tolerance*, is often used. In other fields, these indexes are not commonly used, however. To make this paper accessible to readers with different backgrounds I will state all conditions in terms of r. In economic terms, r is the elasticity of x^r.[4] In consumer demand theory it reflects the elasticity of substitution (Nicholson, 2005).

Another popular parametric family of utility is the exponentional family, also called the family of constant absolute risk aversion (CARA). It results if we apply the power family to e^x instead of x. Hence, the results of this paper can be applied to this family through the transformation described.

[3] A general drawback of the term relative risk aversion is that it cannot be used well for applications outside decision under risk. A serious drawback of this term within decision under risk is that risk attitude cannot be equated with utility curvature for nonexpected utility models, because risk attitude also depends on other factors such as probability weighting in prospect theory. Hence, it would be better to replace 'risk aversion' by 'concavity' in such terminologies.

[4] A change in x of δ% generates a change in x^r of $r\delta$% for δ close to 0.

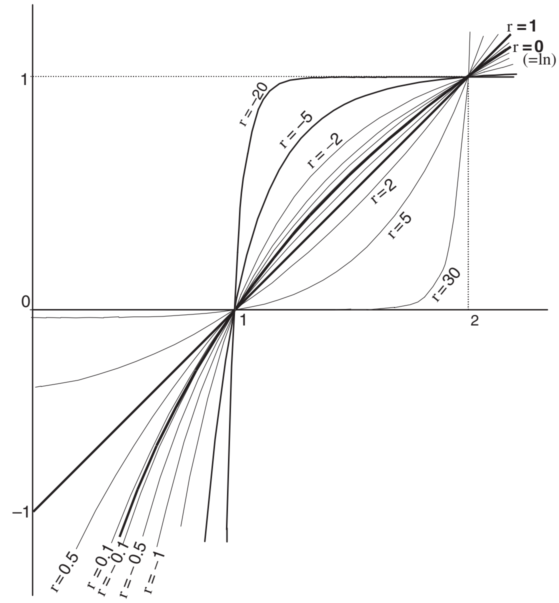

Figure 1. Log-power utility curves (normalized at $x = 1$ and 2)

It is often useful, when comparing interval scales, to choose a common normalization. In the figures hereafter, we normalize the functions considered to be 0 at 1 and 1 at 2. (Any other normalization would give the same conclusions.) That is, we replace $U(x)$ by[5]

$$\frac{U(x) - U(1)}{U(2) - U(1)} \tag{7}$$

and denote this function by $U(x)$ hereafter. Figure 1 depicts the function for various values of r. The figure shows that r is an anti-index of concavity, with lower values of r corresponding with more concave functions. The linear case arises for $r = 1$ and reflects risk neutrality under expected utility. For $r < 1$ the functions are concave and for $r > 1$ they are convex.

Figure 1 demonstrates that the choice of a logarithmic function for $r = 0$ is not *ad hoc*, but is the only natural choice, with this function uniquely embedded between positive and negative r. The curves for $r = 0.1$ and -0.1 are almost indistinguishable from the (fat) logarithmic curve for $r = 0$, indeed. Example 2.1 already illustrated this property. The property can be proved formally by showing that the normalized logarithmic function is the limit of the normalized power functions for r tending to 0, both from above ($r > 0$) and from below ($r < 0$):

$$\lim_{r \to 0} \frac{x^r - c^r}{d^r - c^r} = \frac{\ln(x) - \ln(c)}{\ln(d) - \ln(c)} \quad \text{for all positive } x \text{ and } d > c > 0 \tag{8}$$

A derivation is omitted. A common normalization as in Equation (7), reflecting the interval scale type of these functions, is essential in this limiting result.

Figure 1 also demonstrates that the negative part ($r < 0$, i.e. the CRRA index exceeds 1) naturally belongs to the family. Without this part, only the least-concave half of the family remains. Whereas powers below 0 (CRRA indexes exceeding 1) are well known and are widely used in economics, many

[5] For Equation (4), this means that $a = 1/(2^r - 1)$ and $b = -1/(2^r - 1)$ for the normalized U.

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empirical researchers from other domains overlook the negative part and only use the positive part ($r > 0$) of the family, leading to problems as in Example 2.1. I have seen several attempts to fit power functions to data that failed for this reason, and that, not surprisingly, did not make it into published papers. Figure 1 clarifies how this could happen. All data for which the optimal curve is more concave than logarithmic ($r < 0$) result in such degenerate solutions, erroneously interpreted as r tending to 0. The negative part has also been overlooked in some theoretical analyses, such as in Krantz *et al*. (1971, Section 4.5.3). The latter omission was pointed out and corrected by Miyamoto (1983).

SCALING OUTPUTS $U(x)$ FOR INPUTS $x \approx 0, x = 0$, AND $x < 0$

The extreme behavior of the power functions near $x = 0$ can cause problems. This section describes this extreme behavior and the problems caused. Some pragmatic remedies are suggested to avoid, or at least reduce, the problems, to the extent possible.

Scaling outputs $U(x)$ for $x > 0$

At first we continue to assume that the domain of interest concerns only positive $x, x > 0$. For $r > 0$, the power functions are bounded below, where $ax^r + b$ tends to b as x tends to 0, but they are unbounded above. For $r = 0$ the power function – which then is the logarithmic function – is unbounded above and below, tending to $-\infty$ if x tends to 0. For $r < 0$, the power functions are bounded above with $-ax^r + b$ tending to b as x tends to ∞, and unbounded below, with limit $-\infty$ as x tends to 0.

For $0 < r < 1$, the functions themselves are bounded near $x = 0$ but their derivatives are extreme near $x = 0$, tending to ∞ as x tends to 0. The following example illustrates the extreme behavior of negative powers near $x = 0$.

Example 4.1 (Extreme Behavior near $x = 0$ for $r = -1$)
Assume expected utility for $x > 0$, with $U(x) = 1 - 1/x$ ($r = -1$). Then receiving \$2 for sure is preferred to a 50–50 prospect yielding either \$1 or \$$M$, no matter how large M is. The utility of M never attains the level 1, which is the level required to yield indifference. Likewise, for any large M, say a million dollars, receiving a small amount s for sure, such as two cents, is preferred to a 50–50 prospect yielding M or $s/2$ (one cent). □

In medical applications, behavior as illustrated in the example may be found for subjects with an extreme dislike of risking death in the very short term. In such a case, negative powers are appropriate. For many applications, however, such extreme behavior near $x = 0$ for $r \leq 0$ may be undesirable. We first present some pragmatic considerations. Problems as described do not arise on intervals $[c, d]$ with c remote from 0, where a pragmatic threshold can be $c/d \geq 0.1$.[6] If more concavity is desired than positive r can generate on such intervals, then negative r can be considered; see Example 2.1. Such cases often arise in macroeconomics and finance, where the power (CRRA) family is most commonly used. Because large inputs x, remote from 0, are common in these fields, elevated parts of the curve become relevant, where the curvature is less pronounced. A theoretical explanation of the latter claim will be given later. To obtain a proper level of curvature, a negative r then has to be used, and this is indeed common practice in these fields (Bliss and Panigirtzoglou, 2004). The anomalies near $x = 0$ described in Example 4.1 are not important in such applications. The following example illustrates the case.

[6] Given that scale changes of x are immaterial as we will see later, the normalized c/d is a more plausible measure of distance from 0 than c itself.

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Example 4.2
Consider the prospect $(0.5 : x, z)$ yielding x with probability 0.5 and z otherwise, and its certainty equivalent $y \sim (0.5 : x, z)$. Assume expected utility with $0.5U(x) + 0.5U(z) = U(y)$, and $r = 0.94$, generating weak risk aversion. For $x = 400$ and $z = 200$ we obtain $y = 299$, with a risk premium $300 - 299 = 1$.

Next assume an increase in wealth by 100 000, so that $x = 100\,400$ and $z = 100\,200$. To have the same risk premium of 1, with certainty equivalent $y = 100\,299$, we now have to take $r = -19$. Thus, $r = -19$ generates similar risk premiums on the outcome interval $[100\,200, 100\,400]$ as $r = 0.94$ does on the outcome interval $[200, 400]$. □

We next consider the case where the direct vicinity of $x = 0$ is important and should be included in the analysis, but not $x = 0$ itself. For the purpose of data fitting, attention may first be restricted to a subdomain $[c, d]$ for some $c > 0$. If the data suggest less concavity than logarithmic, attention can be restricted to positive powers r. If the subdomain $[c, d]$ suggests strong concavity and relevance of the case $r \leq 0$, as in Example 2.1, then for x near 0 a truncation of the functions at some low negative level may be considered. Alternatively, $U(x + \varepsilon) - U(\varepsilon)$ instead of $U(x)$ may be used if an $\varepsilon > 0$ can be found that is small enough to generate empirical distortions of an acceptably small size, but with ε large enough to avoid the extreme behavior near $x = 0$. Such rescalings of inputs are further discussed later.

Scaling outputs $U(x)$ for $x > 0$ and $x = 0$

The problems just discussed are aggravated if not only values x near 0 but also $x = 0$ itself is to be included in the analysis. Then for $r \leq 0$ the power function is not defined, i.e. it is $-\infty$, at $x = 0$. In situations where the extreme evaluation $U(0) = -\infty$ is implausible and $U(0)$ should have a finite value, the negative powers are automatically ruled out. This happens, for instance, if the scaling $U(0) = 0$ is appropriate. This scaling is commonly assumed in the health domain (Bleichrodt *et al.*, 1999; Miyamoto and Eraker, 1988; Stiggelbout *et al.*, 1994), and also in prospect theory (Tversky and Kahneman, 1992). Then attention can be restricted to positive powers without further ado. If other parts of the domain strongly suggest that $r \leq 0$ is relevant, then a pragmatic solution as suggested in the preceding subsection cannot be avoided. Indeed, in macroeconomics and finance, functions $U(x + \varepsilon)$ have often been considered with ε large and interpreted as initial wealth (Friend and Blume, 1975; Meyer and Meyer, 2005).

In health applications, negative powers (reflecting extreme aversion to dying with $U(0) = -\infty$) generate some undesirable implications for the standard gamble method for measuring utility and quality of life. The following example clarifies this point.

Example 4.3 (Standard gamble utility measurement)
In the standard gamble method for measuring utility, a favorable outcome x, say living your life expectancy in perfect health, is chosen as well as an unfavorable outcome z. For z often immediate death is chosen ($z = 0$). Next intermediate outcomes y, say living y years in perfect health and then dying, are considered, where $x > y > z$. Intermediate outcomes can also concern impaired health states such as in quality-of-life measurements. Next a probability p is measured such that the probability distribution $(p : x, 1 - p : z)$, yielding the good outcome x with probability p and the bad outcome z with probability $1 - p$, is equally preferred as receiving the intermediate outcome y for sure. Expected utility is commonly assumed, implying $U(y) = pU(x) + (1 - p)U(z)$. Normalization $U(x) = 1$ and $U(z) = 0$ then yields the convenient $U(y) = p$.

With $z = 0$ in the standard gamble, it has often been found that subjects are not willing to accept any $p < 1$, indicating that they are not willing to take any risk of dying. This behavior can be modeled through $U(0) = -\infty$, as with negative powers. No useful information can then be obtained about the

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utility of y.[7] In such cases, it is better to adopt standard gamble measurements with outcomes $z > 0$ being more favorable than immediate death (Bleichrodt *et al.*, 2002).

In many applications in health, very bad health states play no role. Then only 'large' inputs, remote from $x = 0$, are relevant, and the anomalies near $x = 0$ of Example 4.1 are not important, similarly as this occurs in finance and macroeconomics. □

Scaling outputs $U(x)$ for $x > 0, x = 0$, and $x < 0$

The problems discussed above are further aggravated if both positive and negative x are to be included in the domain. In the health domain this happens, for instance, if death is modeled as $x = 0$, but quality of life worse than death is considered, and is to be turned, for instance, into utility through power transformations (Dolan, 1997; Revicki and Kaplan, 1993; Robinson and Spencer, 2006; Stiggelbout and de Haes, 2001). Note here that quality of life, and not life duration, is the input of the power function. Before discussing the problems mentioned, we will give some definitions.

The power family can be defined for $x < 0$ through $U(x) = -U(-x)$ with U as above. For a simultaneous definition on the positive and negative domains, with 0 included in the domain and only positive powers considered, we can set $U(x) = x^r$ for $x > 0$ and $U(x) = -(-x)^s$ for $x < 0$ and $r > 0, s > 0$. This definition was used in prospect theory (Tversky and Kahneman, 1992).

If positive and negative x have to be considered jointly, then it is probably better to exclude $r \leq 0$ and $s \leq 0$. The extreme behavior near $x = 0$ then takes place in the center of the domain. All aforementioned pragmatic solutions, such as truncations, then concern not only utility near $x = 0$, but they crucially affect every trade-off between positive and negative x. Such major implications generated by heuristic pragmatic modifications should be treated with caution. With both positive and negative x present, a negative power r or s generates an infinite distance between gains and losses. Such a phenomenon is not empirically plausible, so that negative r and s should then not be expected to occur. Vendrik and Woltjes (2007) used a function $U^+(x + \varepsilon) - U^+(\varepsilon)$ for gains and a function $U^-(x - \varepsilon) - U^-(-\varepsilon)$ for losses, for some $\varepsilon > 0$ (they took $\varepsilon = 1$), to avoid the infinite derivatives at $x = 0$, and to avoid the problems of defining loss aversion generated by infinite derivatives.

SCALING THE INPUTS

Where Assumption 3.1 dealt with the scaling of the outputs $U(x)$, this section deals with the scaling of the inputs x.

Changing the units of inputs

In this subsection we consider changes of the unit of x (multiplications by positive factors). For the power functions, no empirical implication is affected if all inputs are multiplied by a common positive factor. For example, if the inputs are no longer expressed as number of years but as number of months, then we can continue to use the same function U and no empirical description or prescription will be affected. To see this point, first consider choices between prospects for money under expected utility. Then no choice will change if all stakes are multiplied by 12 because this simply implies that all utilities and expected utilities are multiplied by the same factor 12^r and this does not affect any ordering of prospects. Similarly, there is no change if we change numbers of dollars into numbers of cents, multiplying all inputs by 100.

[7] A resort to 'non-standard' real numbers, allowing for the value $-\infty$, could be considered here (Stroyan and Luxemburg, 1976). We will not pursue this point.

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In general, if we multiply all inputs by a factor $\sigma > 0$, then, for $r \neq 0$, all utilities x^r or $-x^r$ are multiplied by σ^r (homogeneity of degree r). Because U is an interval scale (Assumption 3.1), no empirical implication is affected. For $r = 0$, and logarithmic utility, multiplying all inputs by a factor $\sigma > 0$ implies that $\ln(\sigma)$ is added to all utilities. Again, no empirical implication is affected because U is an interval scale. The invariance just demonstrated is known as CRRA for decision under risk and as homotheticity in consumer theory. It is well known that no utility functions other than the power functions satisfy the aforementioned invariance. That is, the power family is appropriate (inappropriate) if and only if the invariance condition is appropriate (inappropriate).

In health economics, power functions are commonly used for the utility of life duration (Abellán *et al.*, 2006). The invariance with respect to input units then implies a 'constant proportional trade-off' condition, with preferences over pairs (y, Q) (living y years in health state Q) not affected by changes in the time unit. This condition is crucial for the validity of the commonly used time-trade-off method for measuring quality of life (Gold *et al.*, 1996; Miyamoto and Eraker, 1988; Pliskin *et al.*, 1980). We do not elaborate on this point.

Changing the levels of inputs

Unlike changes of the unit of the inputs, changes of the level of the inputs, through substitutions $x \rightarrow x + \tau$, are empirically relevant. As we have seen in the preceding section, the power functions display extreme behavior near $x = 0$. Further, the functions are not readily extended to values $x < 0$. Such phenomena are typical of $x = 0$ and show that changes in the level of inputs have empirical implications. The decision as to which physical input is to be modeled as $x = 0$ should be made deliberately, where any extreme behavior of the functions at $x = 0$ is to be matched as much as possible by empirical phenomena at the corresponding physical input. For life duration, $x = 0$ usually refers to immediate death, which, indeed, often exhibits extreme behavior.

Meyer and Meyer (2005) pointed out that information about the scaling of $x = 0$ is often hard to find in empirical papers.[8] The choice of $x = 0$ has been discussed in some economic papers. Often $x = 0$ is then taken to model ruin or level of starvation. Here, indeed, extreme phenomena can be expected similar to the mathematical properties of power utility and similar to such phenomena for the death outcome in health. Then inputs $x = y + \tau$ are considered where τ designates initial wealth and y designates the change with respect to initial wealth. Cohen and Einav (2007) took annual income as proxy for initial wealth, and then found a median power between $r = 0$ and 1, but mean estimates well below $r = -45$. In general, there have been debates about whether, for example, human capital and housing should be included in τ. Rosenzweig and Wolpin (1993) examined insurance for bullock farmers in India using the power utility family, and for each household modeled the minimum consumption needed to survive as $x = 0$.

Most commonly, it is not ruin or level of starvation, but the *status quo* that is taken as $x = 0$ for power utility (Barsky *et al.*, 1997; Harrison *et al.*, 2006; Holt and Laury, 2002). For decision under risk, extreme behavior is indeed often observed at the *status quo*, and power utility may fit such behavior well. Cubitt *et al.* (2001, pp. 401–402) pointed out the remarkably good fit of power utility with a power of approximately 0.3 in individual experiments for monetary outcomes, which is hard to reconcile with expected utility in terms of final wealth. It suggests that in experiments subjects use their *status quo* as reference point $x = 0$ and model inputs as changes with respect to this reference point, irrespective of final wealth. Such a modeling of inputs is central in prospect theory (Kahneman and Tversky, 1979;

[8] Similarly, for exponential (CARA) utility, the location of $x = 0$ need not be specified, but it should always be stated explicitly what the unit of the inputs is. In papers that report estimates of CARA (with an index defined in Equation (9)) it is, again, often hard to find out what unit of the inputs was assumed, unfortunately.

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Tversky and Kahneman, 1992), and was supported by Rabin (2000, pp. 1288–1289); we will not discuss prospect theory here.

In general, there is interest in the two-parameter family $(x+\tau)^r$ with $\tau \geq 0$ simply an additional free parameter that does not have a particular empirical interpretation, and that also serves as an anti-index of concavity (see later; Harrison *et al.*, 2007, p. 452). This two-parameter family may be more natural than the power family. For nonzero τ these functions, obviously, do not exhibit CRRA, but decreasing relative risk. This is empirically desirable for decision under risk. The one-parametric subfamily for $r = 0, \ln(x+\tau)$, did receive some attention, and is known as the logarithmic family. It is popular for introspective measurements of happiness (van Praag and Ferrer-i-Carbonell, 2004; Vendrik and Woltjes, 2007). Some studies considered negative τ, which is possible if only values of x remote from 0 have to be considered. For instance, Stone–Geary utility functions consider $(x-\sigma)^r$ where $\sigma > 0$ ($\sigma = -\tau$) is the minimum consumption level needed to survive or it is habitual consumption. We will not study the families $(x+\tau)^r$ further in this paper for nonzero τ, but will restrict ourselves to the commonly made assumption that $x = 0$ is fixed with no τ involved, this being the most widely used family.

The extra importance of the scaling of inputs if power functions do not fit data perfectly well

If the power family fits data perfectly well, or is assumed in a theoretical model, then, as explained before, the unit of the inputs need not be specified. In practice, however, when using the power family to fit data empirically, the fit will never be perfect, and often considerable and systematic deviations can be expected. Then power utility can still be used for pragmatic reasons, in the absence of a better and more tractable alternative. Different power parameters may then be optimal for different subparts of the domain of the data. Example 4.2 illustrates this point. When reporting a measurement of r there, it is very desirable to also report the outcome domain from which this value of r was derived, be it [200, 400] or [100 200, 100 400] or otherwise. In general, it is desirable to report not only where $x = 0$ is located, but to report also the units of the inputs considered and the exact domain of measurement.

In macroeconomics and finance, large amounts of money are considered and optimal powers r are usually negative, i.e. $1-r$, the index of relative risk aversion, usually exceeds 1 (Bliss and Panigirtzoglou, 2004). Values of $1-r$ around 2 are commonly used in these fields (referenced by Carlsson *et al.*, 2005, and Chen and Huang, 2007, fourth section). Dominguez and Frankel's (1993) study of effects of central bank intervention on the risk premium estimated a value of $1-r$ well over 100.

Kaplow (2005) examined the income elasticity of a statistical life, concerning the small amounts we pay to obtain small reductions in risk of loss of life, such as when buying safety belts. He assumed expected utility, and provided evidence that the income elasticity of a statistical life is typically around 0.5. He then showed that this value can be equated with the CRRA index of the utility of income, and presented this finding as a paradox in view of the CRRA index of utility of income (or consumption) of approximately 2 typically assumed in macroeconomics. However, in individual choice experiments where inputs typically range between \$0 and \$200, powers r between 0 and 1 usually fit data best (Cubitt *et al.*, 2001; Tversky and Kahneman, 1992). This agrees well with the aforementioned income elasticity of a statistical life, which implied a power 0.5. Thus, Kaplow's (2005) paradox can be resolved if the domain of inputs is specified.

ANALYSIS THROUGH THE ARROW–PRATT INDEX

At first sight, the natural limit of x^r for $r \downarrow 0$ may seem to be x^0, i.e. the constant function $U(x) = 1$. This conclusion is not correct because it takes utility at an absolute level. For interval scales, neither the level (intercept) nor the unit (slope) is relevant, but the degree of curvedness is relevant (referring to orderings

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of ratios of differences). Several papers independently observed the importance of the *concavity index*

$$-\frac{U''(x)}{U'(x)} \tag{9}$$

to capture curvedness (de Finetti, 1952; Arrow, 1971). In particular, Pratt's (1964) independent introduction is deep and revealing. He showed that the concavity index captures all information relevant to interval scales. The concavity index is often called the Arrow–Pratt index or the index of absolute risk aversion. The index is constant for the exponential (CARA) family of utility. Calculating the index for our family yields

$$-\frac{r(r-1)x^{r-2}}{rx^{r-1}} = \frac{1-r}{x} \quad \text{for } r > 0 \text{ (Equation (1))}$$

$$\frac{x^{-2}}{x^{-1}} = \frac{1}{x} = \frac{1-r}{x} \quad \text{for } r = 0 \text{ (Equation (2))}$$

and

$$-\frac{r(r-1)x^{r-2}}{rx^{r-1}} = \frac{1-r}{x} \quad \text{for } r < 0 \text{ (Equation (3))}$$

We conclude that

$$\text{the Pratt}-\text{Arrow concavity index for the power family is } (1-r)/x \tag{10}$$

for all r. The same results follow, obviously, for Equations (4)–(6). The concavity index shows once more that the functions in Equations (1)–(3) belong together and constitute one natural family. The degree of concavity decreases in r, with a natural smooth transition at $r = 0$. It confirms that the curvature of x^r converges to that of $\ln(x)$ for r tending to 0 both from above and from below.

The concavity index also shows that the concavity of power functions becomes less extreme for large inputs x. This explains why in macroeconomics, where large x are considered, small and negative values of r are commonly used to generate reasonable degrees of concavity, whereas in individual choice theory, where moderate inputs x are considered, values of r between 0 and 1 are more common. Example 4.2 illustrated this phenomenon. The (absolute) risk aversion generated for outcomes from $[200, 400]$ by $r = 0.94$ was similar there to the risk aversion generated for outcomes from $[100\,200, 100\,400]$ by $r = -19$. The concavity index $(1-r)/x$, with midpoints of intervals taken for x, indeed confirms that $(1-0.94)/300 = 0.0002 \approx (1-(-19))/100\,300 = 0.000199$, which explains the numerical results of Example 4.2.

The concavity index illustrates once more that interpreting r as an anti-index of concavity (or 'risk aversion') without specifying the domain of inputs can be misleading, and we should be careful when comparing values of r derived from different domains of inputs. This was pointed out by Rabin (2000, footnote 10) and Kandel and Stambaugh (1991). There has been much confusion about this point. For empirical evidence regarding r, see Palacios–Huerta and Serrano (2006).

CONCLUSION

The power family provides a rich set of functional forms that can fit data well in many domains, while at the same time preserving tractability. Yet there are some intricacies that have raised misunderstandings, such as the importance of choosing a proper level of inputs $x = 0$, and the natural transition from the positive powers to the logarithm and then to the negative powers. The extreme behavior at $x = 0$ can cause empirical and analytical problems. Remedies for avoiding these problems have been suggested.

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APPENDIX A: EMPIRICAL IMPLICATIONS

The meaning of utility resides in its empirical implications. Strictly speaking, all arguments advanced in the main text were stated in terms of theoretical constructs such as utility graphs and they were not stated directly in terms of empirical primitives. Therefore, this appendix gives an illustration of the claims of the preceding sections, such as about the limiting role of logarithmic utility for $r = 0$, in terms of directly observable preferences. More precisely, we will consider indifference curves in an example.

We consider preferences over pairs (x, y), evaluated through

$$\alpha U(x) + (1 - \alpha)U(y) \quad \text{for some } 0 < \alpha < 1 \tag{A1}$$

with U a power function. The power $r > 0$ specifies the degree of homogeneity, where multiplying all inputs (x and y) by a factor λ multiplies the functional by a factor λ^r. Evaluations as in (A1) occur in decision under risk or uncertainty, where (x, y) is a prospect for which it is uncertain whether it yields \$$x$ or \$$y$, and α is an objective or subjective probability, or a transformed or nonadditive probability as in many nonexpected utility theories (Doctor *et al.*, 2004; Keeney and Raiffa, 1976; Starmer, 2000). Equation (A1) is also used in microeconomics, where (x, y) designates a commodity bundle with a quantity x of one good and a quantity y of another. The representation $\alpha U(x) + (1 - \alpha)U(y)(0 < \alpha < 1)$ with U from the power family is then (assuming the additive separability of (A1)) known as the family of constant elasticity of substitution (Nicholson, 2005). In macroeconomics, x and y can be welfare allocations for two individuals, and the power r is an index measuring the concern for equity (Salanié, 2003, Chapter 4).

Figure A1 depicts indifference classes through (1,1), (2,2), and (3,3) for the case of $\alpha = 0.5$. The indifference curves confirm once again that the logarithmic family is the natural limit of $r = 0$. The indifference curves of the logarithmic family are almost indistinguishable from those for $r = 0.1$ or -0.1 and, therefore, the preferences resulting from these utilities are very similar. Figure A2 displays the indifference curves passing through (2,2) for various values of r.

I next present a mathematical description of the indifference curves that further illustrates some points of this paper. At each point in the plane (x, y), we can calculate the marginal rate of substitution (MRS), i.e. the negative slope of the indifference curve passing through (x, y). It reflects the number of infinitesimal units of y given up to acquire one extra infinitesimal unit of x. A formula for the MRS, known from microeconomics (Nicholson, 2005) and given without derivation, is

$$-\frac{\alpha U'(x)}{(1 - \alpha)U'(y)}$$

The result for the power family is

$$\frac{\alpha r x^{r-1}}{(1 - \alpha) r y^{r-1}} = \frac{\alpha}{1 - \alpha}\left(\frac{x}{y}\right)^{r-1}$$

which also holds for $r = 0$. Again, the MRS for $r = 0$ is the limit of r tending to 0, with a smooth transition at $r = 0$. The indifference curve for $r = 0$ is the limit for r approaching 0, showing once more that the logarithmic function is a natural member of the family.

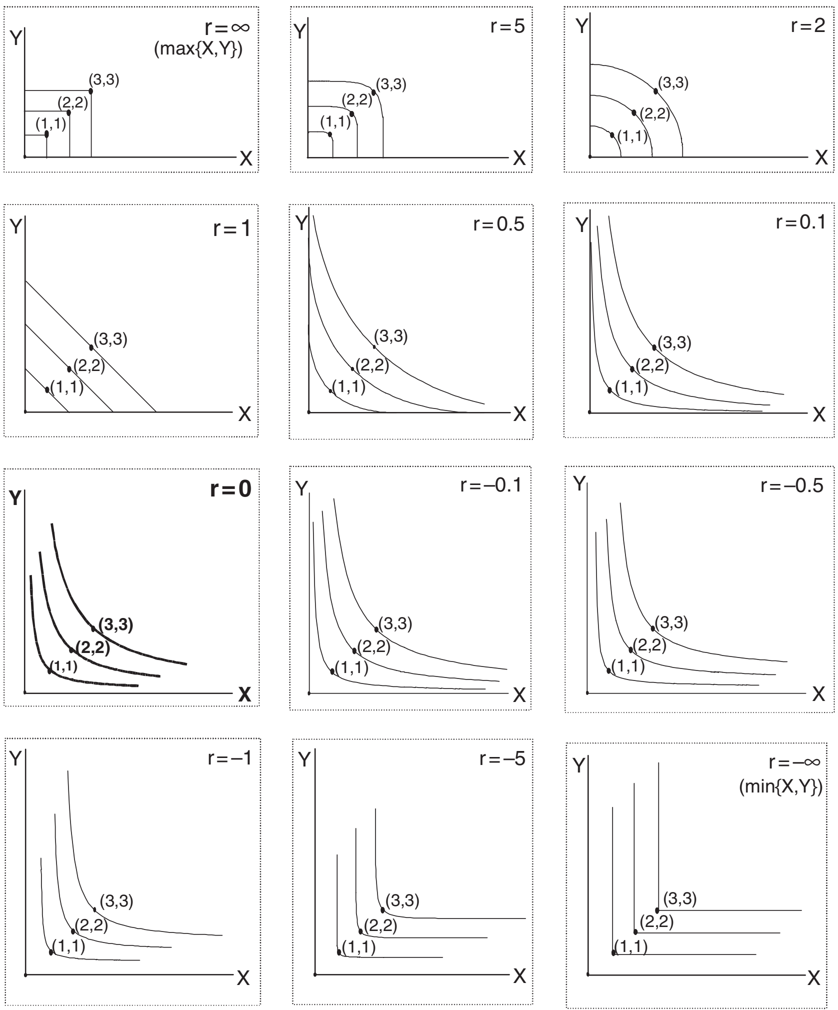

Figure A1. Indifference curves for various powers

APPENDIX B: CALCULATIONS OF INDIFFERENCE CURVES IN APPENDIX A

This appendix presents calculations used for the indifference curves of Figures A1 and A2. For consistency reasons, I will use utility functions normalized at 1 and 2. It is easy to see, and in agreement with the interval scale type of utility, that the same indifference curves would follow for nonnormalized utility functions. The algebra would be simpler then. An indifference curve

$$\alpha \times \frac{X^r - 1^r}{2^r - 1^r} + (1-\alpha) \times \frac{Y^r - 1^r}{2^r - 1^r} = c$$

yields

$$(1-\alpha) \times \frac{Y^r - 1^r}{2^r - 1^r} = c - \alpha \times \frac{X^r - 1^r}{2^r - 1^r}$$

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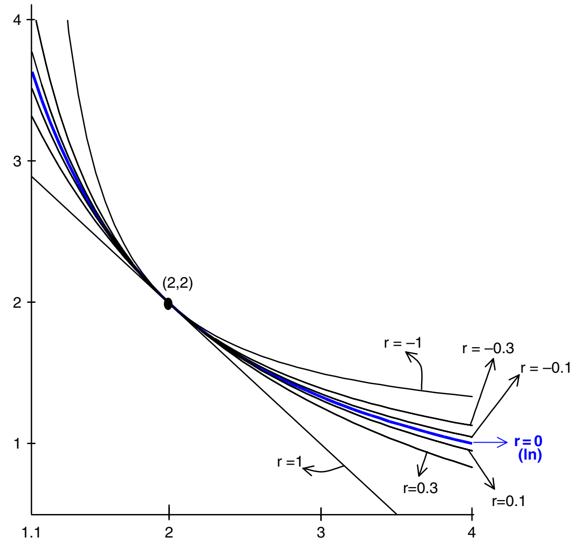

Figure A2. Indifference curves through (2, 2) for power utility with various powers of r

or

$$Y^r - 1^r = \frac{2^r - 1^r}{1-\alpha} \times \left(c - \alpha \times \frac{X^r - 1^r}{2^r - 1^r}\right)$$

or

$$Y^r = \frac{2^r - 1^r}{1-\alpha} \times \left(c - \alpha \times \frac{X^r - 1^r}{2^r - 1^r}\right) + 1^r$$

For the next step we apply the transformation $z \mapsto z^{1/r}$ to the left- and right-hand side of the equality. Whenever $1/r$ is not an integer, the transformation is defined only if the left- and right-hand side are nonnegative. This is no problem for us because we consider $Y \geq 0$ only. Whenever $1/r$ is an odd integer, the step can be made without any problem because the transformation is defined and strictly increasing everywhere. Whenever $1/r$ is an even integer (e.g. for $r = 0.5$, so that $1/r = 2$), problems can arise. The transformation is then defined everywhere, but can equate things with the same absolute value but with a different sign. Hence, it can give positive Y values that should be negative. In this case, graphs based on the following equation should be checked for this problem:

$$Y = \left(\frac{2^r - 1^r}{1-\alpha} \times \left(c - \alpha \times \frac{X^r - 1^r}{2^r - 1^r}\right) + 1^r\right)^{1/r}$$

For $r = 0$ we have

$$\alpha \times \frac{\ln(X) - \ln(1)}{\ln(2) - \ln(1)} + (1-\alpha) \times \frac{\ln(Y) - \ln(1)}{\ln(2) - \ln(1)} = c$$

$$(1-\alpha) \times \frac{\ln(Y) - \ln(1)}{\ln(2) - \ln(1)} = c - \alpha \times \frac{\ln(X) - \ln(1)}{\ln(2) - \ln(1)}$$

$$\ln(Y) - \ln(1) = \frac{\ln(2) - \ln(1)}{(1-\alpha)} \times \left(c - \alpha \times \frac{\ln(X) - \ln(1)}{\ln(2) - \ln(1)}\right)$$

$$\ln(Y) = \frac{\ln(2) - \ln(1)}{(1-\alpha)} \times \left(c - \alpha \times \frac{\ln(X) - \ln(1)}{\ln(2) - \ln(1)}\right) + \ln(1)$$

$$Y = \exp\left(\frac{\ln(2) - \ln(1)}{(1-\alpha)} \times \left(c - \alpha \times \frac{\ln(X) - \ln(1)}{\ln(2) - \ln(1)}\right) + \ln(1)\right)$$

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