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Koopmans' constant discounting for intertemporal choice: A simplification and a generalization

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ABSTRACT

Koopmans provided a well-known preference axiomatization for discounted utility, the most widely used model for maximizing intertemporal choice. There were, however, some technical problems in his analysis. For example, there was an unforeseen implication of bounded utility. Some partial solutions have been advanced in various fields in the literature. The technical problems in Koopmans' analysis obscure the appeal of his intuitive axioms. This paper completely resolves Koopmans' technical problems. In particular, it obtains complete flexibility concerning the utility functions that can be used. This paper, thus, provides a clean and complete preference axiomatization of discounted utility, clarifying the appeal of Koopmans' intuitive axioms.

1. Introduction

Koopmans' (1960; 1972) axiomatization of discounted utility (Samuelson, 1937), the most widely used model for intertemporal choice, is among the most appealing and well-known preference axiomatizations in the literature. The intuitive part of Koopmans' preference axiomatization is exceptionally appealing and efficient. Unfortunately, his analysis is obscured by technical digressions and several inaccuracies. It is, for instance, never stated what the domain of preference is, i.e. which consumption programs are considered, and there is an unanticipated implication of bounded utility (Example 8 in our Appendix A).

This paper corrects the mathematical inaccuracies in Koopmans' analysis, completely resolving the problems of unbounded utility. Not only do we allow for every utility function, but also our domain has maximal flexibility concerning the unbounded programs considered. For every utility function we are free to incorporate (or exclude) every program that generates unbounded utility in the future as long as its discounted utility is well-defined and finite. By resolving Koopmans' technical problems we clarify how appealing and efficient his intuitive axioms are.

2. Koopmans' intuitive preference conditions

X, the set of all conceivable consumptions, is any convex subset of $\mathbb{R}^m$. For example, X may be $\mathbb{R}^m_+$.[1] A *(consumption) program* $x = (x_1, x_2, \ldots)$ yields consumption $x_t \in X$ in *period* $t \in \mathbb{N}$. Preferences over consumption programs are denoted $\succcurlyeq$, with $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ as usual. We will specify the requirements for the preference domain F later.

The main reason for the complications in Koopmans' analysis is that he did not allow for a restricted domain of programs on which the preference relation is defined and on which completeness and other preference conditions are imposed. The basic problem of Koopmans' analysis occurred similarly in Savage's (1954) preference axiomatization of expected utility, another classical result. Savage, like Koopmans, imposed completeness and other axioms on all programs ("acts" in his model). An unforeseen implication, discovered by Fishburn, was that these axioms imply boundedness of utility (Savage, 1972, 2nd edition, footnote on p. 80). Koopmans' axioms have the same problem.

Discounted utility holds on a domain of programs if there exist a *utility function* $u : X \to \mathbb{R}$ and a *discount factor* $0 < \rho < 1$ such that every consumption program x in the domain is evaluated through the well-defined and finite value

$$\sum_{t=1}^{\infty} \rho^{t-1} u(x_t). \tag{1}$$

This summation is called the *discounted utility* (DU) of x. A program is preferred to another if and only if it has the higher discounted

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[1] All results of this paper remain valid if X is a topologically separable connected topological space; see the appendix. It can, for instance, be a set of qualitative health states.

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utility. Our term discounted utility entails what has sometimes been called constant or exponential discounting. The evaluation implies the common assumption of *weak ordering* of $\succcurlyeq$, meaning that $\succcurlyeq$ is *complete* ($x \succcurlyeq y$ or $y \succcurlyeq x$ for all x, y, possibly both) and transitive. For programs for which the summation in Eq. (1) is not defined or is infinite, we continue to use the term DU informally and we say that "DU is not well-defined" or "DU is infinite", as the case may be.

Before defining the preference conditions for our characterization of DU, we introduce some notation. For a program $x = (x_1, x_2, \ldots)$ and a consumption α, αx denotes the program $(\alpha, x_1, x_2, \ldots)$ where the first consumption is α and then the consumptions of x follow, all delayed by one period. The procedure can be repeated, as in $\alpha\beta x = (\alpha, \beta, x_1, x_2, \ldots)$, and so on.

Preferences over consumptions agree with preferences over constant programs in the sense that $\alpha \succcurlyeq \beta$ if and only if $(\alpha, \alpha, \ldots) \succcurlyeq (\beta, \beta, \ldots)$. Koopmans (1972, p. 84) discussed (a small variation of) the following condition. *Monotonicity* holds if $x \succcurlyeq y$ whenever $x_t \succcurlyeq y_t$ for all t, with strict preference $x \succ y$ whenever $x_t \succ y_t$ for some t. In our result this condition need not be imposed because it is implied by the other conditions, mainly stationarity. Like Koopmans (1960, Postulate 2), we assume that period 1 is *sensitive*, i.e. $\alpha x \succ \beta x$ for some α, β, x. This condition avoids triviality and degenerate cases of preferences that are entirely determined by the tail behavior of programs without any concern for the present.

Koopmans assumed the following two intuitive axioms. *Initial-tradeoff independence* holds if

$$\alpha\beta x \succcurlyeq \gamma\delta x \quad \text{if and only if } \alpha\beta y \succcurlyeq \gamma\delta y \tag{2}$$

for all programs x, y and all consumptions $\alpha, \beta, \gamma, \delta$. That is, tradeoffs between today and tomorrow are not affected by future consumption. The condition amounts to separability of the first two periods. A set of periods is *separable* if preferences over consumptions in these periods, while keeping consumptions in other periods fixed, are independent of the level at which those other consumptions are kept fixed. *Joint independence* means that all periods are separable.

Koopmans' second intuitive axiom, *stationarity*, holds if

$$\alpha x \succcurlyeq \alpha y \quad \text{if and only if } x \succcurlyeq y \tag{3}$$

for all programs x, y and all consumptions α. That is, a preference is not affected if a common first consumption is dropped, and the timing of all other consumptions is advanced by one period. By repeated application, it implies that for a preference between two programs all initial periods with common consumption can be dropped, and the first period of different consumption can be taken as the initial period.

Koopmans formulated his intuitive axioms equivalently but slightly differently, with for instance stationarity imposed only for one initial consumption α and then separability of $\{2, 3, \ldots\}$ added, which is equivalent to our stationarity imposed for all initial consumptions α.

3. Preference conditions for unbounded time horizons

In Koopmans' model as well as in ours, programs are infinite-dimensional objects; hence topological considerations can be complex. We avoid such complexities by imposing topological conditions (continuity) only on finite-dimensional subspaces. For two further implications of infinite-dimensional continuity that are used in proofs of other papers, namely constant-equivalence and tail-robustness (defined later), it will be both more appealing and more general to state these as explicit axioms, rather than to derive them from stronger infinite-dimensional topological assumptions.

An *ultimately constant* program x is such that $\alpha = x_t = x_{t+1} = \cdots$ for all $t > T$, for some consumption α and some period T. By $x_T\alpha$, for some general program x, period T, and consumption α, we denote the ultimately constant program $(x_1, \ldots, x_T, \alpha, \alpha, \ldots)$. For each period T, X_T is the set of ultimately constant programs of the form $x_T\alpha$. That is, X_T contains all ultimately constant programs that are constant over all $t > T$. X_T can be considered a $T + 1$ dimensional product space, specified by $T + 1$ tuples $(x_1, \ldots, x_T, \alpha)$. *Ultimate-continuity* holds if $\succcurlyeq$ is continuous on each set X_T.

To extend the discounted utility evaluation to unbounded programs, we will impose the preference relation only on programs with finite discounted utility. This is one of the main respects in which our analysis deviates from Koopmans', who assumed preferences over all programs. We achieve maximal generality regarding the domain of unbounded programs considered by allowing for the set of all programs with (finite) discounted utility, or any subset thereof, in our theorem. To achieve our purpose, we have to solve a mathematical problem that has hampered many papers dealing with infinite-dimensional evaluations, and that is explained next.

A typical example of the aforementioned mathematical problem concerns DeGroot's (1970, Chapter 7) derivation of subjective expected utility. Having derived the evaluation on bounded programs ("acts"), he explicitly used utility and the expected utility functional to define the domain of all programs with finite expected utility (denoted $\mathcal{P}_E$ in his Section 7.10). He then went on to establish the preference axiomatization of expected utility in terms of preference conditions on this extended domain. DeGroot's procedure is undesirable because utility and expected utility are theoretical constructs and are related to observables only in complex ways. Hence, they should not be used explicitly in preference axiomatizations. In fact, if expected utility can be used explicitly in the definition of the domain and in preference conditions, then its preference axiomatization becomes a tautology because we can then simply state expected utility maximization directly as a preference axiom.

A similar problem as in DeGroot's analysis arose in Hübner and Suck's (1993) extension of Koopmans' discounted utility to unbounded programs. They used a condition concerning the interior of circles of convergence that explicitly uses both the discount factor and the utility function, i.e. theoretical constructs.

The problem to be solved is that we should find conditions that imply well-defined and finite discounted utility for the preferences and programs considered, but that are stated entirely in terms of observables (preferences) without any explicit use of utility or discounted utility. Such conditions were used by Harvey (1986, see in particular his C^* definition) for the present context of summation over discrete periods, and by Wakker (1993a) for integrals over general spaces for the context of decision under uncertainty. This paper will combine these two approaches.

A program x satisfies *constant-equivalence* on a domain if there exists an equivalent constant program in that domain. This condition was derived from topological conditions in Diamond (1965, Lemma on p. 172), Harvey (1986, p. 1136 second para) and Koopmans (1960, Eq. 17 and Section 10). The main condition in our analysis is the following.

Definition 1. A program x is *tail robust* if, for all outcomes β: if $x \succ \beta$ ($x \prec \beta$) then there exists a t such that $x_T\beta \succ \beta$ ($x_T\beta \prec \beta$) for all $T \geq t$.

In words, a sufficiently remote future does not affect preference much. In Example 7 of Becker and Boyd (1997, Section 3.3.3), the remote future does affect preference in a way violating tail robustness, demonstrating the necessity of adding such a condition. The two conditions just defined deliver the proper restrictions on preferences and programs considered, entirely in terms of observables. A program will have well-defined finite discounted utility if and only if it satisfies constant-equivalence and tail robustness.

4. A preference axiomatization for discounted utility

In the characterizing Statement (ii) in the following theorem, (a) states usual preference conditions, with continuity only in a simple finite-dimensional version, (b) gives the conditions to ensure well-defined finite discounted utility, and (c) gives Koopmans' intuitive conditions. The interval scale property in the following theorem means that any constant (changing the "level") can be added to utility, and utility can be multiplied by any positive number (changing the "unit").

Theorem 2 (*Preference Axiomatization for Discounted Utility*). *Let $\succcurlyeq$ be defined on a domain F of programs that contains all ultimately constant programs. Then the following two statements are equivalent.*

(i) *Discounted utility holds on F, where the utility function is continuous and not constant.*

(ii) *$\succcurlyeq$ satisfies*
  (a) *weak ordering, sensitivity of the first period, and ultimate-continuity;*
  (b) *constant-equivalence and tail robustness;*
  (c) *initial-tradeoff independence and stationarity.*

Furthermore, the discount factor in Statement (i) *is unique, and utility is an interval scale.* □

Some of the above conditions can be relaxed on particular domains. A program x is *bounded* if there exist consumptions μ, ν such that $\mu \succcurlyeq x_t \succcurlyeq \nu$ for all t.

Observation 3 (*Relaxing Conditions on Particular Domains*). *In Theorem 2, monotonicity is implied by the other conditions. If F contains only ultimately constant programs, then constant-equivalence can be dropped and tail robustness can be replaced by monotonicity. Tail robustness can also be replaced by monotonicity if F contains only bounded programs.* □

The results in the rest of this section illustrate the generality of our approach. Corollary 4 shows that the only restriction for the domain F is that all ultimately constant programs be present. Other than that, we have complete flexibility regarding the domain, and we can incorporate any set of programs with well-defined and finite DU.

Corollary 4 (*Maximal Generality of Domain*). *Assume discounted utility with given u and ρ. Then the set F in Theorem 2 can be any subset of the set of all programs with well-defined and finite DU that contains all ultimately constant programs.* □

Given constant equivalence, our theorem can handle all domains of programs with well defined discounted utility and in that sense is maximally general. It, thus, generalizes results obtained only for particular domains in the literature. Popular domains considered in the literature are, for example, so-called *ω-bounded* sets A_ω. Here ω is a given program, and the set A_ω contains all programs x for which there exists $\lambda > 0$ such that $|x_t| \le |\lambda\omega_t|$ for all t (Becker & Boyd, 1997). Streufert (1990) considered other domains bounded by a production process. If utility u is bounded, then Theorem 2 can handle the domain of all programs, which is more general than the domains A_ω. In addition, for unbounded utility our domain can be more general than any set A_ω.

Given a domain of preference, tail robustness can serve as a restriction on preferences. Conversely, given preferences on some domain, tail robustness can serve as a tool to extend the domain of preference. We present two examples to illustrate these applications. Restricted domains of conceivable consumption as in the following example, taken from Becker and Boyd (1997, Section 3.2, Example 1), naturally arise from limitations on future production or from restrictions on future borrowing. Tail robustness then identifies the preference relations that have well-defined and finite DU on the domain considered. *Homotheticity* means that preference is invariant under multiplication by a positive constant.

Example 5 (*Restricting Preferences through Tail Robustness*). Assume that $\alpha > 1$, and $X = \mathbb{R}_+$ ($m = 1$). F contains all programs x with $\sup_t\{x_t/\alpha^t\} < \infty$. That is, it is A_ω with $\omega_t = \alpha^t$ for all t. The parameter α represents the growth rate of capital in the optimal accumulation model or the growth rate of the endowment in an exchange economy. Assume that all conditions of Theorem 2 are satisfied, so that all programs in F satisfy tail robustness. Assume further that homotheticity holds. Then there exists $s > 0$ such that $u(\beta) = \beta^s$. This follows from Becker and Boyd (1997, p. 81) or Miyamoto and Wakker (1996, Theorem 2), with s necessarily positive because 0 is contained in u's domain. Further, $\rho\alpha^s < 1$, as follows from finiteness of discounted utility of ω. □

In Example 5, tail robustness implies the inequality $\rho\alpha^s < 1$ of Becker and Boyd (1997, Section 3.2, Example 1). That is, impatience (discounting) dominates growth in utility units; see also Streufert's (1993 p. 83) interpretation of his biconvergence condition. Conversely, the inequality implies tail robustness, so that tail robustness provides a preference axiomatization for the inequality to hold. The following observation shows that tail robustness can serve as a tractable tool for constructing a domain of preference on which DU is well-defined and finite.

Example 6 (*Constructing the Preference Domain through Tail Robustness*). Assume that all conditions of Theorem 2 are satisfied. We consider a program x not contained in F, and wonder whether we can incorporate x in the domain of discounted-utility preference by adding an indifference $x \sim (\alpha, \alpha, \ldots)$ for some outcome α, combined with extension through weak ordering. Denote the extended domain as $F' = F \cup \{x\}$. Then the DU representation still holds, with $DU(x)$ well-defined, finite, and equal to $DU(\alpha)$ if and only if x satisfies tail robustness (proved in the appendix). In particular, tail robustness and weak ordering imply all other conditions in Statement (ii) of Theorem 2.

A convenient aspect of the above extension of domain is that we need to verify tail robustness of x only with respect to the ultimately constant programs. That is, tail robustness is satisfied on $F' = F \cup \{x\}$ if and only if it is on $F^{uc} \cup \{x\}$ where F^{uc} denotes the set of ultimately constant programs. This follows immediately from the definition of tail robustness which, apart from the preference of x, involves only ultimately constant programs. □

The example has demonstrated that tail robustness identifies the new programs that can be incorporated. The other preference conditions then automatically follow. To verify whether x can be incorporated into the domain, we only need to relate x to the ultimately constant programs, and need not consider any other unbounded program, which enhances the tractability of the domain construction.

5. Related literature

The efficiency of Koopmans' intuitive axioms, in particular stationarity, is exceptional, mainly because commonly used preference conditions to obtain additively decomposable preferences and their restrictions can be weakened in the presence of stationarity. Thus, for instance, our result generalizes many results in the literature that used stronger separability than of merely {1, 2} as we do,[2] or that first derived general, period-dependent, discounting

[2] See Becker and Boyd (1997, p. 83), Harvey (1986,Theorem 4.a and 4.b, 1995, Theorem 3.2) Hübner and Suck (1993, Theorem 3), Koopmans (1960, Eq. 47, 1972, Proposition 3) Kreps (1977), Bleichrodt and Gafni (1996, Theorem 2.1), Fishburn (1970, Theorem 7.5), Fishburn and Edwards (1997, Theorem 3), Krantz, Luce, Suppes, and Tversky (1971, Theorem 6.15.ii), Meyer (1976, Theorem 9.1) and Wakker (1989, Theorem IV.4.4).

from separate preference conditions and then imposed additional conditions to obtain discounted utility (Bleichrodt & Gafni, 1996, p. 53 ; Fishburn & Edwards, 1997; Harvey, 1986, 1995; Meyer, 1976, p. 480; Wakker, 1989, Corollary IV.4.4).

Dolmas (1995) proposed a partial solution to the problem of domain definition without the use of theoretical constructs (such use was the problem of DeGroot's (1970) analysis). The rest of this paragraph briefly describes Dolmas' approach. He considered domains A_ω defined in Section 4. Although for each given time point utility is still bounded in his approach, overall it can be unbounded. Dolmas imposed continuity with respect to a modified supnorm that made his space of programs homeomorphic (equivalent in a topological sense) to the set of all bounded programs. Through a strengthening of Koopmans' axioms he could then simply obtain a representation homeomorphic to Koopmans'.

We briefly discuss Kreps' (1977) and Streufert's (1990; 1993) partial solutions to the problem of domain definition. Our tail robustness extends Streufert's (1990) biconvergence condition (when redefined for preferences as in Becker & Boyd, 1997, Section 3.3.3) and Kreps' (1977) upper and lower convergent utility to the case where no upper bound (time-dependent and generated by a production function), and no lower bound 0 need to be available for consumption. It similarly extends Streufert's (1993) tail insensitivity to the case of unbounded utility (by not considering every replacement of tails but considering only β-tail replacements). In the special case of an upper and lower bound for consumption and under usual assumptions such as monotonicity, our tail robustness can be seen to be equivalent to Kreps' and Streufert's conditions. These authors did not specify the exact restrictive nature of their preference conditions through necessity results. They allowed for some unbounded programs and utility functions but not for all.

The consumption sets X considered in the literature were mostly less general than the consumption set considered in this paper (Becker & Boyd, 1997, Section 3.3.3; Dolmas, 1995; Epstein, 1983; Harvey, 1986, 1995; Koopmans, 1960, 1972; Meyer, 1976; Streufert, 1990). Some further restrictions in the literature that we generalize include convex preferences over consumption in Becker and Boyd (1997, P3), linear utility in Epstein (1983, where probability distributions generate a convex subset of a linear space), Harvey (1995, Theorem 3.2) and Meyer (1976), restrictions to program pairs that differ in at most finitely many periods (Fishburn & Edwards, 1997; it is implied by the overtaking criterion of Becker and Boyd (1997, Section 3.2)), and restrictions to single consumption (Fishburn & Rubinstein, 1982).

Stationarity is the most critical empirical condition in Koopmans' model. It is not only used to justify his model but also to falsify it. Indeed, empirical violations have raised an interest in generalizations, such as the hyperbolic discounting models. Phelps and Pollak (1968) introduced quasi-hyperbolic discounting, which was popularized in the economic literature by Laibson (1997). Loewenstein and Prelec (1992) introduced generalized hyperbolic discounting. Both generalized and quasi-hyperbolic discounting can be axiomatized through generalizations of stationarity.

The literature on Koopmans' representation has been scattered over journals in economics, management science, and psychology. These papers were written when internet was not yet available, and obtained their extensions of Koopmans' theorem independently. There were virtually no cross references. Thus, our paper also unifies a number of developments in different fields.

We have confined our analysis to the case of impatience ($\rho < 1$), with early consumption preferred to late consumption, which is the most important case. In the case of patience ($\rho \geq 1$), discounted utility will often diverge, for instance for constant programs. Our analysis should then be considerably modified.

6. Conclusion

Many generalizations and applications of Koopmans' discounted utility have been developed. Preference axiomatizations were sometimes obtained as a corollary and by-product, which is also how Koopmans' original preference axiomatization was obtained. This axiomatization has, however, turned out to be the most important contribution of Koopmans' paper, and it is the reason why his paper has become a classic. Authors specialized in Koopmans' work invariably did not focus on his axiomatization but addressed other and more specialized topics, treating discounted utility as a by-product in the same way as Koopmans did. Hence, for discounted utility, no clean, efficient, and general preference axiomatization exists today. This paper has provided such a tool. Resolving the technical problems in Koopmans' analysis, in particular regarding unbounded utility, was the main technical step in achieving our main goal: to clarify and popularize Koopmans' intuitive axioms. We are not aware of any preference condition in the literature that is so appealing and simple to comprehend, and at the same time so powerful in its implications, as Koopmans' stationarity.

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Appendix A. Proofs

The following example prepares for the proof. Similar examples were presented in Becker and Boyd (1997, Section 3.3.2, Example 6), Kreps (1977, p. 48) distinguishing between optimal and unimprovable strategies, and Streufert (1990). The example shows that tail robustness cannot be omitted in Theorem 2. It clarifies Koopmans' (1972) discussion of his monotonicity postulates P5 and P5′.

Example 7. The preference domain F consists of all ultimately constant programs. Evaluations are through

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto \sum_{i=1}^{T} \rho^{i-1} u(x_i) + \rho^T u(\alpha)/(1-\rho) \tag{4}$$

with u continuous and not constant, and $\rho > 1$. For $\rho < 1$ the above evaluation agrees with DU. For $\rho > 1$ as considered here, DU is not defined, but the above evaluation is. To ensure that the above formula is well-defined, we should verify that different ways of splitting up an ultimately constant program into the first part and its constant-tail part lead to the same evaluation. They do because, if $x_{T+1} = \alpha$, then

$$\sum_{i=1}^{T} \rho^{i-1} u(x_i) + \rho^T u(\alpha)/(1-\rho) = \sum_{i=1}^{T+1} \rho^{i-1} u(x_i) + \rho^{T+1} u(\alpha)/(1-\rho),$$

and so on. All conditions of Statement (ii) in Theorem 2, except tail robustness, are satisfied. Besides tail robustness, monotonicity and impatience (preference for early receipt of outcomes with higher u value) are also violated. We obtain an evaluation of α equal to $u(\alpha) + \rho u(\alpha) + \rho^2 u(\alpha)/(1-\rho) = u(\alpha)/(1-\rho)$. It is not increasing, but decreasing, in $u(\alpha)$ because $\rho > 1$. Preferences satisfy a weaker monotonicity condition, i.e. *finite monotonicity*, which means that replacing any finite number of consumptions x_t by other consumptions with higher u value always improves the program. By sensitivity of period 1, u is not constant; and there are γ, β with $u(\gamma) > u(\beta)$. Then $\gamma \prec \beta$. This preference in combination with $\gamma x \succ \beta x$ violate monotonicity. It is like preferring cake to bread for any finite number of days, but preferring bread to cake for an infinite lifetime. Tail robustness would require that $\gamma_T\beta \prec \beta$ should hold for all T sufficiently large. However, by finite monotonicity, $\gamma_T\beta \succ \beta$ for all T.

Even if utility of consumption is bounded, the evaluation of programs is not (cf. $\gamma_T\beta$ for T tending to infinity). Consequently, there does not exist a best program, so that Koopmans' (1972) Postulate P5′ is still violated.

This example is an alternative to the example of Burness (1976, p. 505), who showed that the very existence of a utility indicator need not imply impatience. Our example revealed a more fundamental problem, i.e. a violation of monotonicity. By imposing the supremum norm, all conditions of Burness (1976) are satisfied. □

Proof of Theorem 2. The following proof and, consequently, Theorem 2, are valid for every topologically separable[3] and connected consumption space X. For topological definitions, see Dugundji (1966). For each x, $DU(x) = \sum_{t=1}^{\infty} \rho^{t-1}u(x_t)$ whenever defined. First assume that Statement (i) holds. Tail robustness follows because of convergence of the summations in $DU(x)$. There exists a constant-equivalent for each x, with utility strictly between $limsup(u(x_t))$ and $liminf(u(x_t))$ if these two are different, mainly because of connectedness of $u(X)$. All other conditions in Statement (ii) easily follow.

For the reversed implication, we assume henceforth that Statement (ii) holds. We fix an arbitrary outcome θ throughout, at which many functions below will be normalized to be 0. Initial-tradeoff independence amounts to separability of the set of periods {1, 2}, with a preference $\alpha\beta x \succcurlyeq \gamma\delta x$ independent of x. By stationarity, it implies that a preference $\mu\alpha\beta x \succcurlyeq \mu\gamma\delta x$ is independent of μ and x, i.e. {2, 3} is separable. Similarly, by repeated application of stationarity, separability of all sets $\{i, i+1\}$ follows. Stationarity implies that a preference $\alpha x \succcurlyeq \alpha y$ is the same as between x and y and, hence, is independent of α, so that separability of $\{2, 3, \ldots\}$ follows. Similarly as above, stationarity then implies separability of $\{3, 4, \ldots\}$, and then of all "tail" sets $\{i, i+1, \ldots\}$. Similarly, sensitivity of period 1 and stationarity imply sensitivity of all periods t.

Consider a period $T > 1$. By the separabilities just established, and the other conditions such as sensitivity of at least three periods (in fact all) and continuity, Gorman (1968) (with Vind (1971), showing that connectedness instead of arc-connectedness of domain suffices) implies that we have joint independence with separability of all subsets of components) over every $T + 1$ dimensional space X_T, and an additive evaluation

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto V_{1,T}(x_1) + \cdots + V_{T,T}(x_T) + R_T(\alpha) \quad (5)$$

on each such set, where all functions used are continuous. We may set $V_{j,T}(\theta) = 0 = R_T(\theta)$ for all j. The function

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto V_{1,T+1}(x_1) + \cdots + V_{T,T+1}(x_T) + V_{T+1,T+1}(\alpha) + R_{T+1}(\alpha),$$

obtained from X_{T+1}, is an alternative additive evaluation over X_T. By the usual uniqueness results of additive evaluations, we may, inductively with respect to T, set

$$\text{for all } T, \text{ and all } j \le T, V_{j,T+1} = V_{j,T} \quad \text{and} \quad V_{T+1,T+1}(\alpha) + R_{T+1}(\alpha) = R_T(\alpha). \quad (6)$$

From the first equality it follows that $V_{j,T}$ is independent of T for $j \le T$, and we can drop the subscript T.

By stationarity and consideration of programs γx for a fixed γ, $V_2(x_1) + \cdots + V_{T+1}(x_T) + R_{T+1}(\alpha)$ evaluates the same preferences over X_T as does $V_1(x_1) + \cdots + V_T(x_T) + R_T(\alpha)$. By the usual uniqueness results, there exists a $\rho_T > 0$ such that $V_{j+1} = \rho_T V_j$ for all $j \le T$ and

$$R_{T+1} = \rho_T R_T. \quad (7)$$

$\rho_T = V_2/V_1$ is independent of T because V_1 and V_2 are, and we drop the subscript T. Writing $u = V_1$ and $R = R_2/\rho^2$, we have obtained an evaluation

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto \sum_{i=1}^{T} \rho^{i-1}u(x_i) + \rho^T R(\alpha) \quad (8)$$

with u continuous. Note that consumptions are ordered the same for every period, which comprises most of monotonicity. The last equality in Eq. (6) implies that $\rho^T R(\alpha) = \rho^T u(\alpha) + \rho^{T+1}R(\alpha)$, or

$$R(\alpha) = u(\alpha) + \rho R(\alpha). \quad (9)$$

$\rho = 1$ cannot be: By Eq. (9), then $u(\alpha) = 0$ for all α. Constantness of u violates sensitivity of period 1, and $\rho = 1$ cannot be indeed. Hence,

$$R = u/(1-\rho). \quad (10)$$

$\rho > 1$ cannot be either because it violates tail robustness (and monotonicity); see Example 7. We conclude that, besides $0 < \rho$, also $\rho < 1$. Eq. (10) implies $R(\alpha) = \sum_{i=1}^{\infty} \rho^{i-1}u(\alpha)$. Substituting this in Eq. (8) yields discounted utility for the ultimately constant programs.

To extend the evaluation to general, possibly unbounded, programs, consider a general program x and its constant-equivalent α. If there exists a consumption β with $x \sim \alpha \succ \beta$ then, by tail robustness, $x_T\beta \succ \beta$ for all T sufficiently large. By the evaluation of ultimately constant programs, $DU(x_T\beta) > DU(\beta)$ for all T sufficiently large. Hence, $liminf_T(\sum_{t=1}^{T} \rho^{t-1}u(x_t)) \ge DU(\beta)$. Because this holds for all $\beta \prec \alpha$, the *liminf* also is not less than $DU(\alpha)$. If there exists no β as above, then α is the worst consumption, and by the DU representation and its implication of monotonicity on the ultimately constant programs, the above *liminf* again is not less than $DU(\alpha)$. Similarly, $lim\,sup_T \sum_{t=1}^{T}(\rho^{t-1}u(x_t))$ is not more than $DU(\alpha)$. It follows that $DU(x) = DU(\alpha)$ and, hence, $DU(x)$ evaluates x.

Note that we did not make any assumption about which of the not ultimately constant programs are contained in F, and complete flexibility of domain has been maintained. By standard uniqueness results for additive evaluations (Gorman, 1968; Krantz et al., 1971), the functions $V_{j,T} = \rho^{j-1}u$ are interval scales in additive evaluations as in Eq. (5). This implies the uniqueness result regarding u and $\rho = (V_2(.) - V_2(\theta))/(V_1(.) - V_1(\theta))$. Our result is a special case of an additively decomposable representation on an infinite product space. General results on this topic are in Wakker and Zank (1999). □

Proof of Observation 3. Monotonicity follows from the discounted utility representation. For ultimately constant programs, constant-equivalence was not used in the proof of Theorem 2, and tail-robustness was used only to show that $\rho > 1$ cannot be. The latter is also excluded by monotonicity, as indicated in the proof of Theorem 2; see also Example 7. To show that tail robustness can be replaced by monotonicity for bounded programs, assume that $\mu \succcurlyeq x_t \succcurlyeq \nu$ for all t, and that α is the constant-equivalent of x. Then, by monotonicity, $x_T\mu \succcurlyeq x \sim \alpha \succcurlyeq x_T\nu$ for all T. By discounted utility for ultimately constant programs, $DU(x_T\mu) \ge DU(\alpha) \ge DU(x_T\nu)$ for all T. Because $DU(x_T\mu)$ and $DU(x_T\nu)$ converge to each other, they converge to $DU(\alpha)$. They also converge to $DU(x)$, which, hence, is equal to $DU(\alpha)$ and evaluates x. □

[3] We add the adjective "topological" throughout to distinguish this condition, not defined here, from the separability preference condition.

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