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2 Preference axiomatizations for decision under uncertainty

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Several contributions in this book present axiomatizations of decision models, and of special forms thereof. This chapter explains the general usefulness of such axiomatizations, and reviews the basic axiomatizations for static individual decisions under uncertainty. It will demonstrate that David Schmeidler's contributions to this field were crucial.

2.1. The general purpose of axiomatizations

In this section we discuss some general purposes of axiomatizations. In particular, the aim is to convince the reader that axiomatizations are an essential step in the development of new models. To start, imagine that you are a novice in decision theory, and have an important decision to take, say which of several risky medical treatments to undergo. You consult a decision theorist, and she gives you a first advice, as follows:

1. List all relevant uncertainties. In your case we assume that the uncertainty concerns which of n potential diseases $s_1, \ldots, s_n$ is the one you have.
2. Express your uncertainty about what your disease is numerically through probabilities $p_1, \ldots, p_n$, subjective if necessary.
3. Express numerically how good you think the result is of each treatment conditional upon each disease. Call these numbers utilities.
4. Of the available treatments, choose the one that maximizes expected utility, that is, the probability-weighted average utility.

Presented in this way, the first advice is ad hoc, and will not convince you. What are such subjective probabilities, and how are you to choose them? Similar questions apply to the utility numbers. And, if such numbers can be chosen, why should you take products of probabilities and utilities, and then sum these products? Why not use other mathematical operations? The main problem with the first advice is that its concepts of probabilities and utilities do not have a clear meaning. They are theoretical constructs, which means that they have no meaning in isolation, but can only get meaning within a model, in relation to other concepts.

The decision theorist did not succeed in convincing you, and she now turns to a second advice, seemingly very different. She explains the meaning of transitivity

and completeness of preferences to you, and you declare that you want to satisfy these conditions. She next explains the sure-thing principle to you, meaning that a choice between two treatments should depend only on their results under those diseases where the treatments differ, and not on the results for diseases for which the two treatments give the same results. Let us assume that you want to satisfy this condition as well. Next the decision theorist succeeds in convincing you of the appropriateness of the other preference conditions of Savage (1954). Satisfying these conditions is the decision analyst's second advice.

The second advice is of a different nature than the first. All of its conditions have been stated directly in terms of choice making. Even if you would not agree with the appropriateness of all conditions, at least you can relate to them, and know what they mean. They do not concern strange undefined theoretical concepts. Still, and this was Savage's (1954) surprising result, the two advices turn out to be identical. One holds if and only if the other holds, given a number of technical assumptions that we ignore here. Whereas the second advice seemed to be entirely different from the first, it turns out to be the same. The second advice translates the first advice, which was stated in a theoretical language, into the meaningful language of empirical primitives, that is, preferences. Such translations are called axiomatizations. They reformulate, directly in terms of the observable primitives such as choices, what it means to assume that some theoretical model holds.

A decision model is normatively appropriate if and only if its characterizing axioms are, and is descriptively valid if and only if the characterizing axioms are. Axiomatizations can be used to justify a model, but also to criticize it. Expected utility can be criticized by criticizing, for instance, the sure-thing principle. This is what Allais (1953) did. If a model is to be falsified empirically, then axioms can be of help because they are stated in terms of directly testable empirical primitives.

In applications, we usually do not believe models to hold true perfectly well, and use them as approximations or as metaphors, to clarify some aspects of reality that are relevant to us. We mostly do not actually measure the concepts used in models. For instance, most economic models assume that consumers maximize utility, but we rarely measure consumers' utility functions. The assumption of utility maximization is justified by the belief that for the topics considered, completeness and transitivity of preference are reasonable assumptions. These preference axioms, jointly with continuity, axiomatize the maximization of utility and clarify the validity and limitations thereof.

Axiomatizations are crucial at an early stage of the development of new models or concepts, namely at the stage where setups and intuitions are qualitative but quantifications seem to be desirable. Not only do axiomatizations show how to verify or falsify, and how to justify or criticize given models, but they also demonstrate what are the essential parameters and concepts to be measured or determined. Without axiomatizations of expected utility, Choquet expected utility (CEU), and multiple priors, it would not be clear whether at all their concepts such as utility etc. are sensible concepts, and are at all the parameters to be assessed.

A historical example may illustrate the importance of axiomatizations. For a long time, models were popular that deviated from expected utility by transforming probabilities of separate outcomes, such as those examined by Edwards (1955) and Kahneman and Tversky (1979). These models were never axiomatized, which could have served as a warning signal that something was wrong. Indeed, in 1978, Fishburn discovered that no sensible axiomatization of such models will ever be found because these models violate basic axioms such as continuity and, even more seriously, stochastic dominance. When Quiggin (1982) and Schmeidler (1989, first version 1982) introduced alternative models of nonlinear probabilities, they took good care of providing axiomatic foundations. This made clear what the empirical meaning of their models is, that these models do not contain intrinsic inconsistencies, and that their concepts of utilities and nonlinear probabilities are sensible. Quiggin (1982) and Schmeidler (1989) independently developed the idea of rank-dependence and, thus, were the first to present sound models that allow for a new component in individual decision theory: a subjective decision attitude toward incomplete information (i.e. risk and uncertainty). This new component is essential for the study of decision under incomplete information, and sound models for handling it had been dearly missing in the literature up to that point. I consider this development the main step forward for decision under incomplete information of the last decades. Quiggin developed his idea for decision under risk, Schmeidler for the more important and more subtle domain of decision under uncertainty, which is the topic of this book.

Axioms can be divided into three different classes. First there are the basic rationality axioms such as transitivity, completeness, and monotonicity, which are satisfied by most models studied today. For descriptive purposes, it has become understood during the last decades that these very basic axioms are the main cause of most deviations from theoretical models. For normative applications, these axioms are relatively uncontroversial, although there is no unanimous agreement on any axiom.

The second class of axioms consists of technical axioms, mostly continuity, that impose a richness on the structures considered. For decision under uncertainty, these axioms impose a richness on the state space or on the outcome space. They are usually necessary for obtaining mathematical proofs, and will be further discussed later in this chapter.

The third and final class of axioms consists of the "intuitive" axioms that are most characteristic of the models they characterize. They vary from model to model. For expected utility, the sure-thing principle (which amounts to the independence axiom for given probabilities) is the most characteristic axiom. Most axiomatizations of nonexpected utility models have relaxed this axiom. Many examples will be discussed in the following sections, and in other chapters in this book.

I end this introduction with a citation from Gilboa and Schmeidler (2001), who concisely listed the purposes of axiomatizations as follows:

Meta-theoretical:	Define theoretical terms by observables (and enable their elicitation).
Descriptive:	Define terms of refutability.
Normative:	Do the right thing.

2.2. General conditions for decision under uncertainty

S denotes a *state space*, with elements called *states* (of nature). Exactly one state is true, the others are not true. The decision maker does not know which state is the true one, and has no influence on the truth of the states (no moral hazard). For example, assume that a horse race will take place. Exactly one horse will win the race. Every $s \in S$ refers to one of the horses participating, and designates the "state of nature" that this horse will win the race. Alternative terms for state of nature are state of the world or proposition. An *event* is a subset of S, and is *true* or *obtains* if it contains the true state of nature. For example, the event "A Spanish horse will win" is the set $\{s \in S: s \text{ is Spanish}\}$.

$\mathcal{C}$ denotes the *outcome space*, and $\mathcal{F}$ the set of *acts*. Formally, acts are functions from S to $\mathcal{C}$, and $\mathcal{F}$ contains all such functions. A decision maker should choose between different acts. An act will yield the outcome $f(s)$ for the decision maker where s is the true state of nature. Because the decision maker is uncertain about which state is true, she is uncertain about what outcome will result from an act, and has to make decisions under uncertainty. An alternative term for an act is state-contingent payoffs, and acts can refer to financial assets. Acts can be considered random variables with the randomness not expressed through probabilities but through states of nature. David Schmeidler is known for his concise ways of formulating things. In the abstract of Schmeidler (1989), he used only seven words to describe the above model: "Acts map states of nature to outcomes."

By $\succcurlyeq$, a binary relation on $\mathcal{F}$, we denote the *preference relation* of the decision maker over acts. In decision under uncertainty, we study properties of the quadruple S, $\mathcal{C}$, $\mathcal{F}$, $\succcurlyeq$. A function V *represents* $\succcurlyeq$ if $V: \mathcal{F} \to \mathbb{R}$ and $f \succcurlyeq g$ if and only if $V(f) \geq V(g)$. If a representing function exists, then $\succcurlyeq$ must be a *weak order*, that is, it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all acts f, g) and transitive. Completeness implies *reflexivity*, that is, $f \succcurlyeq f$ for all acts f. We write $f \succ g$ if $f \succcurlyeq g$ and not $g \succcurlyeq f$, $f \sim g$ if $f \succcurlyeq g$ and $g \succcurlyeq f$, $f \prec g$ if $g \succ f$, and $f \preccurlyeq g$ if $g \succcurlyeq f$. For a weak order $\succcurlyeq$, $\sim$ is an *equivalence relation*, that is, it is *symmetric* ($f \sim g$ if $g \sim f$), transitive, and reflexive. Outcomes are often identified with the corresponding constant acts. In this way, $\succcurlyeq$ on $\mathcal{F}$ generates a binary relation on $\mathcal{C}$, denoted by the same symbol $\succcurlyeq$ and identified with the restriction of $\succcurlyeq$ to the constant acts.

Decision under risk refers to the special case of decision under uncertainty where an objective probability measure Q on S is given, and $f \sim g$ whenever f and g generate the same probability distribution over $\mathcal{C}$. Then the only information relevant for the preference value of an act is the probability distribution that the act generates over the outcomes. Therefore, acts are usually identified with the probability distributions generated over the outcomes, and S is suppressed from the model. It is useful to keep in mind, though, that probabilities must be generated by some random process, and that some randomizing state space S is underlying, even if not an explicit part of the model. It is commonly assumed in decision under risk that S is rich enough to generate all probabilities, and all probability distributions. My experience in decision under risk and uncertainty has been that formulations of concepts for the general context of uncertainty are more clarifying and intuitive than formulations only restricted to the special case of risk.

This chapter will focus on axiomatizations for decision under uncertainty, the central topic of this book, and will not discuss axiomatizations for decision under risk. Often, axiomatizations for decision under risk readily follow simply by restricting the axioms of uncertainty to the special case of risk. For example, Yaari's (1987) axiomatization of rank-dependent utility for risk can be obtained as a mathematical corollary of Schmeidler (1989); I will not elaborate on this point.

We will also restrict attention to static models, and will not consider dynamic decision making or multistage models such as examined by Luce (2000) unless serving to interpret static models. Other restrictions are that we only consider individual decisions, and do not examine decompositions of multiattribute outcomes. We will neither discuss topological or measure-theoretic details, and primarily refer to works introducing results and not to follow-up works and generalizations.

The most well-known representation for decision under uncertainty is *subjective expected utility* (SEU). SEU holds if there exists a probability measure P on S, and a *utility function* $U: \mathcal{C} \to \mathbb{R}$, such that $f \mapsto \int_S U(f(s))\,\mathrm{d}P(s)$, the SEU of f, represents preferences. For infinite state spaces S, measure-theoretical conditions can be imposed to ensure that the expectation is well defined for all acts considered. For the special case of decision under risk, P has to agree with the objective probability measure on S under mild richness assumptions regarding S, contrary to what has often been thought in the psychological literature. In general, P need not be based on objective statistical information, and may be based on subjective judgments of the decision situation in the same way as U is. P is, therefore, often called a *subjective probability measure*.

SEU implies *monotonicity*, that is, $f \succcurlyeq g$ whenever $f(s) \succcurlyeq g(s)$ for all s, where furthermore $f \succ g$ if $f(s) = \alpha \succ \beta = g(s)$ for outcomes α, β and all s in an event E that is "nonnull" in some sense. E being nonnull means that the outcomes of E can affect the preference value of an act, in a way that depends on the theory considered and that will not be formalized here.

The most important implication of SEU is the *sure-thing principle*, discussed informally in the introduction. It means that a preference between two acts is not affected if, for an event for which the two acts yield the same outcome, that common outcome is changed into another common outcome. The condition holds true under SEU, because an event with a common outcome contributes the same term to the expected-utility integral of both acts, which will cancel from the comparison irrespective of what that common outcome is. Savage (1954) introduced this condition as his P2. He did not use the term sure-thing principle for this condition alone, but for a broader idea. The term is, however, used exclusively for Savage's P2 nowadays. In a mathematical sense, the sure-thing principle can be equated with separability from consumer demand theory, although Savage developed his idea independently. The condition can be derived from principles for dynamic decisions (Burks, 1977: chapter 5; Hammond, 1988), a topic that falls outside the scope of this chapter.

The sure-thing principle is too weak to imply SEU. For instance, for a fixed partition $(A_1, \ldots, A_n)$ of S, and acts $(A_1: x_1; \ldots; A_n: x_n)$ yielding x_j for each $s \in A_j$, the sure-thing principle amounts to an additively decomposable representation $V_1(x_1) + \cdots V_n(x_n)$, under some technical assumptions discussed later. This representation is strictly more general than the SEU representation $P(A_1)U(x_1) + \cdots + P(A_n)U(x_n)$, for instance if $V_2 = \exp(V_1)$. It can be interpreted as state-dependent expected utility (Karni, 1985). Therefore, additional conditions are required to imply the SEU model. The particular reinforcements of the sure-thing principle depend on the particular model chosen, and are discussed in the next section.

2.3. Conditions to characterize subjective expected utility

The most desirable characterization of SEU, or any model, would concern an arbitrary set of preferences over acts, not necessarily a complete set of preferences over a set $\mathcal{F}$, and would give necessary and sufficient conditions for the preferences considered to be representable by SEU. Most important would be the case of a finite set of preferences, to truly capture the empirical and normative meaning of models such as SEU. Unfortunately, such general results are very difficult to obtain. For SEU, necessary and sufficient conditions for finite models were given by Shapiro (1979). These conditions are, however, extremely complex, and amount to general solvability requirements of inequalities for mathematical models called rings. They do not clarify the intuitive meaning of the model. Therefore, people have usually resorted to continuity conditions so as to simplify the axiomatizations of models. These continuity conditions imply richness of either the state space or the outcome space. Difficulties in using such technical richness conditions are discussed by Krantz *et al.* (1971: section 9.1) and Pfanzagl (1968: section 9.5). The following discussion is illustrated in Table 2.1.

The most prominent model with richness of the state space is Savage (1954). Savage added an axiom P4 to the sure-thing principle, requiring that a preference for betting on one event rather than another is independent of the stakes of the bets. The richness of the state space was ensured by an axiom P6 requiring arbitrarily fine partitions of the state space to exist, so that the state space must be atomless.

Decision under risk can be considered a special case of decision under uncertainty where the state space is rich, because it is commonly assumed that all probabilities can be generated by random events. Other than that, there have not been many derivations of SEU with a rich state space. Most axiomatizations have imposed richness structure on the outcome space, to which we turn in the rest of this section.

We start with approaches that assume convex subsets of linear spaces as outcome space, with linear utility. In these approaches, outcomes are either monetary, with $\mathcal{C} \subset \mathbb{R}$ an interval, or they are probability distributions over a set of prizes. The sure-thing principle is reinforced into linearity with respect to addition ($f \succcurlyeq g \Rightarrow f + c \succcurlyeq g + c$ for acts f, g, c, where addition is statewise), or mixing ($f \succcurlyeq g \Rightarrow \lambda f + (1 - \lambda)c \succcurlyeq \lambda g + (1 - \lambda)c$ for acts f, g, c,

Table 2.1 Axiomatizations and their structural assumptions

	SEU	*CEU*	*PT*	*Multiple priors*
Continuous state space	Savage (1954)	Gilboa (1987)*		
U linear in money	de Finetti (1931, 1937)	Chateauneuf (1991)		Chateauneuf (1991)
U linear in probability mixing, 2-stage	Anscombe and Aumann (1963)	Schmeidler (1989)		Gilboa and Schmeidler (1989)
Canonical probabilities	Raiffa (1968), Sarin and Wakker (1997)	Sarin and Wakker (1992)	Sarin and Wakker (1994)	
Continuous U, tradeoff consistency	Wakker (1984)	Wakker (1989)	Tversky and Kahneman (1992)	
Continuous U, multisymmetry	Nakamura (1990)	Nakamura (1990)	×	
Continuous U, act-independence	Gul (1992)	Chew and Karni (1994), Ghirardato *et al.* (2003)	×	Ghirardato *et al.* (2003); Casadesus-Masanell *et al.* (2000)

Notes

× Such an extension is not possible, because the required certainty equivalents are not contained in most of the sign-comonotonic sets.

* Required more modifications than only comonotonic restrictions.

where mixing is statewise, and under continuity can be restricted to $\lambda = \frac{1}{2}$). Both of these approaches characterize SEU with a linear utility function. The additive approach was followed by de Finetti (1931, 1937) and Blackwell and Girshick (1954: theorem 4.3.1 and problem 4.3.1). For the mixture approach, Anscombe and Aumann (1963) provided the most appealing result. For earlier results on mixture spaces, see Arrow (1951: 431–432). In addition to the axioms mentioned, these works used weak ordering, monotonicity (this, together with additivity, is what de Finetti's book-making amounts to), and some continuity (existence of "fair prizes" for de Finetti, continuous mixing for the mixture approaches). In the mixture approaches, the linear utility function is interpreted as an expected utility functional for the probability distributions over prizes, and acts are two-stage: In the first stage, the uncertainty about the true state of nature is resolved yielding a probability distribution over prizes, in the second stage the probability distribution is resolved, finally leading to a prize. This approach assumes that the two stages are processed through backwards induction ("folding back"). The second-stage probabilities could also be modeled through a rich product state space, but for this survey the categorization as rich outcomes is more convenient.

An alternative to Anscombe and Aumann's (1963) approach was customary in the early decision-analysis literature of the 1960s (Raiffa, 1968: chapter 5). As in Anscombe and Aumann (1963), a rich set of events with objectively given probabilities was assumed present, with preferences over acts on these events governed by expected utility. However, these events were not part of a second stage to be resolved after the events of interest, but they were simply a subset of the collection of events considered in the first, and only, stage. Formally, this approach belongs to the category that requires a rich state space. To evaluate an arbitrary act $(A_1: x_1; \ldots; A_n: x_n)$, where no objective probabilities are given for the events A_j, a *canonical representation* $(E_1: x_1; \ldots; E_n: x_n)$ is constructed. Here each event E_j does have an objective probability and is equally likely as event A_j in the sense that one would just as well bet \$1 on E_j as on A_j. It is assumed that such canonical representations can be constructed and are preferentially equivalent. In this manner, SEU is obtained over all acts. Sarin and Wakker (1997) formalized this approach. Ramsey (1931) can be interpreted as a variation of this canonical approach, with his "ethically neutral" event an event with probability half, utility derived from gambles on this event, and the extension of SEU to all acts and events not formalized.

Returning to the approach with rich outcome sets, more general axiomatizations have been derived for continuous instead of linear utility. Then $\mathcal{C}$ can, more generally, be a connected topological space. For simplicity, we continue to assume that $\mathcal{C}$ is a convex subset of a linear space. Pfanzagl (1959) gave an axiomatization of SEU when restricted to two-outcome acts. He added a bisymmetry axiom to the sure-thing principle. Denote by $CE(f)$ a certainty equivalent of act f, that is, an outcome (identified with a constant act) equivalent to f. For events A, M with complements A^c, M^c, bisymmetry requires that

$$(A\colon CE(M\colon x_1; M^c\colon y_1); A^c\colon CE(M\colon x_2; M^c\colon y_2)) \\ \sim (M\colon CE(A\colon x_1; A^c\colon x_2); M^c\colon CE(A\colon y_1; A^c\colon y_2)).$$

For arbitrary finite state spaces S, Grodal (1978) axiomatized SEU with continuous utility using a mean-groupoid operation (a generalized mixture operation derived from preference) developed by Vind. These works were finally published in Vind (2003). Wakker (1984, 1993) characterized SEU for continuous utility using a tradeoff consistency technique based on conjoint measurement theory of Krantz *et al.* (1971) and suggested by Pfanzagl (1968: end of remark 9.4.5). The basic axiom requires that

$$(\mathbf{A}_1\colon \alpha; A_2\colon x_2; \ldots; A_n\colon x_n) \preccurlyeq (\mathbf{A}_1\colon \beta; A_2\colon y_2; \ldots; A_n\colon y_n),$$
$$(\mathbf{A}_1\colon \gamma; A_2\colon x_2; \ldots; A_n\colon x_n) \succcurlyeq (\mathbf{A}_1\colon \delta; A_2\colon y_2; \ldots; A_n\colon y_n),$$

and

$$(A_1\colon v_1; \ldots; A_{n-1}\colon v_{n-1}; \mathbf{A}_n\colon \alpha) \succcurlyeq (A_1\colon v_1; \ldots; A_{n-1}\colon v_{n-1}; \mathbf{A}_n\colon \beta)$$

imply

$$(A_1: v_1; \ldots; A_{n-1}: v_{n-1}; \mathbf{A}_n: \gamma) \succcurlyeq (A_1: v_1; \ldots; A_{n-1}: v_{n-1}; \mathbf{A}_n: \delta),$$

where $(A_1, \ldots, A_n)$ can be any partition of S.

By renumbering, similar conditions follow for outcomes $\alpha, \beta, \gamma, \delta$ conditional on all pairs of events A_i, A_j.

Nakamura (1990) used multi-symmetry, a generalization of Pfanzagl's (1959, 1968) bisymmetry to general acts, to characterize SEU with continuous utility for finite state spaces. Similar conditions had appeared before in decision under risk (Quiggin, 1982; Chew, 1989). Chew called the condition event commutativity. Consider a partition $(A_1, \ldots, A_n)$ and a "mixing" event M with complementary event M^c. *Multisymmetry* requires that

$$(A_1: CE(M: x_1; M^c: y_1); \ldots; A_n: CE(M: x_n; M^c: y_n))$$
$$\sim (M: CE(A_1: x_1; \ldots; A_n: x_n); M^c: CE(A_1: y_1; \ldots; A_n: y_n)).$$

Multisymmetry implies that $(x_1, \ldots, x_n)$ is separable in $(A_1: CE(M: x_1; M^c: c_1); \ldots; A_n: CE(M: x_n; M^c: c_n))$. This implication is called *act-independence*, and was introduced by Gul (1992). Formally, the condition requires that

$$(A_1: x_1; \ldots; A_n: x_n) \succcurlyeq (A_1: y_1; \ldots; A_n: y_n)$$

implies

$$(A_1: CE(M: x_1; M^c: c_1); \ldots; A_n: CE(M: x_n; M^c: c_n))$$
$$\succcurlyeq (A_1: CE(M: y_1; M^c: c_1); \ldots; A_n: CE(M: y_n; M^c: c_n)).$$

Gul showed that this condition suffices to characterize SEU with continuous utility for finite state spaces, under the usual other assumptions. Gul used an additional symmetry requirement that was shown to be redundant by Chew and Karni (1994).

Using bisymmetry axioms for two-outcome acts, Ghirardato *et al.* (2003a) defined a mixture operation that can be interpreted as an endogeneous analog of the mixture operation used in Anscombe and Aumann (1963). They used it also to derive nonexpected utility models discussed in the next section.

Characterizations of properties of utility such as concavity have mostly been studied for decision under risk, and less so for decision under uncertainty. Also for uncertainty, utility is concave if and only if the subjective expected value of an act is always preferred to the act (Wakker, 1989; proposition VII.6.3.ii). This result is more difficult to prove than for decision under risk because not all probabilities need to be available, and is less useful because the subjective expected value is not directly observable, in the same way as subjective probabilities are not. More interesting for uncertainty is that utility is concave if and only if preferences are convex with respect to the mixing of outcomes, that is, if $f \succcurlyeq g$ then $\frac{1}{2}f + \frac{1}{2}g \succcurlyeq g$ where outcomes are mixed statewise (Wakker, 1989: proposition VII.6.3.iv). This condition has the advantage that it is directly observable.

2.4. Nonexpected utility models

This section considers models deviating from SEU.

Abandoning basic axioms. Models abandoning completeness (Bewley, 1986; Dubra *et al.*, 2004), transitivity (Fishburn, 1982; Loomes and Sugden, 1982; Vind, 2003), or continuity (Fishburn and LaValle, 1993) will not be discussed. We will only discuss models that weaken the sure-thing principle. In this class, we will not discuss betweenness models (Chew, 1983; Dekel, 1986; Epstein, 1992). These models have been examined almost exclusively for risk, with statements for uncertainty only in Hazen (1987) and Sarin and Wakker (1998), and have nowadays lost popularity. We will neither discuss quadratic utility (Chew *et al.*, 1991), which has been stated only for decision under risk.

Choquet expected utility. The first nonexpected utility model that we discuss is rank-dependent utility, or Choquet expected utility (CEU) as it is often called when considered for uncertainty. We assume a utility function as under SEU, but instead of a subjective probability P on S we assume, more generally, a *capacity* W on S. W is defined on the collection of subsets of S with $W(\emptyset) = 0$, $W(S) = 1$, and $C \supset D \Rightarrow W(C) \geq W(D)$. $\succcurlyeq$ is represented by $f \mapsto \int_S U(f(s))\,dW(s)$, the CEU of f, defined next. Assume that $f = (E_1: x_1; \ldots; E_n: x_n)$. The integral is $\sum_{j=1}^{n} \pi_j U(x_j)$ where the π_js are defined as follows. Take a permutation ρ on $\{1, \ldots, n\}$ such that $x_{\rho(1)} \geq \cdots \geq x_{\rho(n)}$. $\pi_{\rho(j)}$ is $W(E_{\rho(1)} \cup \cdots \cup E_{\rho(j)}) - W(E_{\rho(1)} \cup \cdots \cup E_{\rho(j-1)})$; in particular, $\pi_{\rho(1)} = W(E_{\rho(1)})$.

An important concept in CEU, introduced by Schmeidler (1989), is comonotonicity. Two acts f and g are *comonotonic* if $f(s) > f(t)$ and $g(s) < g(t)$ for no states s, t. A set of acts is *comonotonic* if every pair of its elements is comonotonic. Comonotonicity is an important concept because, as can be proved, within any comonotonic subset of $\mathcal{F}$ the CEU functional is an SEU functional (with numbers such as the above $\pi_{\rho(j)}$ playing the role of probabilities). It is, therefore, obvious that a necessary requirement for CEU is that all conditions of SEU hold within comonotonic subsets. Such restrictions are indicated by the prefix comonotonic, leading to the comonotonic sure-thing principle, etc. It is more complex to demonstrate that these comonotonic restrictions are also sufficient to imply CEU, but this can be proved in many circumstances. The third column of Table 2.1 gives the axiomatizations of CEU.

Prospect theory. Original prospect theory, introduced by Kahneman and Tversky (1979), assumed nonlinear probability weighting but had theoretical problems, and was defined only for risk, not for uncertainty. Only when Schmeidler (1989) introduced a sound model for nonlinear probabilities, could a model of prospect theory be developed that is theoretically sound and that also deals with uncertainty (Tversky and Kahneman, 1992). We define it next.

Under prospect theory, one outcome, called the *reference outcome*, plays a special role. Outcomes preferred to the reference outcome are gains, outcomes preferred less than the reference outcome are losses. The main deviation from other theories is that in different decision situations the decision maker may choose different reference points, and remodel her decisions accordingly. Although there is much empirical evidence for such procedures, formal theories to describe them have not yet been developed. We will therefore restrict attention, in this theoretical chapter, to one fixed reference point. For results on varying reference points, see Schmidt (2003). With a fixed reference point, prospect theory generalizes CEU and SEU in that it allows for a different capacity, W^-, for losses than for gains, where the gain capacity is denoted as W^+. Under prospect theory we define, for an act f, f^+ by replacing all losses of f by the reference outcome, and f^- by replacing all gains of f by the reference outcome. Our notation f^- deviates from mathematical conventions that, for real-valued functions f, take f^- as a positive function, being our function f^- multiplied by -1. For general outcomes, however, such a multiplication cannot be defined, which explains our definition. The *prospect theory* (PT) of an act f is

$$\mathrm{PT}(f) = \mathrm{CEU}(f^+) + \mathrm{CEU}(f^-)$$

where $\mathrm{CEU}(f^+)$ is with respect to W^+ and $\mathrm{CEU}(f^-)$ is with respect to the dual of W^-, assigning $1 - W^-(A^c)$ to each event A (A^c denotes complement). Two acts f, g are *sign-comonotonic* if they are comonotonic and, further, there is no state s such that of $f(s)$, $g(s)$ one is a gain and the other a loss. A set of acts is *sign-comonotonic* if any pair of its elements is sign-comonotonic. Sign-comonotonicity plays the same role for PT as comonotonicity for CEU. Within any sign-comonotonic set, PT agrees with SEU and, therefore, all conditions of SEU are satisfied within sign-comonotonic sets. A more difficult result, that can be proved in several situations, is that PT holds as soon as the sign-comonotonic conditions of SEU hold, that is, the restrictions of these conditions to sign-comonotonic subsets of acts. Axiomatizations of PT are given in the fourth column of Table 2.1.

Properties of utility and capacities under CEU and PT. Specific properties of utilities and capacities have been characterized for CEU, and for PT alike. Schmeidler (1989) demonstrated, in his CEU model with linear utility, that the capacity is *convex* ($W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$) if and only if preferences are convex. Chateauneuf and Tallon (2002) generalized this result by showing that, under differentiability assumptions, preferences are convex if and only if both utility is concave and the capacity W is convex. Wakker (2001) gave necessary and sufficient conditions for convexity of the capacity, without restricting the form of utility other than being continuous. Tversky and Wakker (1995) characterized a number of other conditions of capacities, such as bounded subadditivity, that are often found in experimental tests of prospect theory.

Multiple priors. Another popular deviation from expected utility is the multiple priors model. As in SEU, it assumes a utility function U over outcomes. It deviates by not considering one fixed probability measure, but a set of probability measures. Say C is such a set of probability measures over S. Then an act f is evaluated by $\min_{P \in C} \mathrm{SEU}_P(f)$, where SEU_P is taken with respect to P. This defines the *multiple priors model*. It was first characterized by Gilboa and Schmeidler (1989) in an Anscombe–Aumann setup where outcomes designate probability distributions over prizes, evaluated through a linear utility function (an expected utility functional). In a comprehensive paper, Chateauneuf (1991) obtained the same characterization independently, also for linear utility, but with linearity relating to monetary outcomes. For two-outcome acts, the multiple priors model coincides with CEU utility, so that the common generalization of these two models that imposes the representation only on two-outcome acts can serve as a good starting point (Ghirardato and Marinacci, 2001).

The axiomatization of the multiple priors model requires convexity of preference, implying that a representing functional is quasi-concave. Mainly independence with respect to constant functions: $f \succcurlyeq g \Rightarrow \lambda f + (1-\lambda)c \succcurlyeq \lambda g + (1-\lambda)c$ for acts f, g, c, both in the Gilboa–Schmeidler approach and in Chateauneuf's approach, where c is required to be constant, ensures that the representing functional is even concave. A functional is concave if and only if it is the minimum of dominating linear functions, which, under appropriate monotonicity, must be expected utility functionals. Thus, the multiple priors model results.

The axiomatization of multiple priors for continuous instead of linear utility has been obtained by Casadesus-Masanell *et al.* (2000) who used both bisymmetry-like and tradeoff-consistency-like axioms, and Ghirardato *et al.* (2003a) who used bisymmetry-like axioms to define an endogeneous mixture operation. A less conservative extension of the multiple priors model is the α-Hurwicz criterion, where acts are evaluated by α times the minimal SEU plus $1-\alpha$ times the maximal SEU over C. It was axiomatized by Ghirardato *et al.* (2003b).

Probabilistic sophistication. We finally discuss probabilistic sophistication. The derivation of SEU can be divided into two steps. In the first step, uncertainty is quantified through probabilities and the only relevant aspect for the preference value of an act is the probability distribution that it generates over the outcomes. In the second step, the probability distribution over outcomes is evaluated through expected utility. Probabilistic sophistication refers to the first of these steps without imposing expected utility in the second step. A first characterization was given by Machina and Schmeidler (1992), with an appealing generalization in Epstein and LeBreton (1993). The main axiom is de Finetti's (1949) additivity: If you rather bet on A than on B, then you also rather bet on $A \cup D$ than on $B \cup D$ for any event D disjoint from A and B. Under appropriate richness of the event space, this axiom implies that there exists a probability measure P on the events such that you rather bet on A than on B if and only if $P(A) \geq P(B)$ (for a review, see Fishburn, 1986). Additional assumptions then guarantee that two different acts that generate the same probability distribution over outcomes are equivalent, which implies probabilistic sophistication.

2.5. Conclusion

For all models discussed, axiomatizations provided a crucial step in the beginning of their developments, when it was not entirely clear what the right subjective parameters and their quantitative rules of combination were. It is remarkable that prospect theory could be modeled in a sound way only after Schmeidler (1989) had developed the first axiomatization of decision under uncertainty with nonlinear decision weights.

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