Proofs of the Claims in Table 2 of

Wakker, Peter P. (2005), "Decision-Foundations for Properties of Nonadditive Measures for General State Spaces or for General Outcome Spaces," *Games and Economic Behavior* 107–125.

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In Table 2 on p. 117 of Wakker (2005), I claimed some of the equivalences displayed in the following table. This comment gives proofs of these equivalences. Let $W = w \circ \lambda$ on [0,1], where λ denotes the Lebesgue measure. The domain (whose elements are called *events*) of W and λ can be either the algebra of all finite unions of intervals, or the Borel/Lebesgue sigma algebras. $w(0) = 0$ and $w(1) = 1$. [C,D] denotes the set of events E with $W(C) \le W(E) \le W(D)$, which is equivalent to the notation in the paper.

TABLE. Equivalences between properties of W and w.

W	increasing w.r.t. set inclusion	null invariant*	solvable*	convex*	concave*	convex on *[C,D]***	concave on *[C,D]***
w	nondecreasing	strictly increasing*	continuous*	convex*	concave*	convex on $[\lambda(C), \lambda(D)]$**	concave on $[\lambda(C), \lambda(D)]$**

The results with an asterisk * assume that w is nondecreasing. The results with asterisks ** assume that w is strictly increasing.

In the paper, I claimed the right five equivalences, only for w strictly increasing. The results in Section 5 were based on this table and on the results derived before for uncertainty. In this note I prove the claims in the table except the first equivalence, the one in the second column, which is left to the reader.

LEMMA 1. Assume that w is nondecreasing. Then W satisfies null invariance if and only if w is strictly increasing.

PROOF.

1. Assume that w is strictly increasing, and that A is nonnull. Then $d(A,H) > 0$ for some H. It implies $W(A \cup H) - W(H) > 0$, so that $W(A \cup H) > W(H)$, $w(\lambda(A \cup H)) > w(\lambda(H))$, and $\lambda(A \cup H) > \lambda(H)$, finally implying that $\lambda(A) + \lambda(H) > \lambda(H)$. We

conclude that $\lambda(A) > 0$. Because of this and strict increasingness of w we have, for all H' disjoint from A,

$W(A \cup H') = w(\lambda(A \cup H')) = w(\lambda(A) + \lambda(H')) > w(\lambda(H')) = W(H')$. It follows that $d(A,H') > 0$ for all H', and null invariance holds.

2. Assume that W satisfies null invariance, and that $1 \geq a + b > b \geq 0$. To prove is that $w(a+b) > w(b)$.

If, for contradiction, w(a) were zero, then for $A = [0,a[$, $W(A) = w(a)$ would be zero. By nondecreasingness of w, every event E with $\lambda(E) \leq a$ would have $W(E) = w(\lambda(E)) \leq w(\lambda(A)) = W(A) = 0$, i.e. it would be null. But then for a partition $[p_j,p_{j+1}]$, $j = 1, ..., n$, of $[0,1[$ with $p_{j+1} - p_j < \lambda(A)$ for all j, all events in the partition would be null and, because of null invariance, $W(p_j+ \cdots +p_1) - W(p_{j-1}+ \cdots +p_1)$ would be zero for all j. This would, finally, imply $W([0,1[) = 0$, contradicting $W([0,1[) = 1$. We conclude that $w(a) > 0$ and $W(A) > 0$ for $A = [0,a[$.

The event A is nonnull and, hence, $W(A \cup B) > W(B)$ for every disjoint B. Take $B= [a,b+a]$. Then $w(b+a) = W(A \cup B) > W(B) = w(b)$, which is what was to be proved.

□

LEMMA 2. Let w be nondecreasing. Then W is solvable if and only if w is continuous.

PROOF.

1. Assume that w is continuous. Assume that $B \subset D$, and let $W(B) < \gamma < W(D)$, that is, γ is between $w(\lambda(B))$ and $w(\lambda(D))$. Because w is continuous it satisfies the intermediate value property, and there must exist $p \in [0,1]$ between $\lambda(B)$ and $\lambda(D)$ such that $w(p) = \gamma$. It is well-known that for each $B \subset D$ and p between $\lambda(B)$ and $\lambda(D)$ there exists C with $B \subset C \subset D$ and $\lambda(C) = p$. This C satisfies $B \subset C \subset D$ and $W(C) = w(\lambda(C)) = w(p) = \gamma$. W is solvable.

2. Assume, for contradiction, that W is solvable, but that w is not continuous. Because w is nondecreasing, there must be a $0 < \gamma < 1$ that is in a "jump" of the graph of w in the

sense that $w(p) = \gamma$ for no p. Therefore, $W(C) = w(\lambda(C)) \neq \gamma$ for all C. Taking $\emptyset \subset [0,1[$ and $W(\emptyset) < \gamma < W([0,1[)$, we see that W does not satisfy solvability.
□

LEMMA 3. Let w be strictly increasing. W is convex on [C,D] if and only if w is on $[\lambda(C), \lambda(D)]$.

PROOF. Asterisks will be used in the proof of concavity in Lemma 4, and should be ignored for Lemma 3.

1. Assume that w is convex* on $[\lambda(C), \lambda(D)]$. Consider $A \subset H \subset H'$ with

$$W(C) \leq W(A) \leq W(A \cup H') \leq W(D), \tag{1}$$

i.e., because w is strictly increasing,

$$\lambda(C) \leq \lambda(A) \leq \lambda(A \cup H') \leq \lambda(D). \tag{2}$$

We have

$d(A,H) = W(A \cup H) - W(H) = w\big(\lambda(A \cup H)\big) - w\big(\lambda(H)\big) =$

$w\big(\lambda(A) + \lambda(H)\big) - w\big(\lambda(H)\big) \leq^*$

$\big[$because w is convex* on $[\lambda(C), \lambda(D)]$ and $\lambda(H') \geq \lambda(H)\big]$

$w\big(\lambda(A) + \lambda(H')\big) - w\big(\lambda(H')\big) =$

$w\big(\lambda(A \cup H')\big) - w\big(\lambda(H')\big) = W(A \cup H') - W(H') = d(A,H')$.

It implies that d(A,H) is increasing* in its second argument on [C,D], which is equivalent to convexity* of W on [C,D], or the requirement that $W(F \cup G) + W(F \cap G) - W(F) - W(G) \geq^* 0$ on [C,D].

2. Assume that W is convex* on [C,D]. We prove that

$w(a+b') - w(b') \geq^* w(a+b) - w(b)$ whenever $b' \geq b$ and

$$\lambda(C) \leq b \leq a+b' \leq \lambda(D). \tag{3}$$

Define $B' = [0,b'[$, $B = [0,b[$, and $A = [b',b'+a[$. A is disjoint from B' and, hence, from B. Because w is nondecreasing (we do not need here that w is strictly increasing), (3) implies

$$W(C) \leq W(B) \leq W(A \cup B') \leq W(D). \tag{4}$$

W being convex* on [C,D] implies that

$d(A,B') \geq^* d(A,B)$, i.e.

$W(A \cup B') - W(B') \geq^* W(A \cup B) - W(B)$.

The left-hand side is

$w(\lambda(A \cup B')) - w(\lambda(B')) = w(a+b') - w(b')$,

the right-hand side is

$w(\lambda(A \cup B)) - w(\lambda(B)) = w(a+b) - w(b)$.

Indeed, $w(a+b') - w(b') \geq^* w(a+b) - w(b)$, and we are done.

□

The following lemma is perfectly dual to the preceding one.

LEMMA 4. W is concave on $[\lambda(C), \lambda(D)]$ if and only if w is on $[\lambda(C), \lambda(D)]$.

PROOF. The lemma can be proved by applying the preceding lemma to the duals of W and w. A direct proof can be obtained from the proof of Lemma 3 by reversing all inequalities with asterisks, replacing all words convex with asterisks by the word concave, and replacing the word increasing with an asterisk by the word decreasing.

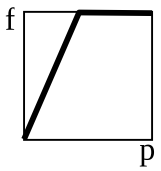
□

We only used strict increasingness of w in the proofs of Lemmas 3 and 4 for the implication (1) $\Rightarrow$ (2). This step is not needed for the unrestricted equivalences of convexity/concavity of W and w on the whole domain.

COROLLARY 5. Assume that w is nondecreasing. Then w is convex [concave] if and only if W is. Convexity [concavity] of W on [C,D] implies convexity [concavity] of w on $[\lambda(C),\lambda(D)]$. □

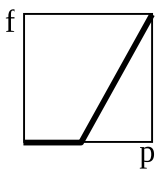
For the derivation of convexity of W on [C,D] from convexity of w on $[\lambda(C),\lambda(D)]$, the only thing that can go wrong if w is not strictly increasing, is if there is an event D' with $\lambda(D') > \lambda(D)$ but $W(D') = W(D)$, i.e., if w is flat on an interval to the right of $\lambda(D)$. The following example illustrates this point.

EXAMPLE. Let $S = [0,1[$, $X = \mathbb{R}$, U is the identity, λ is the Lebesgue measure, and $W(E) = w(\lambda(E))$ where $w(p) = 2p$ for $0 \leq p \leq 1/2$,



w(p) = 1 for all 1/2 < p ≤ 1. w and W are not convex. Take C = ∅ and D = [0,1/2[. Then W(D) = 1 and [C,D] contains all events. W is not convex on [C,D]. Nevertheless, w is convex on [λ(C), λ(D)] = [0,1/2]. The difficulty arises because [C,D] contains events with λ exceeding λ(D). □

For concavity of W on [C,D], the only thing that can go wrong if w is not strictly increasing, is if there is an event C' with λ(C') < λ(C) but W(C') = W(C), i.e., if w is flat on an interval to the right of λ(C). The following figure illustrates this point.



Because of examples of this kind, I assumed null invariance for the results for uncertainty.

References

Wakker, Peter P. (2005), "Decision-Foundations for Properties of Nonadditive Measures for General State Spaces or for General Outcome Spaces," *Games and Economic Behavior* 50, 107–125.