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Decision-foundations for properties of nonadditive measures: general state spaces or general outcome spaces

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Abstract

This paper characterizes properties of chance attitudes (nonadditive measures). It does so for decision under uncertainty (unknown probabilities), where it assumes Choquet expected utility, and for decision under risk (known probabilities), where it assumes rank-dependent utility. It analyzes chance attitude independently from utility. All preference conditions concern simple violations of the sure-thing principle. Earlier results along these lines assumed richness of both outcomes and events. This paper generalizes such results to general state spaces as in Schmeidler's model of Choquet expected utility, and to general outcome spaces as in Gilboa's model of Choquet expected utility.

JEL classification: D81; C60

Keywords: Risk attitude; Rank-dependent; Comonotonic; Choquet expected utility; Prospect theory; Pessimism; Optimism; Sure-thing principle

1. Introduction

As emphasized by Keynes (1921) and Knight (1921), objective probabilities of uncertainties are rarely known in economics. This is contrary to, for instance, the medical field, where extensive statistical data is often available. De Finetti (1937) and Savage

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(1954) made famous contributions to the measurement of uncertainty. They showed that Bayesian subjective probabilities can often substitute for unknown objective probabilities. Allais' (1953) and Ellsberg's (1961) examples showed, however, that there are empirical, and according to some also normative, problems with the Bayesian models of de Finetti and Savage.

Allais (1953) proposed non-Bayesian models for decision under risk. (In this paper, *risk* refers to the case of known objective probabilities.) Unfortunately, his models, while psychologically well-founded, were intractable because they had too many parameters that were, accordingly, unidentifiable. Kahneman and Tversky (1979) provided a major breakthrough with their (original) prospect theory. It was the first theory that combined theoretical tractability with empirical realism, and that could make predictions about something considered unpredictable up to that point: irrational decision making. Original prospect theory was formulated for decision under risk only. Soon after came the influential contribution of Machina (1982), who showed that nonexpected utility can still give positive predictions about first- and second-order conditions at economic optima. His model was, again, restricted to decision under risk.

Ellsberg (1961) showed that deviations from Bayesianism for unknown probabilities can exhibit phenomena of a nature essentially different than those for decision under risk. Nevertheless, decision theorists focused on decision under risk up to the early 1990s. This is remarkable because of the importance of unknown probabilities, which had been widely understood. The reason for this seeming neglect of an important topic is that for a long time no-one was able to formulate a sound non-Bayesian model for uncertainty.

It had long been understood that, besides an attitude towards outcomes, also an attitude towards uncertainty (chance attitude) is important for decision under uncertainty. Schmeidler (1989, first version 1982) was the first to formalize such an attitude and, thus, was the first to provide a sound non-Bayesian model for uncertainty. He used nonadditive measures (capacities) to capture his intuition that utility alone cannot model all of decision attitudes under uncertainty. Only with Schmeidler's idea available, could a sound version of prospect theory be developed (Tversky and Kahneman, 1992) that, importantly, could also be applied to decision under uncertainty.

Schmeidler's basic intuition was that capacities of events and their complements should sum to less than one so as to designate a lack of information about probabilities. This was combined with a conservative, pessimistic, decision attitude towards such a lack of information, modeled through convex capacities. Schmeidler gave a preference axiomatization of such capacities, and convexity has been the most-studied property of capacities.

A restriction of Schmeidler's (1989) analysis was that it needed linear utility of outcomes, as in Anscombe and Aumann (1963), in the following manner. Outcomes are probability distributions over nonrisky outcomes called prizes, and preferences over outcomes are governed by expected utility. Thus, a two-stage resolution of uncertainty results, where the basic uncertainty of interest, regarding the true state of nature, is resolved in the first stage, yielding a particular outcome, and in the second stage the probability distribution of the outcome is resolved, finally yielding a prize. Backwards induction is assumed for the two-stage optimization. A difficulty is that backward induction is problematic for nonexpected utility models (Machina, 1989).

Preference axiomatizations of Schmeidler's model that relax the restrictions of the Anscombe and Aumann two-stage approach were left to two of his PhD students. Gilboa (1987) obtained such an extension but still needed a richness assumption on the state space. Wakker (1989) also obtained such an extension but needed a richness assumption on the outcome space.

Preference axiomatizations for properties of capacities without the restriction of linear utility such as in the Anscombe and Aumann two-stage approach were first given by Tversky and Wakker (1995). They focused on bounded subadditivity, and did not consider convexity. Wakker (2001) characterized convexity, and a number of related conditions, independently of utility. These two references relied heavily on richness assumptions. They needed both the richness of the state space as in Gilboa (1987), and the richness of the outcome space as in Wakker (1989, 1993), requiring both of these sets to be infinite. Further, these results were restricted to real-valued outcomes. Relaxing these restrictions is the purpose of this paper. We allow for general, possibly finite, outcome spaces or for general, possibly finite, state spaces (but not both). In the latter case, we also allow for nonmonetary outcomes as long as the utility space contains a nondegenerate interval, such as under connected-continuity. Thus, this paper provides necessary and sufficient conditions for convexity, and some other properties, of capacities for:

(a) the generalization of Schmeidler's (1989) model to continuous instead of linear utility;
(b) Gilboa's (1987) generalization of Savage (1954) to nonadditive measures.

It is, in general, desirable to minimize the structural restrictions used for a number of reasons. First, contrary to what has sometimes been thought, structural assumptions are not merely technical, but they add empirical content to the axioms, and the difficulty is that it is not clear what this content is (Ghirardato and Marinacci, 2001; Köbberling and Wakker, 2003; Krantz et al., 1971, Section 9.1; Pfanzagl, 1968, Section 6.6). Second, in many applications, structural richness of the outcome set is not natural, and it is desirable to avoid it. In many medical applications, the only conceivable outcomes concern a limited number of health states. Measurement methods, and preference conditions to explain or test qualitative properties, that require consideration of artificial continua of outcomes, then are not realistic. Similarly, we often face only a finite number of uncertainties (democratic or republican victory). Then techniques that require continuous extensions of the uncertainties through, say, repeated outcomes of tosses of a coin, as in Savage's (1954) approach, are less suited.

This paper will also show how the results obtained for decision under uncertainty imply corresponding results for decision under risk. Some recent characterizations of related properties of capacities, provided in the literature, are discussed in Appendix B.

2. Notation and definitions

S denotes a *state space*, endowed with an algebra $\mathcal{A}$ of subsets called *events*. X is an *outcome space*, endowed with a binary relation, the *preference relation*, denoted $\succcurlyeq$. $(A_1, x_1; \dots; A_n, x_n)$ denotes a function from S to X that assigns x_j to each $s \in A_j$,

$j = 1, \ldots, n$, where $A_1, \ldots, A_n$ are events partitioning S. Such a function is called an *act*. $\succcurlyeq$, the *preference relation*, is a binary relation on the set of all acts. The notation $f \succcurlyeq g$, and $\succ$ (*strict preference*), $\sim$ (*indifference* or *equivalence*), $\preccurlyeq$, and $\prec$ is as usual. V *represents* $\succcurlyeq$ if V is from the set of acts to $\mathbb{R}$ and $f \succcurlyeq g \Leftrightarrow V(f) \geqslant V(g)$. Outcomes are identified with constant acts. Preferences over outcomes agree with preferences over constant acts, and are also denoted by $\succcurlyeq$. We restrict attention to simple (finite-valued) acts for simplicity of presentation. Infinite-valued acts can easily be incorporated in the analysis because then all the following axiomatizations follow by considering the simple acts.

We assume *Choquet expected utility* (CEU) throughout this paper. That is, $U : X \to \mathbb{R}$ is the *utility function*, W is the *capacity* on S (W is defined on $\mathcal{A}$ with $W(\emptyset) = 0$, $W(S) = 1$, and $C \supset D \Rightarrow W(C) \geqslant W(D)$), and $\succcurlyeq$ is represented by $f \to \int_S U(f(s))\,dW(s)$, the CEU of f, which is defined as follows. Let $d(\mathbf{A}, H) = W(\mathbf{A} \cup H) - W(H)$. It is implicit in this notation that $\mathbf{A}$ and H are disjoint. $d(\mathbf{A}, H)$ will be the *decision weight* of event $\mathbf{A}$ in what follows. Consider $f = (E_1, x_1; \ldots; E_n, x_n)$. The CEU of f is $\sum_{j=1}^{n} \pi_j U(x_j)$ where the π_js are defined as follows. Let ρ be a permutation on $\{1, \ldots, n\}$ such that $x_{\rho(1)} \succcurlyeq \cdots \succcurlyeq x_{\rho(n)}$. The *decision weight* $\pi_{\rho(j)}$ of outcome $x_{\rho(j)}$ is $d(E_{\rho(j)}, E_{\rho(1)} \cup \cdots \cup E_{\rho(j-1)})$. The permutation reflects the *ranking positions* of the events, i.e. the favorableness of their outcome relative to the outcomes obtained under other events. Event $E_{\rho(1)}$ is ranked highest, and event $E_{\rho(j)}$ is ranked below $E_{\rho(1)} \cup \cdots \cup E_{\rho(j-1)}$.

A set of acts is *comonotonic* if every pair f and g of its elements is *comonotonic* ($f(s) \succ f(t)$ and $g(s) \prec g(t)$ for no s, t). *Decision under risk* concerns the special case where a probability measure P is given on $\mathcal{A}$, S is rich enough to generate all simple (finite-valued) probability distributions $(p_1, x_1; \ldots; p_n, x_n)$ over X, and all acts that generate the same probability distribution over X are equivalent. Choquet expected utility then reduces to Quiggin's (1981) *rank-dependent utility* (Wakker, 1990). Section 5 gives formal definitions.

W is *convex* if $W(C) + W(D) \leqslant W(C \cup D) + W(C \cap D)$ for all events C, D. W is *concave* if the reversed inequality holds. Convexity holds if and only if $d(\mathbf{A}, H)$ is nondecreasing in H, concavity if and only if it is nonincreasing. We will also be interested in capacities that have properties on particular subdomains. Of special interest are *cavex* capacities, i.e. capacities that are concave for unlikely events and convex for likely events. Such a phenomenon can be interpreted as insensitivity towards changes in likelihood, i.e. a cognitive deviation from Bayesianism reflecting lack of understanding of uncertainty without necessarily a bias towards favorable or unfavorable outcomes. Tversky and Fox (1995) and Wakker (2004) argued for the importance of such a property.

As a preparation, we define event C to be *revealed more likely* than event D, denoted $C \succcurlyeq D$, if there exist outcomes $h \succ \ell$ such that $(C, h; S - C, \ell) \succcurlyeq (D, h; S - D, \ell)$. This gives a behavioral way to elicit that $W(C) \geqslant W(D)$. By $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ we denote the asymmetric, symmetric, and reversed parts of this relation, which are related to the corresponding inequalities of W. For events $C \preccurlyeq D$, $[C, D]$ denotes the set of events E for which $C \preccurlyeq E \preccurlyeq D$. W is *convex on* $[C, D]$ if $d(\mathbf{A}, H \cup \mathrm{I}) \geqslant d(\mathbf{A}, H)$ whenever $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathrm{I} \preccurlyeq D$. The likelihood bounds ensure that the condition only concerns the behavior of W on $[C, D]$ because all the arguments of W relevant to $d(\mathbf{A}, H \cup \mathrm{I})$ and $d(\mathbf{A}, H)$ are in $[C, D]$. The smallest of these, H, is revealed to be more likely than C, and the largest, $H \cup \mathbf{A} \cup \mathrm{I}$, is revealed to be less likely than D. Then so are the relevant arguments of W in between, $H \cup \mathbf{A}$ and $H \cup \mathrm{I}$. W is *concave on* $[C, D]$ if $d(\mathbf{A}, H \cup \mathrm{I}) \leqslant d(\mathbf{A}, H)$ whenever $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathrm{I} \preccurlyeq D$.

The use of the same symbol $\succcurlyeq$ for preferences over acts, outcomes, and events, is based on the interpretation of outcomes as constant acts and of events as indicator-function acts, yielding a predesigned good outcome for an event and a predesigned bad outcome for the complementary event. The act-interpretation of events is less common than that of outcomes, but is also natural and was propagated by de Finetti (1974, Section 3.1.4).

Figure 4 of Wakker (2001) indicates, for monetary outcomes, a general method of obtaining preference axiomatizations of properties of W. Instead of a figure, we give a verbal explanation hereafter, extended to general outcomes.

For the intuition of the method, consider a convex capacity. Such a capacity can be characterized through particular violations of the sure-thing principle, as follows. Imagine that in an indifference between two comonotonic acts, a common outcome ℓ conditional upon an event I is improved into a common outcome $m \succ \ell$. The sure-thing principle would require that such an improvement of a common outcome does not affect the indifference. We, however, consider violations of the sure-thing principle generated by a pessimistic non-Bayesian attitude. Imagine that the two improved acts are again comonotonic, that event **A** was ranked above event I before the improvement but is ranked below after, and that no ranking positions of other events were affected. In other words, **A** yielded outcomes between ℓ and m, and the other events did not. Because **A** is ranked lower after the improvement, it becomes more important for a pessimist. Hence, if an act yielded a better outcome under **A** in the indifference before the improvement, then this act will be preferred after the improvement. In summary, improving a common outcome such that event **A** becomes ranked lower makes event **A** more important, and the preference for the better act under **A** will increase more than the preference for the other act.

The preference conditions in this paper are based on the above intuition, and are all necessary for the corresponding conditions of capacities. Recognizing this intuition in the preference conditions below provides insights into the general technique. The particular forms of the preference conditions that are sufficient to imply the corresponding conditions for capacities depend on the particular richness assumptions made in particular models.

3. A continuum of outcomes and general events

The richness assumption in this section assumes a nondegenerate interval in the utility space, and is satisfied if $X = \mathbb{R}$ or if X is a general connected topological space and utility is continuous and nonconstant. In Schmeidler (1989), X can be taken as a convex subset of a linear space, designating probability distributions over prizes. This section, therefore, concerns the extension of Schmeidler's (1989) analysis to continuous instead of linear utility. It, likewise, extends Chateauneuf's (1991) analysis, who considered linear utility for real-valued outcomes (money).

Example 3.5 shows that complications can still arise for a continuum of outcomes if there are only two nonnull events. Hence, we rule that case out too. As a preparation, we define null events in the Savage sense. Event E is *null* if any two acts that agree outside of E are equivalent. Otherwise E is *nonnull*. *Monotonicity* means that $f \succ g$ whenever, for

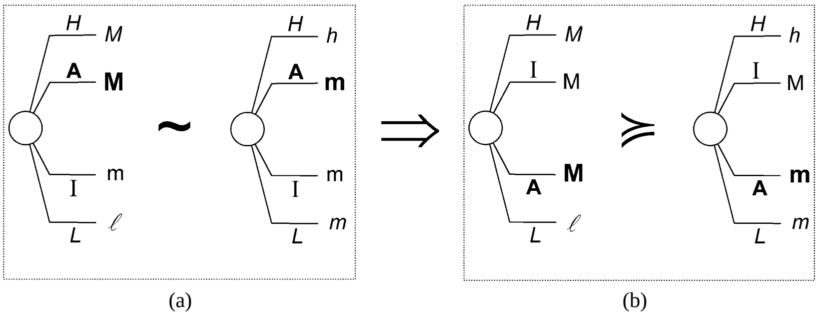


Fig. 1. Pessimism for a continuum of outcomes ($h \succcurlyeq M \succcurlyeq m \succcurlyeq \ell$).

some outcomes $m \succ \ell$, $f = m$ and $g = \ell$ on a nonnull event and, further, $f = g$ outside that event. Under CEU, this version of monotonicity holds if and only if W satisfies *null invariance*, meaning that $d(E, H) = 0$ for some H if and only if it is for all H.

Assumption 3.1 (*Continuum-of-outcomes*). CEU holds, the range of U contains a nondegenerate interval, there exist three disjoint nonnull events, and $\succcurlyeq$ satisfies monotonicity.

A continuum of outcomes without restriction to linear utility appears in the CEU characterizations by Bleichrodt and Miyamoto (2003), Köbberling and Wakker (2003), and several earlier papers. In Fig. 1(a), the left circle designates the act $(H, M$; **A**; **M**, I; m; $L, \ell)$; other acts are illustrated similarly. Formally, *outcome-pessimism* holds if the implication of Fig. 1 holds whenever $h \succcurlyeq M \succcurlyeq m \succcurlyeq \ell$ and $\{H, \mathbf{A}, \mathrm{I}, L\}$ partitions the state space. We first explain the notation used, then the idea underlying the condition.

Throughout this paper, h denotes a *high* outcome, ℓ a *low* outcome, and M and m *medium* outcomes, with $h \succcurlyeq M \succcurlyeq m \succcurlyeq \ell$. H denotes an event yielding a high outcome, L an event yielding a low outcome, and I an event yielding the same outcome for two acts being considered so that I is, in a way, *irrelevant* for the choice between the acts. Finally, **A** denotes the event whose change in decision weight, generated by a change in the ranking position, is used to *assess* what the property of the capacity is.

In Fig. 1(a), **A** yields a higher outcome for the left act, which provides an argument for the left choice. H and L provide counterarguments that, apparently, exactly offset **A**'s argument. The intuition of the preference in Fig. 1(b) was explained at the end of the preceding section. Because of the richness of outcomes, we can construct the configuration of Fig. 1(a) for sufficiently many events $H, \mathbf{A}, \mathrm{I}, L$ to imply convexity of the capacity. *Outcome-optimism* is defined similarly but with the reversed preference $\preccurlyeq$ in Fig. 1. These conditions hold on $[C, D]$ if the restriction $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathrm{I} \preccurlyeq D$ is added.

Lemma 3.2. *Under Assumption* 3.1*, the capacity is convex* (*concave*) *on* $[C, D]$ *if and only if* $\succcurlyeq$ *exhibits outcome-pessimism* (*outcome-optimism*) *on* $[C, D]$.

Theorem 3.3. *Under the continuum-of-outcomes Assumption* 3.1:

(i) *the capacity is convex* *if and only if* $\succcurlyeq$ *exhibits outcome-pessimism*;

(ii) *the capacity is concave* *if and only if* $\succcurlyeq$ *exhibits outcome-optimism*;
(iii) [*cavexity*] *the capacity is concave on* $[\emptyset, C]$ *and convex on* $[C, S]$

if and only if

$\succcurlyeq$ *exhibits outcome-optimism on* $[\emptyset, C]$ *and outcome-pessimism on* $[C, S]$.

The following remark provides a simplification for finite state spaces. The resulting generalization of Theorem 3.3 was stated in the unpublished Section VI.11 of Wakker (1986).

Remark 3.4. If the state space S is finite, then the preference conditions for convex and concave capacities in Lemma 3.2 and Theorem 3.3 can be restricted to singleton events **A** and I.

Example 3.5 (*Continuum of outcomes, two states, but W not convex*). This example is Example VI.11.5 from Wakker (1986). Let $S = \{1, 2\}$, $X = \mathbb{R}$, let $1/2 < W(\{1\}) = W(\{2\}) < 1$, and let U be the identity. Then W is not convex. However, the indifference in Fig. 1(a) automatically implies a weak preference, and even an indifference, in Fig. 1(b): of **A** and I at least one must be empty because otherwise the indifference in Fig. 1(a) cannot hold. It can, similarly, be seen that any indifference in Fig. 4(a) of Wakker (2001) implies indifference in Fig. 4(b) there, so that no variation of this general method can be used in this example.

4. A continuum of events and general outcomes

Solvability of the capacity W means that for each pair of events $B \subset D$ and $W(B) < \gamma < W(D)$, there exists an event C such that $B \subset C \subset D$ and $W(C) = \gamma$. It is the richness condition for the state space needed in the analyses of Savage (1954) and Gilboa (1987). An expression of this assumption for CEU directly in terms of preference conditions, necessary for preference axiomatizations as presented in the present paper, is given by Gilboa (1987). In Gilboa's model, as in Theorem 4.3 below, more than two equivalence classes of outcomes are needed. For less than three such equivalence classes, W in CEU is unique only up to strictly increasing transformations, and convexity is not a meaningful condition (Example 4.4). Three nonequivalent outcomes give uniqueness of the capacity, and suffice for the result below. This result applies, for instance, to the often-studied probability triangle that consists of all probability distributions over three fixed outcomes.

Assumption 4.1 (*Continuum-of-events*). CEU holds with a solvable capacity and at least three nonequivalent outcomes.

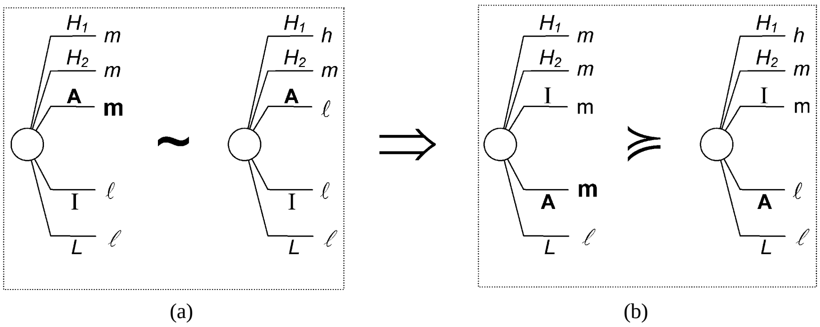


Fig. 2. Pessimism for a continuum of events ($h \succcurlyeq m \succcurlyeq \ell$).

Event-pessimism means that the implication of Fig. 2 holds whenever $h \succcurlyeq m \succcurlyeq \ell$ and $\{H_1, H_2, \mathbf{A}, \mathrm{I}, L\}$ partitions the state space.[1] The condition reflects the intuition described at the end of Section 2. Because of the richness of events, we can construct many partitions $\{H_1, H_2\}$, and obtain the configuration of Fig. 2(a) for sufficiently many events $\mathbf{A}$, I, L, to ensure convexity of the capacity. *Event-optimism* holds if the implication in Fig. 2 holds with reversed preference $\preccurlyeq$ instead of $\succcurlyeq$ in Fig. 2(b), and has similar interpretations. The conditions just defined hold *on* $[C, D]$ if their respective implications are restricted to the case $C \preccurlyeq H_1 \cup H_2 \preccurlyeq H_1 \cup H_2 \cup \mathbf{A} \cup \mathrm{I} \preccurlyeq D$.

The following lemma prepares for Theorem 4.3.

Lemma 4.2. *Assume the continuum-of-events Assumption* 4.1. *Then the capacity is convex* (*concave*) *on* $[C, D]$ *if and only if* $\succcurlyeq$ *exhibits event-pessimism* (*event-optimism*) *on* $[C, D]$.

Theorem 4.3. *Under the continuum-of-events Assumption* 4.1:

(i) *the capacity is convex if and only if* $\succcurlyeq$ *exhibits event-pessimism*;
(ii) *the capacity is concave if and only if* $\succcurlyeq$ *exhibits event-optimism*;
(iii) [*cavexity*] *the capacity is concave on* $[\emptyset, C]$ *and convex on* $[C, S]$

if and only if

$\succcurlyeq$ *exhibits event-optimism on* $[\emptyset, C]$ *and event-pessimism on* $[C, S]$.

Wu and Gonzalez (1999, Questions 6.1 and 6.2) tested event-pessimism empirically. On May 19, 1995 they asked participants about the Dow Jones Industrial Average close on June 30, 1995 (D; close of 4341). The design was between-subjects, with 70 participants answering each question. Table 1 describes the events.

[1] It suffices to impose the requirement only for some, instead of for all, outcomes $h \succ m \succ \ell$. Nothing more is used in the proofs. For real outcomes, $(h =)\ M \geqslant m \geqslant (\ell =)\ 0$ could have been taken, as in Wakker (2001).

Table 1
Events in Wu and Gonzalez (1999)

$D < 4200$	$4200 \leqslant D < 4250$	$4250 \leqslant D \leqslant 4300$	$4300 \leqslant D < 4600$	$4600 \leqslant D$
H_1	**A**	H_2	I	L

Payments were $h = \$300$, $m = \$150$, and $\ell = \$0$. Safe choice decreased from 48% in Fig. 2(a) to 35% in Fig. 2(b). The difference was nonsignificant, but suggested a violation of event-pessimism.

Example 4.4 (*Continuum-of-events, two outcomes, but W not convex*). $S = [0, 1[$, there are only two outcomes, 0 and 1, λ is the Lebesgue measure, U is the identity, and $W(\mathbf{A}) = w(\lambda(\mathbf{A}))$ where $w : [0, 1] \to [0, 1]$ is strictly increasing. Then the following statements are equivalent:

(i) $f \succcurlyeq g$;
(ii) $W\{s \in S\colon f(s) = 1\} \geqslant W\{s \in S\colon g(s) = 1\}$;
(iii) $\lambda\{s \in S\colon f(s) = 1\}) \geqslant \lambda(\{s \in S\colon g(s) = 1\}$.

Thus, the preference relation could also be represented by expected utility with λ as probability measure, and the indifference in Fig. 4(a) of Wakker (2001), or any Figure (a) in this paper, automatically implies indifference in Fig. 4(b) of Wakker (2001), or any corresponding Figure (b) in this paper. Consequently, all convexity conditions of this paper are satisfied. It is elementarily verified that W is convex if and only if w is. Hence, W can be nonconvex by taking any nonconvex w, for example $w(p) = \sqrt{p}$.

5. Decision under risk: given probabilities

This section considers decision under risk. An outcome set X is given and the set of all *lotteries*, i.e., simple probability distributions over X. Lottery $(p_1, x_1; \ldots; p_n, x_n)$ assigns probability p_j to outcome x_j, $j = 1, \ldots, n$. Probabilities are nonnegative numbers that sum to 1. The lottery $(1, x)$ is identified with the outcome x. $\succcurlyeq$ now denotes the preference relation over lotteries.

Let us next see how risk can be considered to be a special case of uncertainty, following Wakker (1990). For decision under risk, we take the set $S = [0, 1[$ as the state space, with $\mathcal{A}$ the usual Borel sigma-algebra generated by intervals. It also suffices to let A be the algebra generated by intervals, which consists of all finite unions of intervals. S is endowed with the Lebesgue probability measure λ. This probability measure assigns to each interval its length and is naturally extended to the other sets in A. Each simple probability distribution $(p_1, x_1; \ldots; p_n, x_n)$ can be identified with the act assigning x_j to the interval $[p_1 + \cdots + p_{j-1}, p_1 + \cdots + p_j[$, for $j = 1, \ldots, n$. All acts that generate the same probability distribution over outcomes are equivalent. Hence, preferences over acts correspond with preferences over probability distributions and results for uncertainty immediately apply to risk.

For risk, CEU reduces to *rank-dependent utility* (RDU). We define RDU formally and use the following notation: $P(U \geqslant t) = \sum_{j:U(x_j)\geqslant t} p_j$, for a lottery $P = (p_1, x_1; \ldots; p_n, x_n)$, $U : X \to \mathbb{R}$, and $t \in \mathbb{R}$.

(1) A function $U : X \to \mathbb{R}$ is given, the *utility function*.
(2) A *probability transformation* w is given on [0, 1], i.e., $w : [0, 1] \to [0, 1]$ with
   (i) $w(0) = 0$,
   (ii) $w(1) = 1$, and
   (iii) w is strictly increasing.
(3) $\succcurlyeq$ is represented by $P \to \int_{\mathbb{R}_-} [w(P(U \geqslant t)) - 1]\,\mathrm{d}t + \int_{\mathbb{R}_+} w(P(U \geqslant t))\,\mathrm{d}t$, the *rank-dependent utility* (RDU) of P.

We do not require w to be continuous, so as to have this condition optional. We neither impose restrictions on U. It is well known that, for a probability distribution $(p_1, x_1; \ldots; p_n, x_n)$, the RDU value is equal to

$$\sum_{j=1}^{n} \pi_j U(x_j)$$

where the *decision weights* π_j are defined by $\pi_{\rho(j)} = w(p_{\rho(1)} + \cdots + p_{\rho(j)}) - w(p_{\rho(1)} + \cdots + p_{\rho(j-1)})$ for a permutation ρ with $x_{\rho(1)} \succcurlyeq \cdots \succcurlyeq x_{\rho(n)}$, in agreement with the definitions in Section 2.

That RDU is a special case of CEU can be seen by defining the capacity $W = w \circ \lambda$ on the state space $S = [0, 1[$. Verification is left to the reader. Figure 3 illustrates convexity of the probability transformation function. It adapts Fig. 3 of Wakker (2001) to the context of risk. This π_s will later serve as decision weights for the lotteries in Fig. 4. Table 2 lists equivalent properties of the transformation w and the capacity W. The equivalences regarding convexity and concavity are most easily proved by examining whether decision

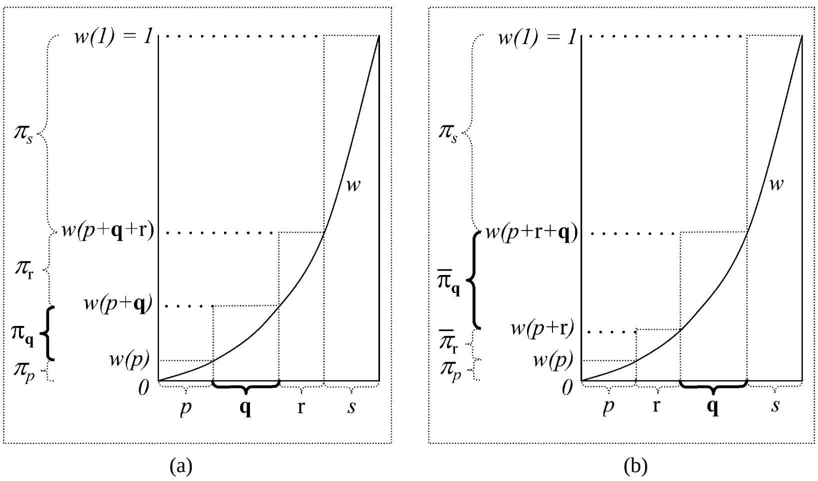
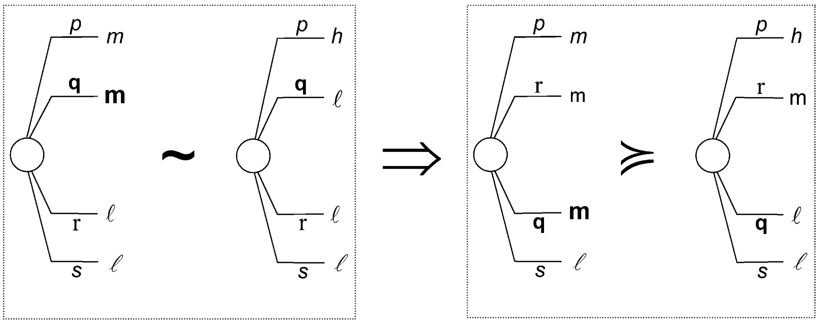


Fig. 3. $\bar{\pi}_q \geqslant \pi_q$ for convex probability transformations.

Table 2
Equivalent properties of the probability transformation w and the capacity $W = w \circ \lambda$ on [0, 1[

W	Solvable	Convex	Concave	Convex on $[C, D]$	Concave on $[C, D]$
w	Continuous	Convex	Concave	Convex on $[\lambda(C), \lambda(D)]$	Concave on $[\lambda(C), \lambda(D)]$



(a) (b)

Fig. 4. Pessimism for risk ($M \geqslant m \geqslant 0$).

weights are increasing or decreasing in the second argument. These and the other proofs are, again, left to the reader.

Theorems for decision under uncertainty can immediately be translated to risk. The method is simple: One attaches probabilities to all events and then writes out the probability distributions. The following assumption adapts Assumption 5.1 of Wakker (2001) to risk.

Assumption 5.1 [*Continuum-of-outcomes-and-risk*]. RDU holds with outcome set $X = \mathbb{R}$, utility continuous and strictly increasing, and w continuous.

Assumption 5.1 is usually satisfied in the literature on decision under risk. By the following probability substitutions, Fig. 4 adapts Fig. 2 of Wakker (2001) to risk: $p = \lambda(H)$, $\mathbf{q} = \lambda(\mathbf{A})$, $\mathrm{r} = \lambda(\mathrm{I})$, $s = \lambda(L)$. *Pessimism* holds on $[a, b]$ if the implication of Fig. 4 holds whenever $h \geqslant m \geqslant \ell$ and $p \geqslant a$, $p + \mathbf{q} + \mathrm{r} \leqslant b$ (i.e., $s \geqslant 1 - b$). Then indeed all relevant arguments of w are from $[a, b]$. *Optimism* on $[a, b]$ is defined similarly, with $\preccurlyeq$ instead of $\succcurlyeq$ in Fig. 4(b). "On [0, 1]" is often omitted. The following lemma follows immediately from the described substitutions, Table 2, Lemma 5.3 of Wakker (2001), and the substitutions $a = W(C)$, $b = W(D)$. Hence, no proof is given. Wakker (2001, end of Section 5) described the following two results informally.

Lemma 5.2. *Under the continuum-of-risk Assumption* 5.1*, the probability transformation is convex* (*concave*) *on* $[a, b]$ *if and only if* $\succcurlyeq$ *exhibits pessimism* (*optimism*) *on* $[a, b]$.

Pessimism has been tested in many empirical studies; see Examples A.3–A.4 of Wakker (2001). The condition has mostly been considered for $s = 0$, where it tests upper subadditivity, i.e., the certainty effect. This is one of the best-confirmed phenomena in the field (MacCrimmon and Larsson, 1979; Conlisk, 1989). Still, exceptions exist (Starmer, 1992; described in Wakker, 2001, Example A.4). The most important implications of Lemma 5.2 are gathered in the following theorem, where (iii) describes the prevailing empirical pattern for b equal to, approximately, $1/3$.

Theorem 5.3. *Under the Continuum-for-risk Assumption* 5.1*,*

(i) *the probability transformation is convex if and only if* $\succcurlyeq$ *exhibits pessimism*;
(ii) *the probability transformation is concave if and only if* $\succcurlyeq$ *exhibits optimism*;
(iii) [*cavexity*] *the probability transformation is concave on* $[0, b]$ *and convex on* $[b, 1]$

if and only if

$\succcurlyeq$ *exhibits optimism on* $[0, b]$ *and pessimism on* $[b, 1]$.

Because convexity and concavity have been tested extensively in the probability triangle, where Assumption 5.1 is not satisfied, the preference conditions for risk without a continuum of outcomes are presented next. We generalize Assumption 5.1 by allowing for general outcomes and by relaxing the assumptions on utility and probability transformation. The preference conditions presented next have been obtained from those for decision under uncertainty by the following probability substitutions in Fig. 2: $\lambda(H_1) = p_1$, $\lambda(H_2) = p_2$, $\lambda(\mathbf{A}) = \mathbf{q}$, $\lambda(\mathrm{I}) = \mathrm{r}$, $\lambda(L) = s$. Further, $\lambda(C) = a$ and $\lambda(D) = b$ is chosen. The next lemma follows from these substitutions, Table 2, and Lemma 4.2.

Observation 5.4. Assume RDU, with continuous w and at least three nonequivalent outcomes. Then

(i) w is convex on $[a, b]$ if and only if $(p_1 + p_2 + \mathbf{q}, m; \mathrm{r} + s, \ell) \sim (p_1, h; p_2; m, \mathbf{q} + \mathrm{r} + s, \ell)$ implies $(p_1 + p_2 + \mathrm{r} + \mathbf{q}, m; s, \ell) \succcurlyeq (p_1, h; p_2 + \mathrm{r}, m; \mathbf{q} + s, \ell)$ whenever $h \succcurlyeq m \succcurlyeq \ell$, $p_1 + p_2 \geqslant a$, and $p_1 + p_2 + \mathbf{q} + \mathrm{r} \leqslant b$.
(ii) Concavity of w is characterized by replacing the first preference $\succcurlyeq$ in (i) by a reversed preference $\preccurlyeq$.

All other results for uncertainty immediately imply the corresponding results for risk by similar translations. For brevity, these related results are not made explicit. Wu and Gonzalez (1996, 1998; see Appendix B) characterized convex and concave probability transformations by means of preference conditions of the same nature. They used more restrictive technical assumptions.

The simplest characterization of convex probability transformations under RDU can be obtained by using betweenness as a benchmark. It is well known that RDU reduces to expected utility, with linear probability transformation, if betweenness holds. It turns out that probability transformation is convex under RDU if and only if quasiconvexity holds with respect to probabilistic mixing, and concave if and only if quasiconcavity holds (Wakker, 1994, Theorem 25(e); Prelec, 1998). For this simple result there is, unfortunately, no easy analog in uncertainty.

Table 3
Summary of results

		W is convex	W is concave	W is cavex
Continuum of events (e.g. *risk*)	Continuum of outcomes (e.g. $\mathbb{R}$, U cont)	Wakker (2001), Theorem 5.2(i) (*risk: Th. 5.3i*)	Wakker (2001), Theorem 5.2(ii) (*risk: Th. 5.3ii*)	Wakker (2001), Theorem 5.4 (*risk: Th. 5.3iii*)
General events (e.g. S finite)	Continuum of outcomes (e.g. $\mathbb{R}$, U cont)	Theorem 3.3(i) (*risk:* [a])	Theorem 3.3(ii) (*risk:* [a])	Theorem 3.3(iii) (*risk:* [a])
Continuum of events (e.g. *risk*)	General outcomes (e.g. X finite)	Theorem 4.3(i) (*risk:* [b])	Theorem 4.3(ii) (*risk:* [b])	Theorem 4.3(iii) (*risk:* [b])
General events (e.g. S finite)	General outcomes (e.g. X finite)	[c]	[c]	[c]

Theorems characterize the property of the capacity in a column given the structural assumptions of the row.
[a] means: dropped for brevity.
[b] means: through Observation 5.4.
[c] means: open research question.

6. Summary and conclusion

Table 3 summarizes the results of this paper. These results concern chance attitude (capacities and probability transformations) and have been derived independently of utility. A central feature of rank-dependent models, i.e. the separation of chance attitude and utility, has thus been maintained in the analysis.

Acknowledgments

I first read a paper by David Schmeidler in 1983, when my Dutch supervisor Stef Tijs gave me the 1982 version of what later became David's famous 1989 *Econometrica paper*. The supervisor of my master's thesis, Wim Vervaat, gave me mathematical literature on nonadditive measures (e.g., by Bernd Anger). Stef arranged a visit for me to David in Tel Aviv that took place in the end of 1984. During this visit, David proposed that I extend a representation technique of my PhD dissertation to his model. While I immediately understood that I was lucky to meet this fascinating creative mind, I could not foresee in those days how extremely lucky I was, and how extremely great and fascinating this creative mind was. Now, 20 years later, we all know.

In addition, the editor and an anonymous referee made useful comments.

Appendix A. Proofs

We will use ordered partitions, denoted $(E_1, \ldots, E_n)$, assuming the rank-ordering with best outcomes for $E_1, \ldots,$ and worst outcomes for E_n. When no misunderstandings can arise, the term "ordered" is suppressed. Consider a partition $(E_1, \ldots, E_n)$ and the decision weights $d(E_j, E_1 \cup \cdots \cup E_{j-1})$ of events in that partition. For the decision weight of $E_j \cup E_{j+1}$ in that partition we have

$$d(E_i \cup E_{i+1}, E_1 \cup \cdots \cup E_{i-1}) = d(E_i, E_1 \cup \cdots \cup E_{i-1}) + d(E_{i+1}, E_1 \cup \cdots \cup E_i), \tag{A.1}$$

which entails a kind of additivity. Figure 3 in Wakker (2001) gives examples of Eq. (A.1). The equation will often be used without explicit mention.

A.1. Proof of necessity of the preference conditions in all results

This always follows from substitution of CEU. It also follows from Theorem 4.1 of Wakker (2001) and the reasoning preceding it, with the obvious adaptations for nonmonetary outcomes ($x \sim y$ instead of $x = y$, $x \succ y$ instead of $x > y$, etc.). □

Henceforth, only sufficiency of the preference conditions needs to be established. In the following proofs, the following notation, introduced by Wakker (2001, B.2), will be used. For a partition $(E_1, \ldots, E_n)$,

$$(E_1, \ldots, \{E_i; E_{i+1}\}, \ldots, E_n) \quad \text{means that}$$
$$d(E_i, E_1 \cup \cdots \cup E_{i-1}) \leqslant d(E_i, E_1 \cup \cdots \cup E_{i-1} \cup E_{i+1}). \tag{A.2}$$

To understand the idea, note that the first decision weight concerns the rank-ordering $(E_1, \ldots, E_i, E_{i+1}, \ldots, E_n)$, and the second the rank-ordering with E_i and E_{i+1} interchanged, i.e., where E_{i+1} has "passed by" E_i in ranking. The sum of the decision weights of E_i and E_{i+1} is the same in both rankings, being the left-hand side of Eq. (A.1) each time. Equation (A.2) entails that E_{i+1} loses decision weight to its neighbor E_i if E_{i+1} passes by E_i in the rank-ordering. Convexity of W is equivalent to $(E_1, \ldots, \{E_i; E_{i+1}\}, \ldots, E_n)$ for all E_i in all partitions, and it is also equivalent to the condition $(H, \{\mathbf{A}; \mathrm{I}\}, L)$ for all four-fold partitions. These are different ways of saying that a decision weight $d(\mathbf{A}, H)$ increases in its second argument and, therefore, is below $d(\mathbf{A}, H \cup \mathrm{I})$. The following observation follows from substitution of CEU, or from Theorem 4.1 of Wakker (2001).

Observation A.1. If the equivalence and preference of Figs. 1–2 hold, then $(H, \{\mathbf{A}; \mathrm{I}\}, L)$ (with $H = H_1 \cup H_2$ in Fig. 2).

A.2. Proof of sufficiency in Lemma 3.2 for convexity

We assume outcome-pessimism on $[C, D]$ and derive convexity of the capacity on $[C, D]$. We derive $(H, \{\mathbf{A}; \mathrm{I}\}, L)$, for $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathrm{I} \preccurlyeq D$. The restrictions of outcome-pessimism on $[C, D]$ apply to these events and, hence, the implications of Fig. 1 can be used hereafter.

Case 1. $W(H) > 0$. Then we can take outcomes $h \succcurlyeq M \succ m \sim \ell$ such that the indifference in Fig. 1(a) holds. We can use Observation A.1 because pessimism on $[C, D]$ does imply the preference in Fig. 1(b) (which is a special case of Fig. 4(b) in Wakker, 2001). Hence, $(H, \{\mathbf{A}; \mathrm{I}\}, L)$.

Case 2. $d(L, H \cup \mathbf{A} \cup \mathrm{I}) > 0$. Then we can take outcomes $h \sim M \succ m \succcurlyeq \ell$ such that the indifference in Fig. 1(a) holds. By Observation A.1, $(H, \{\mathbf{A}; \mathrm{I}\}, L)$.

Case 3. $W(H) = 0 = d(L, H \cup \mathbf{A} \cup \mathrm{I})$. Then an indifference such as in Fig. 1(a) cannot be obtained and more complicated constructions must be used. The following lemma states the implications of monotonicity that are needed in the proof. (It, therefore, also shows ways to relax the monotonicity requirement.)

Lemma A.2. *Either* $\mathbf{A}$ *can be partitioned into* A_1 *and* A_2 *such that* $d(A_1, H) > 0$ *and* $d(A_2, H \cup A_1) > 0$*, or* I *can be partitioned into* I_1 *and* I_2 *such that* $d(\mathrm{I}_2, H \cup \mathbf{A} \cup \mathrm{I}_1) > 0$ *and* $d(\mathrm{I}_1, H) > 0$.

Proof. There are three disjoint nonnull events, say B, C, D. Under CEU, we have

- If E is null, then so is any subset of E.

Further, because of monotonicity:

- If E and G are null, then so is $E \cup G$.
- If E is nonnull and is partitioned, then at least one of the elements of the partition is nonnull.

Hence, of the partition $\{B \cap H, B \cap \mathbf{A}, B \cap \mathrm{I}, B \cap L\}$ of B, at least one must be nonnull and it must be $B \cap \mathbf{A}$ or $B \cap \mathrm{I}$. Similarly, $C \cap \mathbf{A}$ or $C \cap \mathrm{I}$ must be nonnull and $D \cap \mathbf{A}$ or $D \cap \mathrm{I}$ must be nonnull. Thus, of $\mathbf{A}$ and I, at least one must contain two nonnull subsets. From that and monotonicity, all claims in the lemma follow. *QED*

Case 3a. $\mathbf{A}$ can be partitioned into A_1 and A_2 with $d(A_1, H) > 0$ and $d(A_2, H \cup A_1) > 0$. Now $(H \cup A_1, \{A_2; \mathrm{I}\}, L)$ by case 1, i.e., I loses weight when passing by A_2 in rank-ordering. $d(A_2, H \cup A_1) > 0$ implies that $d(A_2, H \cup A_1 \cup \mathrm{I}) > 0$, because of monotonicity (without monotonicity, it could be derived from the fact that I has lost decision weight while passing by A_2). Therefore, $(H, \{A_1; \mathrm{I}\}, A_2 \cup L)$ by case 2. I loses weight when passing by A_2 and also when passing by A_1 and, therefore, $(H, \{\mathbf{A}; \mathrm{I}\}, L)$ follows.

Case 3b. I can be partitioned into I_1 and I_2 with $d(\mathrm{I}_2, H \cup \mathbf{A} \cup \mathrm{I}_1) > 0$ and $d(\mathrm{I}_1, H) > 0$. $(H, \{\mathbf{A}; \mathrm{I}_1\}, \mathrm{I}_2 \cup L)$ follows from $d(\mathrm{I}_2, H \cup \mathbf{A} \cup \mathrm{I}_1) > 0$ and case 2. Further, $(H \cup \mathrm{I}_1, \{\mathbf{A}; \mathrm{I}_2\}, L)$ follows from $d(\mathrm{I}_1, H) > 0$ and case 1. In other words, the decision weight of $\mathbf{A}$ increases both if I_1 passes it by and if I_2 does so. $(H, \{\mathbf{A}; \mathrm{I}\}, L)$ follows. □

A.3. Proof of sufficiency in Lemma 3.2 for concavity

Although this case is not perfectly dual to the convex case (because of the ranking of $\mathbf{A}$ above I in Fig. 1(a)), the proof is nevertheless similar. Case 1 ($W(H) > 0$) and case 2 ($d(L, H \cup \mathbf{A} \cup \mathrm{I}) > 0$) are completely the same. In the remaining case 3, we can either partition $\mathbf{A}$ into nonnull A_1, A_2, in which case I can pass by A_2 while gaining decision weight by case 1 and can then pass by A_1 while gaining decision weight by case 2, or we can partition I into nonnull $\mathrm{I}_1, \mathrm{I}_2$, in which case **A** loses decision weight when I_1 passes it by because of case 2 and **A** loses decision weight by case 1 when I_2 passes it by. □

A.4. Proof of Remark 3.4

Because finiteness of S excludes continuum-of-states, this remark is relevant only for continuum-of-outcomes. An informal, verbal proof is given. Assume two disjoint events **A** and I, with **A** rank-ordered directly above I, and consider the change in decision weight of **A** as I passes by **A** in rank-ordering. This change in rank-ordering can be obtained through many "elementary changes" in rank-ordering, where in each elementary change one state of I passes by one state of **A** in rank-ordering. More precisely, there will be $|\mathbf{A}| \times |\mathrm{I}|$ such elementary changes. Each such elementary change in rank-ordering increases the decision weight of the element of **A**. At the end, all elements of I have passed by all elements of **A** and all elements of **A** have gained decision weight during the process. The decision weight of **A** is, by additivity (Eq. (A.1)), the sum of the decision weights of its elements and, hence the total decision weight of **A** has increased. □

A.5. Proof of sufficiency in Lemma 4.2 for convexity

We assume event-pessimism and derive convexity of the capacity. Under CEU, all that is relevant about outcomes for determining preference is their utilities. Therefore, we may as well replace outcomes by their utility values, i.e., we may assume that outcomes are utilities. There are three nonequivalent outcomes. We rescale them, and may assume that they are $1+\mu$, 1, and 0 for some positive μ. We use event-pessimism and Fig. 2 with the choices $h = 1+\mu$, $m = 1$, and $\ell = 0$. Note that, for CEU differences in Figs. 2(a) and 2(b), event H_1 always delivers a utility difference μ multiplied by its decision weight and event **A** a utility difference 1 multiplied by its decision weight. We write $H = H_1 \cup H_2$.

Case 1. $W(H) > 0$. The result is first proved for events **A** whose decision weight is so small, relative to that of H, that we can find an event $H_1 \subset H$ as in Fig. 2(a).

Case 1(a). $d(\mathbf{A}, H) \leqslant W(H) \cdot \mu$. Because of solvability, we can take $H_1 \subset H$ such that the indifference in Fig. 2(a) holds (taking $W(H_1) = d(\mathbf{A}, H)/\mu$, the CEU difference between the acts in Fig. 2(a) is $W(H_1)\mu - d(\mathbf{A}, H) = 0$). Because of event-pessimism and Observation A.1, $(H, \{\mathbf{A}; \mathrm{I}\}, L)$.

Case 1(b). $d(\mathbf{A}, H) > W(H) \cdot \mu$. Because of solvability, we can partition **A** into $(A_1, \ldots, A_n)$ such that all A_j have decision weight smaller than $W(H)\mu$ in $(H, A_1, \ldots, A_n, \mathrm{I}, L)$. From this and case 1(a) it follows that I loses decision weight when it passes by A_n, also when it passes by $A_{n-1}, \ldots,$ and, finally, also when it passes by A_1. Hence, I has less decision weight in $(H, \mathrm{I}, A_1, \ldots, A_n, L)$ (hence, in $(H, \mathrm{I}, \mathbf{A}, L)$) than in $(H, A_1, \ldots, A_n, \mathrm{I}, L)$ (hence, in $(H, \mathbf{A}, \mathrm{I}, L)$). From this, $(H, \{\mathbf{A}; \mathrm{I}\}, L)$ follows.

Case 2. $W(H) = 0$. If $d(\mathbf{A}, H) = 0$ then the result is trivial, and if $d(\mathbf{A}, H) > 0$ then the case is proved exactly as case 2(b) in the proof of Theorem 5.2(i) in Wakker (2001) (when in that proof case 2(a) there is used, now use case 1 of this proof). Note that the continuum structure of the outcome set was not used there.

Convexity of W has been established for all cases. □

A.6. Proof of sufficiency in Lemma 4.2 for concavity

This case is not dual to the case of convexity, because events H_1 and H_2 do not play a symmetric role. We assume event-optimism and derive concavity of the capacity.

Case 1. $W(H) > 0$. This case is analyzed exactly as for convexity and pessimism, with inequalities regarding the decision weights of I reversed.

Case 2. $W(H) = 0$. This case is analyzed exactly as case 2 in the proof of Theorem 5.2(ii) in Wakker (2001) (with ℓ instead of 0). Note again that the continuum structure of the outcome set was never used there. □

Appendix B. Recent alternative characterizations

Wu and Gonzalez used conditions as in this paper. Sometimes they used strict preferences and strict inequalities instead of weak. This distinction is ignored here. Wu and Gonzalez' (1996) convexity and concavity conditions are as in Observation 5.4. Wu and Gonzalez' (1998) concavity and convexity conditions I can be restricted to the case $p' = 0$ and then are again the conditions in Observation 5.4. Their concavity and convexity Condition II can be restricted to the case $p' + q' + \varepsilon = 1$ and then are similar to the conditions in Observation 5.4. Wu and Gonzalez' (1999) concavity and convexity conditions are event optimism and pessimism. Wu and Gonzalez (1998) showed how their preference conditions are related to fanning in and fanning out in the probability triangle.

Abdellaoui (2002) found a way to use the tradeoff technique of Wakker (1989) in a dual way, turning it into a general tool to characterize properties of probability transformations. The axioms in Fig. 4 and Observation 5.4 of this paper are special cases of the axioms in Abdellaoui's Corollary 14 and Theorem 16. His results, therefore, follow as corollaries of Observation 5.4. Abdellaoui further provided comparative results. Schmidt (2003) applied Abdellaoui's technique to prospect theory.

The above references all characterized properties of capacities independently of utility. Several alternative preference conditions have been proposed in the literature that did not separate restrictions for capacities and for utility. Recent references include Schmidt and Zank (2003), Kast and Lapied (2003), Chateauneuf et al. (2003), and Chateauneuf et al. (2002). Conditions for nonempty cores are in Chateauneuf and Tallon (2002) and Ghirardato and Marinacci (2002). An appealing result appears in Chateauneuf and Tallon (2002), who characterized concave utility plus convex capacities for monetary outcomes through quasi-concavity with respect to the mixing of outcomes. Chateauneuf et al. (2003), finally, characterized capacities that are nonlinear only at the impossible and universal event, assuming linear utility.

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