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# The Utility of Gambling Reconsidered

ENRICO DIECIDUE
*INSEAD, Decision Sciences, CRDM, Fontainebleau, France*

ULRICH SCHMIDT
*Institut für Finanzmarkttheorie, Universität Hannover, Germany*

PETER P. WAKKER* P.P.Wakker@uva.nl
*Department of Economics, University of Amsterdam, The Netherlands*

## *Abstract*

The utility of gambling, which entails an intrinsic utility or disutility of risk, has been alluded to in the economics literature for over a century. This paper demonstrates that any utility of gambling almost unavoidably implies a violation of fundamental rationality properties, such as transitivity or stochastic dominance, for static choices between gambles. This result may explain why the utility of gambling, a phenomenon so widely discussed, has never been formalized in the economics literature. The model of this paper accommodates well-known deviations from expected utility, such as the Allais paradox and the coexistence of gambling and insurance, while minimally deviating from expected utility.

**Keywords:** certainty effect, utility of gambling, risk aversion, nonexpected utility

**JEL Classification:** D81, C60

The utility of gambling entails that people process risky choice options in a different manner than safe ones, because of an intrinsic utility for the presence or absence of risk. This phenomenon is intuitively convincing and easy to state, and has been alluded to throughout the history of risky choice. Strangely enough, theoretical studies of the phenomenon are virtually absent. This paper discusses the history of the phenomenon, provides a theoretical study motivated by new developments in decision theory, and explains the almost complete absence of theoretical models so far.

The outline of this paper is as follows. Section 1 describes history and motivation. Some basic definitions of the utility of gambling are discussed in Section 2. The most common way of describing the utility of gambling is too general for predictive purposes, which may have contributed to the lack of theoretical studies. Section 3 proposes a theoretical model and gives a preference foundation, building on, and combining, earlier results by Fishburn (1980) and Schmidt (1998). The preference foundation shows how to test the model, and how to measure its parameters. Section 4 demonstrates, in the classical von Neumann-Morgenstern set-up with no restrictions imposed on outcomes, that the formal

*To whom correspondence should be addressed.

model for the utility of gambling necessarily violates stochastic dominance. This result extends Schmidt's (1998) Proposition 1 to general outcomes and utility. We further extend this result to models with more deviations from expected utility than only the utility of gambling.

Section 5 demonstrates how the utility of gambling distorts traditional measurements of risk attitudes, and how such distortions can be avoided. Section 6 considers various phenomena that violate expected utility. These phenomena have been explained through various nonexpected utility models. The novelty of our explanation, compared with earlier ones, lies in its efficiency with only a minimal deviation from expected utility. Section 6 further illustrates violations of stochastic dominance. Section 7 describes the related literature in some detail. Section 8 concludes, and proofs are in the Appendix.

## 1. History and motivation

### *History of the utility of gambling*

Throughout the history of utility theory, authors have alluded to the utility of gambling (Golec and Tamarkin, 1998, p. 220; Harsanyi, 1978, p. 224; Schlee, 1992; Vickrey, 1945, p. 328; etc.) It underlies much of the commonly observed risk aversion (Royden, Suppes, and Walsh 1959), and has distorted empirical investigations of risk attitudes (Conlisk, 1993). The phenomenon was mentioned by von Neumann and Morgenstern (1944, pp. 28, 629, 632), and was considered the main cause of deviation from expected utility in the economics literature up to 1944. Conlisk (1993) gave a survey, citing extensive empirical evidence. A humorous normative criticism of the utility of gambling comes from Marshall (1890): " . . . and we are thrown back upon the induction that pleasures of gambling are in Bentham's phrase "impure"; since experience shows that they are likely to engender a restless, feverish character, unsuited for steady work as well as for the higher and more solid pleasures of life" (p. 843 in the 1920 ed.). Surprisingly, only three papers have provided preference theories for the utility of gambling, namely, Fishburn (1980), Luce and Marley (2000), and Schmidt (1998).

### *Virtual absence of theory because of violation of stochastic dominance*

Observation 7 and Remark 8 below demonstrate that, to model the utility of gambling in static decisions, a rationality requirement as basic as stochastic dominance has to be violated. Example 11 illustrates this phenomenon. Such violations have usually been considered normatively undesirable. For many years, systematic violations of stochastic dominance were also considered descriptively undesirable because they seem to be implausible. In addition, descriptive models should satisfy some minimal rationality requirements in order to be tractable and to permit theoretical derivations, and stochastic dominance was usually considered to be one such rationality requirement. This explains the virtual absence of theoretical models of the utility of gambling. Throughout this paper, we maintain the assumption of transitivity. In a follow-up to this paper, Bleichrodt and Schmidt (2002) demonstrated that the utility of gambling can be explained, alternatively, not by giving up stochastic dominance, but by giving up transitivity (see Section 5).

The demonstrated violation of stochastic dominance allows for a new speculation on a text in von Neumann and Morgenstern (1944): "... concepts like a 'specific utility of gambling' cannot be formulated free of contradiction on this level. ... But anybody who has seriously tried to axiomatize that elusive concept, will probably concur with it" (p. 28, including footnote 3). Possibly, when alluding to a contradiction, von Neumann and Morgenstern foresaw a violation of something as basic as stochastic dominance.[1] Tversky (1967) also suggested that an axiomatization of the utility of gambling is impossible (p. 198). The first empirical test of expected utility in the literature, Mosteller and Nogee (1951), emphasized the importance of establishing empirical predictions for the utility of gambling: "Indeed, the writers would prefer to defer discussion of this point until a way of testing arguments about it is provided" (p. 402).

*Recent interest in basic violations of rationality*

It has become increasingly well understood that models of decision making, to be descriptively accurate, have to incorporate basic violations of rationality, based primarily on findings from the psychological literature. The preference reversal effect (Lichtenstein and Slovic, 1971; Lindman, 1971) was a first signal of rationality violations at a very basic level. Framing effects (Tversky and Kahneman, 1981) provided another signal. Tversky and Kahneman (1986, Problem 7) developed a clever example where violations of stochastic dominance can be generated systematically for the majority of subjects. Starmer and Sugden (1993), Bateman et al. (1997), Birnbaum and Navarrete (1998), Starmer (1999), and others extended this example.

*Impact on the measurement of risk attitude*

Under expected utility, a special preference for riskless outcomes is defined as risk aversion and modeled through concave utility. Several generalizations have been proposed, such as the certainty effect (Kahneman and Tversky, 1979). For a review, see Starmer (2000). These "nonexpected utility" generalizations have in common that the special preference for riskless options is smooth. There is no categorical difference between risky and riskless options, but there is a gradual one. If gambles are risky but close to riskless (all outcomes have almost the same utility) then their evaluation is also close to the riskless evaluation.

The utility of gambling as modeled in this paper entails a more drastic transition from certainty to risk, which is abrupt and discontinuous. Such a transition is psychologically plausible. People do perceive a categorical difference between risky and riskless. As soon as a sure outcome is changed into a risky gamble, no matter how small the risk, new emotions are triggered, and people turn to a different evaluation procedure, the one for risky choices. Fiorillo, Tobler, and Schultz (2003, Endnote 29) wrote, in a study of brain activities during uncertain behavior of two monkeys: "In addition, any attempt to explain gambling behavior must address the fact that gambling is common at all probabilities (except $P = 0$ or 1, by definition) ... Thus, dopamine could contribute to the appeal of gambling in general."

*Risky versus riskless utility*

Different evaluations for risky and riskless options have been proposed in other contexts. For instance, Dyer and Sarin (1982), and many other papers, proposed that expected utility with a utility function $u$ be used for risky choices, but a different function $v$, called value function, be used for riskless choices. Riskless evaluations concern, for instance, intertemporal, welfare, or strength-of-preference judgments. This distinction between utility and value has been widely, though not universally, accepted.

Our model takes the difference between risky and riskless evaluations one step further than the value/utility model. For a preference between a riskless outcome $x$ and a risky gamble $P$, the value/utility model assumes that both $x$ and $P$ be evaluated through $u$ and its expectation. Our model proposes that, already in the choice between $x$ and $P$, the value of $x$ be $v(x)$, and not $u(x)$.

*Efficient explanation of the Allais paradox*

The utility of gambling accommodates the Allais paradox, while minimally deviating from expected utility. As long as no riskless options are considered, expected utility is completely satisfied. Only when riskless options are present does the model deviate from expected utility, allowing for special preferences for riskless options such as in the certainty effect.

*Efficient explanation of some other phenomena*

Most of the deviations from expected utility, known in the literature, are based on a special role for sure outcomes. An example is the coexistence of gambling and insurance. In all of these cases, our model provides an efficient explanation, alternative to other nonexpected utility models. Elaborations of examples are in Section 6. As a price to pay for its simplicity, our model cannot accommodate more complex deviations from expected utility such as subproportionality (Kahneman and Tversky, 1979), a deviation from expected utility that also pertains to choices without riskless options.

## 2. Basic ways to model the utility of gambling

In informal discussions, the utility of gambling is often alluded to in general terms that correspond to the following formalization. A preference $x \succeq P$ between a sure outcome $x$ and a risky gamble $P$ holds if and only if the following inequality holds:

$$W(x) \geq W(P) - C(x, P). \tag{1}$$

$W$ is a preference functional from some risk theory—it will be expected utility in this paper. $C$ does not refer to an intrinsic value of $x$ or $P$ but, instead, describes a holistic cost of the presence or absence of gambling. Costs of gambling may contribute to risk aversion such as exhibited in insurance, etc. Negative costs, and a positive utility of gambling, are found in public lotteries, horse race betting, gambling in casinos, risky investments, and entrepreneurial activities.

Without further restrictions, the gambling-utility model does not have any implication for preferences and even permits intransitivities. For instance, we can set $C(x, P) + W(x) - W(P)$ equal to 1 whenever $x$ is preferred to $P$, equal to 0 whenever $x$ is indifferent to $P$, and equal to $-1$ whenever $P$ is preferred to $x$. Thus, we can accommodate any arbitrary, even intransitive, preference relation. Tversky (1967) may have had this point in mind when he wrote in pessimistic terms about the utility of gambling: "In spite of its apparent appeal, this approach does not yield testable predictions" (p. 198). To generate empirically meaningful predictions, restrictions have to be imposed on $C$.

Two approaches have been considered to ensure transitivity and extend the model to other choices than between a risky and a riskless gamble. In the first approach, $C$ depends only on $P$; in the second approach, $C$ depends only on $x$. In the first approach, the cost can be written as $C(P)$, and transitivity is satisfied with $W(P) - C(P)$ the value of the gamble $P$, and $C$ the holistic cost of gambling. In the second approach, the cost can be written as $C(x)$, and transitivity is satisfied with $W(x) + C(x)$ the value of the riskless outcome $x$, and $C$ the benefit of certainty. At the end of the appendix, we will demonstrate that, in a mathematical sense, the second approach can always be rewritten as a special case of the first approach, with the cost function $C(P)$ linear in $P$.

As is common in studies of the utility of gambling, we assume in the main text that preferences between risky gambles agree with expected utility (Davidson, Suppes, and Siegel, 1967, p. 186). The only deviation from expected utility that is considered is due to the utility of gambling. In the first approach, where $C$ depends on $P$, the model is still too general if any such dependence is permitted. Irrespective of $W$, the model can accommodate almost any transitive relation by defining $C$ accordingly. Therefore, Fishburn (1980), the only decision-theoretic work on the first approach that we are aware of, added further restrictions. As will be shown in Section 7, these imply that his model becomes a special case of the second approach. Observation 7, which demonstrates the violation of stochastic dominance for the second approach, therefore also applies to Fishburn's model.

Our paper follows the second approach. Risky gambles $P$ will be evaluated by expected utility with respect to a von Neumann-Morgenstern utility function $u$. Riskless outcomes $x$ are evaluated by a riskless function $v(x)$ (equal to $W(x) + C(x)$ in the notation of Eq. (1)). A more accurate term for the second approach would be the utility of certainty. We will, however, adhere to the term utility of gambling because it is common in the literature.

Our model can have a normative interpretation if it models transaction costs. For instance, if a sure outcome $x$ could be collected right away but for any risky gamble a contract would have to be signed to settle conditional agreements, then $C$ could designate the cost of the contract. Another example arises if riskless outcomes can be consumed right away, but risky outcomes can be consumed only after the resolution of some uncertainty that affects the value of the outcomes.

## 3. Theory

This section provides the theoretical background for the claims made in this paper. In particular, a preference foundation is given, demonstrating how the model can be tested through observable choices. As is common in discussions of the utility of gambling, we confine our attention to decision under risk with given probabilities. A *gamble* $(p_1, x_1; \ldots; p_n, x_n)$ yields *outcome* $x_j$ with probability $p_j$, $j = 1, \ldots, n$. We only consider gambles that yield finitely many outcomes (for varying $n \in I\!N$). Probabilities are nonnegative and sum to 1. $\mathcal{C}$ denotes the set of all conceivable outcomes. We do not impose any condition on $\mathcal{C}$, and it may be any arbitrary set containing health states, commodity bundles, monetary rewards, etc. *Preferences* over gambles are denoted by $\succeq$, with $\succ$ (strict preference), $\sim$ (indifference), $\preceq$ (weak reversed preference), and $\prec$ (strict reversed preference) as usual.

Throughout, any outcome $x$ is identified with the corresponding riskless gamble $(1, x)$. Thus, the set of all riskless (safe) gambles is identified with the outcome set $\mathcal{C}$. The remaining gambles are called *risky*. These are all gambles $(p_1, x_1; \ldots; p_n, x_n)$ with $x_i \neq x_j$ for some $i, j$ with $p_i > 0$ and $p_j > 0$. An *evaluation* or *representation* $V$ is a function on the gambles that determines preference, which means that $P \succeq Q$ if and only if $V(P) \geq V(Q)$.

*Definition 1.* The *gambling-utility model* holds if there exists a *utility function* $u : \mathcal{C} \to I\!R$, a *cost function* $c : \mathcal{C} \to I\!R$, a *value function* $v = u - c$, and an evaluation $V$ that assigns to each risky gamble $(p_1, x_1; \ldots; p_n, x_n)$ its $u$ expectation $p_1 u(x_1) + \cdots + p_n u(x_n)$, and to each outcome $x$ its value $v(x)$.

At this point, we do not yet impose restrictions on the relations between $v$ and $u$. These functions may order outcomes differently. Relations between these functions will be considered in the next section. In this section, we study preference conditions for the above model. *Weak ordering* means that $\succeq$ is *complete* ($P \succeq Q$ or $Q \succeq P$ for all gambles $P, Q$) and transitive. For the independence condition, we consider mixtures of gambles. For gambles $P$ and $Q$ and $0 \leq \lambda \leq 1$, the *mixture* $\lambda P + (1 - \lambda)Q$ is the gamble assigning probability $\lambda P(x) + (1 - \lambda)Q(x)$ to each outcome $x$. $P$ and $Q$ can be risky or riskless. The main weakening of expected utility is that independence, defined next, is imposed only on the risky gambles, as in Fishburn (1980) and Schmidt (1998).

*Definition 2.* *Gambling independence* holds if

$$P \succ Q \text{ implies } \lambda P + (1 - \lambda)R \succ \lambda Q + (1 - \lambda)R$$

for all risky $P, Q, R$ and $0 < \lambda < 1$.

The axiom implies adherence to expected utility except when the certainty effect can play a role. To obtain real-valued evaluations, Archimedean axioms have to be imposed. These are usually complicated and of a technical nature. Hence, the following axiom can be skipped by readers not interested in mathematical details. In our model, the Archimedean axiom is more complicated than under expected utility because the riskless gambles are treated separately. Our axiom will generalize Fishburn's (1980) Archimedean axioms. In Schmidt (1998), the Archimedean axiom is implied by topological assumptions. Condition (i) in our Definition 3 will restrict the traditional Archimedean axiom of expected utility to risky $P$ and $R$ so as to mix only risky gambles. It covers the *regular* outcomes, i.e. outcomes that are not superior or inferior to all risky gambles. Additional conditions have to be formulated in the presence of nonregular outcomes, i.e. outcomes that are either more preferred than all risky gambles or less preferred than all risky gambles. These conditions are specified in (ii) and (iii).

*Definition 3*. The *Archimedean Axiom* holds if:

(i) For all gambles $Q$ and all risky gambles $P$, $R$, if $P \succ Q \succ R$, then $\lambda R+(1-\lambda)P \succ Q$ and $Q \succ \mu P + (1-\mu)R$ for some $0 < \lambda < 1$ and $0 < \mu < 1$.
(ii) If, for some outcome $x$, $x \succ S$ ($x \prec S$) for all risky gambles $S$, then $\mu$ ($\lambda$) in Statement (i) can be chosen independently of $P$ ($R$).[2]
(iii) There exists a countable subset $D$ of outcomes that is order-dense in the sense that, for every preference $x \succ y$, $x \succeq d \succeq y$ for some $d \in D$.

It follows from substitution that all conditions formulated above are necessary for the gambling-utility model. They are also sufficient, as the following theorem shows.

**Theorem 4.** *Let $\succeq$ be a preference relation on the set of all finite probability distributions over a set $\mathcal{C}$. Then preferences are evaluated by a gambling-utility model if and only if $\succeq$ satisfies* (i) *weak ordering*, (ii) *gambling independence*, *and* (iii) *the Archimedean axiom.*

The uniqueness in the theorem is standard, but its formulation is complex because of the different functions and their interactions. In short, $u$ is, as usual, unique up to unit and origin of measurement, and $v$ shares the same unit and origin except for outcomes that are more or less preferred than all risky gambles. For the latter, $v$ is ordinal.

*Observation 5*. The uniqueness results for Theorem 4 are as follows. The functions $u$ and $v$ can be replaced by other functions $u^*$ and $v^*$ if and only if:

(i) There exist a real $\tau$ and a positive $\sigma$ such that $u^* = \tau + \sigma u$ ($u$ is unique up to unit and origin), and $v^* = \tau + \sigma v$ for all outcomes indifferent to some gamble.
(ii) If outcomes exist that are preferred to all risky gambles then $v$ is ordinal there, and for these outcomes $v^*$ is a strictly increasing transform of $v$ that exceeds all values of $u^*$.
(iii) If outcomes exist that are less preferred than all risky gambles, then $v$ is ordinal there, and for these outcomes $v^*$ is a strictly increasing transform of $v$ that is below all values of $u^*$.

Further, $c$ is replaced accordingly, by $c^* = v^* - u^*$.

## 4. Stochastic dominance

The analysis in the preceding section did not impose any restriction on the relations between $u$ and $v$. This section considers the case where $u$ and $v$ order outcomes in the same manner. This condition is called *ordinal equivalence* and is formally defined as $u(x) \geq u(y)$ if and only if $v(x) \geq v(y)$. It was discussed by Ellsberg (1954, following Eq. (7.2)). It is well-known that ordinal equivalence holds if and only if $u(x) = f(v(x))$ for some strictly increasing function $f$. The condition is natural for monetary outcomes, with higher amounts preferred to lower amounts both under $v$ and under $u$, and was assumed by Fishburn (1980).

For general outcomes, e.g. multiattribute outcomes or commodity bundles, ordinal equivalence is not self-evident because the tradeoffs made between commodities may be different under risk than under certainty. For example, chronic health states are two-dimensional outcomes, with one dimension specifying a health state and the other the duration of that health state. Subjects may prefer (blind, 25 years) to (full health, 20 years) but may prefer the riskless gamble (1/2, (full health, 20 years); 1/2, (full health, 20 years)) to the more complex gamble (1/2, (full health, 20 years); 1/2, (blind, 25 years)). Such discrepancies have often been found when measuring quality of life through the "time-tradeoff method," a method that uses riskless preferences of the former kind, and the "standard-gamble method," which uses risky preference of the latter kind (Miyamoto and Eraker, 1988, pp. 17–18; Lenert et al., 1997).

Bleichrodt and Pinto (2002) observed a direct violation of ordinal equivalence. Participants preferred death to a severely impaired health state following stroke. However, if these outcomes resulted with probability .25 (.75 probability of full recovery), then the preferences reversed.

In the absence of ordinal equivalence, there is no natural way to impose, or even define, stochastic dominance. We now turn to a preference condition that does ensure ordinal equivalence and allows for a natural way to define stochastic dominance. *Gamble monotonicity* holds if the replacement of an outcome by a preferred outcome always leads to a preferred gamble, assuming that both gambles are risky. Formally, for all $0 < \lambda < 1$, outcomes $x$, $y$, and risky gambles $P$, $x \succeq y$ if and only if $\lambda x + (1 - \lambda)P \succeq \lambda y + (1 - \lambda)P$. This is similar to Fishburn's (1980) Axiom A2a. The restriction to risky gambles $P$ serves to avoid confounding with the utility of gambling.

*Observation 6.* Assume the gambling-utility model. Ordinal equivalence of $u$ and $v$ holds if and only if gamble monotonicity holds.

Under ordinal equivalence, the riskless ordering of outcomes is relevant to risky choice, and stochastic dominance becomes meaningful. In our model, it is trivially satisfied for risky gambles because there expected utility holds. For riskless gambles, the condition becomes nontrivial because of the utility of gambling. The formulation of the condition chosen here highlights the relations to and differences from ordinal equivalence.

*Stochastic dominance* holds if, for all outcomes $x$, $y$, $0 < \lambda < 1$, and gambles $P$, $x \succeq y$ implies $\lambda x + (1 - \lambda)P \succeq \lambda y + (1 - \lambda)P$. The condition considers the improvement of one outcome $y$ into a better outcome $x$. By repeated application, it can deal with any number of improvements of outcomes, and is equivalent to other formulations of monotonicity. It is also equivalent, on our domain of finite probability distributions, to traditional formulations of stochastic dominance in terms of the pointwise dominance of distribution functions. The main difference with gamble monotonicity is that all $P$ are permitted, including riskless $P$. For $P = x$ or $P = y$ interaction with the utility of gambling arises. It will turn out that the condition in fact excludes any utility of gambling and reduces the model to expected utility. This was demonstrated for monetary outcomes by Schmidt (1998, Proposition 1).

*Observation 7.* Under the gambling-utility model, stochastic dominance holds if and only if expected utility is satisfied (i.e. $v = u$ can be chosen).

Although nonexpected utility models for risky gambles are not the topic of this paper, we give the following remark, proved in the appendix, to demonstrate that expected utility is not essential for the above result.

*Remark 8.* Observation 7 can be extended to most nonexpected utility representations on risky gambles, including rank-dependent utility (Quiggin, 1982) and prospect theory (Tversky and Kahneman, 1992).

Given the discontinuous evaluation of riskless options and the denseness of the risky evaluations, the result of Observation 7 is not surprising in a mathematical sense. This makes it the more surprising that this violation had not been generally known before, given the extensive discussions of the utility of gambling throughout the history of risk theory. Assuming that stochastic dominance is normatively desirable, Observation 7 shows that the gambling-utility model is not normative. The interest of the model is descriptive and lies in its psychological plausibility.

## 5. Applications

This section assumes the gambling-utility model. We discuss the empirical measurement of the primitives, some restrictions, and applications to other contexts than decision under risk.

### *Eliciting u and v from preferences*

Any method for measuring traditional von Neumann-Morgenstern utility functions that avoids riskless gambles can be used to measure the function $u$ in our model. Such methods have been advocated precisely to avoid certainty effects (McCord and de Neufville, 1986; Davidson and Suppes, 1956, p. 266; Officer and Halter, 1968, pp. 259, 272). The gambling-utility model provides a formal support for such measurements.

We next turn to the measurement of $v$. If $x \sim P$ for a risky gamble $P$ with expected utility $\mu$, then $v(x) = \mu$. If there are outcomes $x$ that are strictly preferred to each gamble, then $u$ is bounded (see the proof of Theorem 4 in the appendix). On this set of strictly preferred outcomes $x$, the function $v$ is ordinal with the extra restriction that it should exceed all $u$

values. Similarly, for outcomes $x$ strictly less preferred than all risky gambles, $v$ is ordinal with the extra restriction that it should be below all $u$ values.

*Intransitivity instead of violation of stochastic dominance*

Imagine a choice between a sure outcome $x$ and a risky gamble $P$, such that each outcome of $P$ is strictly preferred to $x$, but the gambling-utility model assigns a higher value to $x$. As shown in Observation 7, such violations of stochastic dominance, or reversed violations with a preference for a dominated gamble, exist. In general, when will a subject, having to choose directly from the two aforementioned options, really forego a gamble that can only bring better outcomes? The answer depends on the context. Many violations of stochastic dominance concern framings of problems in which the stochastic dominance is opaque and subjects are not aware of it. Transaction costs can be another reason for the utility of gambling. Examples were given in Section 2. A reversed example, where the pleasure of gambling induces a preference for a gamble with inferior prizes, can be observed in casinos (Example 11).

More rational explanations of the gambling effect can be based on calculation costs. The subject may want to lay down his future plans and then forget about it, and simply not even think about possible future profits if they are uncertain and small. Whenever there are such concrete reasons underlying the cost function, it seems plausible that the sure outcome $x$ is chosen, and stochastic dominance, defined in a narrow sense, is violated.

The case is different if the utility of gambling is the result of irrationalities and simplistic decision heuristics. If there is a clear dominance, then it is plausible that subjects go by that dominance and not by the gambling-utility model. Only if there is no clear dominance is subjects' behavior described by the gambling-utility model. Subjects may prefer $x$ to some gamble $Q$ and $Q$ to $P$, but $P$ dominates $x$ and they prefer $P$ to $x$ in a direct choice. Such editing (Kahneman and Tversky, 1979) entails violations of transitivity, but constitutes an empirically plausible variation of the gambling-utility model. Bleichrodt and Schmidt (2002) analyzed this approach, and Loomes and Sugden (1982) gave normative arguments for violating transitivity.

## 6. Accommodating violations of expected utility

This section presents a number of known deviations from expected utility that motivated the development of many nonexpected utility models, and shows that the gambling-utility model can accommodate them. We also illustrate a violation of stochastic dominance.

*Example 9* (Allais paradox; certainty effect). Assume that $v(x) = \ln(x + 10{,}000) - \ln(10{,}000)$, for the riskless value function $v$. We have normalized the function to be 0 at $x = 0$. \$10,000 may be the money in the bank account of the agent. The agent's negative utility of gambling is proportional, with $u(x) = 0.95v(x)$, for $x \geq 0$. The gambling-utility model implies the following evaluations (where $M$ is the abbreviation for million), $V(0.89, \$0; 0.10, \$5M; 0.01, \$0) = 0.59 > 0.48 = V(0.89, \$0; 0.10, \$1M; 0.01, \$1M)$, and the former, riskier, gamble is preferred.

Consider replacing the common 0.89-probability outcome \$0 by \$1$M$ in both gambles, after which the second, safe, gamble becomes the sure gain \$1$M$. Under expected utility, common outcomes do not affect preference and, therefore, the preference between the gambles should not change under the replacement. Under the gambling-utility model, however, we obtain the following evaluations. $V(0.89, \$1M; 0.10, \$5M; 0.01, \$0) = 4.49 < 4.62 = V(0.89, \$1M; 0.10, \$1M; 0.01, \$1M) = V(\$1M)$. That is, the preference switches from risky to safe, because now the safe gamble is riskless. Such preference switches have been commonly found in experiments (Starmer, 2000), and have led to the development of nonexpected utility models.

*Example 10* (Gambling and insurance). Friedman and Savage (1948) proposed, under expected utility, a utility function that is concave for moderate wealth levels which explains insurance, and has a convex region for high levels of wealth which explains gambling, possibly followed by a concave region again. Markowitz (1952) demonstrated that such a utility function has many unrealistic implications. Rank-dependent utility and prospect theory provide more plausible explanations. They explain gambling and insurance by the overestimation of small probabilities, both the small probability of gaining a high prize and the small probability of incurring a loss. Our model provides an alternative solution. These phenomena can also be explained by a negative utility of gambling ($u(x) < v(x)$) for small and moderate wealth levels, but a positive utility of gambling ($u(x) > v(x)$) for high wealth levels. Both aforementioned explanations fit the psychology of people who, for a gamble with a main prize of \$1 million, enjoy the hope of gaining this large prize, and do not pay much attention to any other aspect of the gamble.

*Example 11* (Violating stochastic dominance by gambling on odd and even in roulette). In roulette, people bet on a number between 0 and 36 that is drawn randomly. An anomaly that can be observed in casinos is that people sometimes bet a stake $X$ on odd and, simultaneously, a stake $X$ on even. Such a bet returns the stake $2X$ for all numbers except 0. For zero, which is not included among the even numbers in roulette, there is no return and the stake $2X$ is lost. The corresponding gamble is $(1/37, -2X; 36/37, 0)$, which is stochastically dominated by not betting, i.e. $(1, 0)$. Such behavior can be accommodated by a riskless value function $v(x) = x$ and constant joy of gambling $u(x) = x + 1$. Then the simultaneous betting on odd and even has a value $1 - 2X/37$ which, for small $X$, exceeds zero. For small stakes, the simultaneous bet on odd and even is preferred to not betting, even though it is stochastically dominated by the latter, because of the pleasure of gambling.

## 7. Related literature

This section discusses related works on the utility of gambling in some detail. An extensive discussion of the history can be found in Pope (1995). We begin with the decision-theoretic models by Fishburn (1980), Schmidt (1998), and Luce and Marley (2000). The first approach described in Section 2, namely, Eq. (1) with $C$ depending only on $P$, was the starting point in Fishburn (1980). We first discuss his central representation, Theorem 3. Fishburn assumed that not only $W - C$ but also $W$ in isolation represents preferences over risky gambles. This implies that $C$ is a transform of $W$, i.e. the cost of gambling depends only on the preference value of the gamble and not on other characteristics. Further, the risky and riskless functionals order outcomes in the same manner, and $W$ is assumed to be an expected utility functional. These assumptions also underlie the representations (which he called "fragmented", p. 441) in Fishburn's Theorems 1 and 2 and in his other theorems. Under these assumptions, Fishburn's model can be rewritten as a special case of the second approach, with $C$ depending on the outcome $x$. It is a special case because both $W$ and $W - C$ represent preferences over risky gambles, implying that $v$ and $u$ order outcomes in the same manner.[3]

The second approach, with $C$ depending only on $x$, was suggested by Fishburn (1980) and Tversky (1967) and was formalized by Schmidt (1998), a work written independently of Fishburn (1980). Schmidt used the term certainty effect instead of our term utility of gambling. He assumed a separable metric outcome space and imposed continuity and boundedness conditions. Thus, his work does not provide a complete generalization of the von Neumann-Morgenstern expected utility model. Our axiomatization generalizes Schmidt's by permitting general outcomes, as in Fishburn (1980). In this manner, we obtain a genuine generalization of the von Neumann-Morgenstern model.

Luce and Marley (2000) considered decision under uncertainty instead of risk. Uncertainty was described through events for which no probabilities need to be given. Their model can be considered a special case of the first approach to Eq. (1), with the cost $C(x, P)$ depending on $P$ only through the uncertain events used to describe $P$ and not through the outcomes (with their kernel equivalent playing the role of our $W$). Their approach is based on Luce's (2000) paradigm for decision under uncertainty. It provides an interesting alternative to the common, Savagean decision paradigm.[4]

We next discuss some works that formulated versions of the utility of gambling but did not provide preference axioms. Tversky (1967) explicitly pointed out that discrepancies between risky and riskless utilities can be affected by the utility of gambling. He considered single nonzero outcomes and the logarithm of von Neumann-Morgenstern utility. Nonlinear probability weighting was permitted. Tversky did not elaborate formally on his interpretations and experimental measurements, and did not explicate their similarities to and differences from Savage's (1954) expected utility versus Edwards' (1962) risk model. Hence, his model is not discussed further.

Conlisk (1993) considered the first approach ($C$ depending on $P$), for two-outcome gambles with expectation zero. He assumed that $C$ is negative (gambling is valued positively) but remains small, so that it can affect gambles for small outcomes but not for large outcomes. This model could explain risk seeking for small-outcome gambles. Empirical evidence was presented. Conlisk derived plausible implications from assumptions such as concavity on $C$ and the other functions. Note that the Allais paradox cannot be explained by Conlisk's model because $C$ is negative (Conlisk, 1993, end of Section 5). His Section 5 also demonstrated that decreasing proportional risk aversion, often observed empirically, may be explained by a utility of gambling.

Many empirical investigations have suggested that violations of expected utility are generated primarily by boundary effects. These entail drastic changes in the evaluation process when the number of positive-probability outcomes is changed (Conlisk, 1989; Harless and

Camerer, 1994; Sopher and Gigliotti, 1993). Formal models describing such changes in evaluation have been proposed (Humphrey, 1998; Neilson, 1992). Gambles with $n$ positive-probability outcomes are evaluated through expected utility with respect to a utility function $u_n$ that depends on $n$. The often-discussed violation of stochastic dominance by original prospect theory (Kahneman and Tversky, 1979) results from a similar effect (Starmer, 1999). The model of this paper can be considered a special case of the above models, with $u_j$ the same for all $j \geq 2$. This case captures the simplest and strongest effect of the above models. Other models with changes in evaluation if the number of positive-probability outcomes changes are in Luce (2000) and Viscusi (1989).

Le Menestrel (2001) presented another general model, where the evaluation of a gamble can depend on the process generating the gamble. Le Menestrel showed that the utility of gambling is a special case of his general model, and derived some empirical implications.

For health outcomes, Richardson (1994) considered two-outcome gambles and assumed that $C$ consists of a constant utility of gambling plus a term depending on the sure outcome $x$. Other references from the health domain are Bombardier et al. (1982), Gafni and Torrance (1984), Loomes (1993), Stiggelbout et al. (1994). Gafni and his colleagues argued for a systematic difference between risky and riskless options (Gafni and Birch, 1997; Mehrez and Gafni, 1989). Unfortunately, their theoretical derivations turned out to be incorrect (Johannesson, Pliskin, and Weinstein, 1993). The intuition underlying their approach is valuable, and the present paper may help to formalize this intuition.

Several authors have casually alluded to the approach modeled in this paper, adhering to expected utility whenever no paradox is involved and deviating only for some particular paradoxes. For example, Burks (1977), in a remarkable justification of expected utility through dynamic decision principles that preceded Hammond (1988), favored satisfying the model except in choice situations like Allais' paradox. In such paradoxes, he preferred deviating from expected utility (pp. 307/308). His viewpoint is similar to the model characterized in this paper. Observation 7 has demonstrated that such approaches face their own normative problems.

## 8. Conclusion

Throughout the history of risky choice, researchers have been aware of the utility of gambling, which entails that people use a different method for evaluating riskless options than for evaluating risky options. Theoretical models for the phenomenon have been almost absent, probably because of the holistic nature of the utility of gambling and the implied violation of stochastic dominance. Only recently have economists developed an interest in basic violations of rationality conditions such as stochastic dominance, and now formal models can be developed for the utility of gambling.

The model presented in this paper provides a tractable theoretical basis for the utility of gambling, and describes the most efficient deviation from expected utility to explain the Allais paradox that is presently available. Tractable methods for the measurement of its primitives have been presented, in particular for measuring riskless utility at a cardinal level without using new empirical primitives. Investigations of risk attitudes in the literature may have been distorted by the utility of gambling, and corrections for this distortion will be useful.

## Appendix A: Proofs

**Proof of Theorem 4 and Observation 5:** We first assume the gambling-utility model and derive the preference conditions. By $\mathcal{R}$ we denote the set of risky gambles. Weak ordering is immediate, and gambling independence follows from the expected utility representation on $\mathcal{R}$. We next derive the Archimedean axiom. Part (i) follows from linearity of expected utility on $\mathcal{R}$ and by taking $\lambda$ and $\mu$ sufficiently small. Part (iii) follows because $\succeq$ has a real-valued representation, $v$, on $\mathcal{C}$ (Fishburn, 1970, Theorem 3.1). Finally, we turn to part (ii). If, for some outcome $x$, $x \succ S$ for all risky gambles $S$, then $v(x)$ provides an upper bound for $u$, implying that $\mu$ in (i) can be chosen independently from $P$. The case of an $x$ inferior to all risky gambles is treated similarly, now with $v(x)$ a lower bound for $u$. All preference conditions have been verified.

We next assume the preference conditions and derive the gambling-utility model. On $\mathcal{R}$ all preference axioms of expected utility are satisfied and, hence, an expected utility representation can be obtained there, with the utility function denoted $u$. This is proved by Fishburn (1980, p. 438). We briefly sketch the proof. $\mathcal{R}$ is a mixture set on which weak ordering, independence, and the appropriate Archimedean axiom (our Part (i) only for risky $Q$) imply a linear representation by Fishburn (1970, Theorem 8.4). Although no outcomes are contained in $\mathcal{R}$, it is still possible to define $u$ on the outcomes such that the linear functional is expected utility with respect to $u$. For example, $u(y) = \lambda u(x) + (1 - \lambda)u(z)$ can be measured through indifferences $\frac{1}{2}P + \frac{1}{2}y \sim \frac{1}{2}P + \frac{\lambda}{2}x + \frac{1-\lambda}{2}z$ for any risky gamble $P$. The results of Fishburn (1980, p. 438), as well as the indifference just written, imply that $u$ is unique up to unit and origin.

Outcomes $x$ such that $x \succ P$ for all $P \in \mathcal{R}$ are *superior* outcomes, and outcomes $x$ such that $x \prec P$ for all $P \in \mathcal{R}$ are *inferior* outcomes. We first extend the evaluation to outcomes (= riskless gambles) that are neither superior nor inferior. For these outcomes, there exist risky gambles $P$, $Q$ such that $P \succeq x \succeq Q$. It can be derived from (i) of the Archimedean axiom that there exists a risky gamble $R$ indifferent to $x$. This reasoning is similar to Fishburn (1970, C2 in Theorem 8.3). In short, if neither $P$ nor $Q$ can play the role of $R$, then $P \succ Q$, and from gambling independence it follows that $\lambda P + (1 - \lambda)Q \succ \lambda' P + (1 - \lambda')Q$ if and only if $\lambda \succ \lambda'$. Then $\{\lambda \in [0, 1] : \lambda P + (1 - \lambda)Q \succ x\}$ is either of the form $(\mu, 1]$ or $[\mu, 1]$, for some $\mu$. From (i) of the Archimedean axiom, it follows that this set is of the form $(\mu, 1]$ for some $\mu$. Similarly, $\{\lambda \in [0, 1] : \lambda P + (1 - \lambda)Q \prec x\}$ is of the form $[0, \nu)$ for some $\nu$. It then follows that $R = \mu P + (1 - \mu)Q$ can be taken (it can also be seen that $\mu = \nu$). We define $v(x)$ as the expected utility of $R$. This is the only definition of $v$ possible, given $u$, so that the uniqueness of $u$ up to unit and origin extends to $v$ as now defined. We have established the gambling-utility model, and its uniqueness, on the union of all risky gambles and all outcomes that are neither inferior nor superior.

We next consider superior and inferior outcomes. Let there exist a superior outcome $x$. We demonstrate that, by (ii) of the Archimedean axiom, $u$ must be bounded above. Assume that $u$ is not constant. Then we can take two risky gambles $Q$ and $R$ with $Q \succ R$. Assume that there is a risky $P$ with $P \succ Q$; if such a $P$ does not exist then $u$ is bounded above anyhow. Take $\mu$ as in (ii) of the Archimedean axiom. Writing $EU$ for expected utility, it follows that $\mu EU(P) < EU(Q) - (1 - \mu)EU(R)$, which provides an upper bound to

$EU(P)$ and, hence, to $u$. Similarly, if there exist inferior outcomes, then $u$ must be bounded below. By part (iii) of the Archimedean axiom and Fishburn (1970, Theorem 3.1), there exists a function $v^*$ on $\mathcal{C}$ that represents $\succeq$ on $\mathcal{C}$. On the superior outcomes, we can and must let $v$ be any ordinal transform of $v^*$ that exceeds the upper bound of $u$, and, on the inferior outcomes, we can and must let $v$ be any ordinal transform of $v^*$ that is below the lower bound of $u$. This establishes the gambling-utility model, and also the uniqueness results of Observation 5. □

**Proof of Observation 6:** First assume ordinal equivalence of $u$ and $v$. Assume that we replace a positive-probability outcome in a gamble by a strictly preferred outcome in such a manner that the gamble is risky both before and after the substitution. It means that we have replaced the outcome by one with a strictly higher $v$ value and, hence, by ordinal equivalence, by one with a strictly higher $u$ value. Given the positiveness of the probability, the replacement strictly increases the expected utility of the gamble and, therefore, also its preference value. A similar reasoning applies if we replace an outcome by an indifferent outcome, or by an outcome strictly less preferred. From this and weak ordering, gamble monotonicity follows.

Next assume that gamble monotonicity holds. Consider two outcomes $x$, $y$ with $v(x) > v(y)$. This implies $x \succ y$. Let $P$ be any risky gamble. By gamble monotonicity, $0.5x + 0.5P \succ 0.5y + 0.5P$. Substitution of expected utility implies $u(x) > u(y)$. Similarly, $v(x) = v(y)$ implies $u(x) = u(y)$ and $v(x) < v(y)$ implies $u(x) < u(y)$. That is, $v$ and $u$ are ordinally equivalent. Note that we used gamble monotonicity only with $\lambda = 0.5$ in this step. It would, therefore, have sufficed to require the condition only for that $\lambda$. This was actually the formulation used by Fishburn (1980, Axiom A2a). □

**Proof of Observation 7:** It is obvious that expected utility implies stochastic dominance. Hence, we assume stochastic dominance and the gambling-utility model, and derive expected utility.

*Claim 1.* If, for outcome $x$, there exists an outcome $y \succ x$, then $v(x) \leq u(x)$.

**Proof:** Consider a gamble $(p, y; 1 - p, x)$. By stochastic dominance, the gamble is preferred to $x$ and, hence, its expected utility exceeds $v(x)$. For $p$ tending to zero, the expected utility tends to $u(x)$ and, therefore, $u(x)$ exceeds $v(x)$. *QED*

Similarly it can be demonstrated that:

*Claim 2.* If, for outcome $x$, there exists an outcome $y \prec x$, then $v(x) \geq u(x)$.

The two claims show that $u(x) = v(x)$ for all outcomes $x$, except possibly for best or worst outcomes. Consider a best outcome $x$. To avoid triviality, we assume two nonindifferent outcomes. Hence, by Claim 2, $v(x) \geq u(x)$. If $v(x) > u(x)$, then we can simply redefine $v(x) = u(x)$ and $v(y) = u(y)$ for all outcomes $y \sim x$. This definition leads to correct descriptions of all preferences that involve $x$ (or outcomes equivalent to $x$). We have strict preference and a stictly higher value for $x$ than for each gamble assigning positive probability to any outcome strictly worse than $x$. We have, by stochastic dominance, indifference and equal value for $x$ and each gamble assigning probability 1 to outcomes indifferent to $x$. Similarly, for a worst outcome $y$, we can redefine $v(y) = u(y)$. □

**Proof of Remark 8:** The only property of expected utility that was used in the proof of Observation 7 is, besides stochastic dominance, that (a) it uses a utility function for outcomes in a strictly monotonic fashion; (b) it assigns to any degenerate gamble the utility of the corresponding outcome; (c) it is continuous in probability for gambles $(p, y; 1-p, x)$ at $p = 0$ which, by symmetry, also includes $p = 1$. Therefore, a similar result holds for virtually all nonexpected utility theories, including those mentioned in the Remark as soon as their probability transformations are strictly increasing and continuous at $p = 0$ and $p = 1$. Note that Condition (c) imposes a continuity requirement on the functional outside of $\mathcal{R}$. □

*The gambling-utility model with a linear cost of gambling*

In the main body of the paper, we assumed the special case of Eq. (1) with $C$ depending only on the sure outcome $x$. This assumption can be rewritten as a special case of dependency of $C$ on the gamble $P$, given the assumption throughout that $W$ is expected utility. To this end, let $W^*$ be expected utility with respect to $v$ instead of $u$, and for each gamble $P$ define $C^*(P)$ as the expectation of $v - u$. With these substitutions, $W^*(x) = v(x)$ for all $x$ and $W^*(P) - C^*(P) = W(P)$ for all $P$, so that the representation is identical to the original one. Conversely, every case of Eq. (1) with $C$ depending on $P$ in a linear manner can, by inverse substitutions, be carried into a case of Eq. (1), with costs depending only on $x$. Therefore, the second approach in Section 1 can be considered to be a special case of the first approach. □

## Notes

1. In other places, e.g. p. 632, von Neumann and Morgenstern suggested that the reduction of compound lotteries should be abandoned so as to accommodate a utility of gambling. This suggestion concerns multistage gambling and dynamic decisions. Our paper confines its attention to single-stage gambles and the impact of the utility of gambling there. Thus, the reduction of compound lotteries plays no role.
2. Condition (ii) ensures for instance that, if a nonregular outcome $x$ exists that is superior to all risky gambles, then the expected utilities of the risky gambles are bounded above, so that $v(x)$ can take a finite value.
3. In the notation of Fishburn's Theorem 3, both $u$ and $u + \phi$ order preferences over risky gambles. We can, therefore, define a strictly increasing transformation $f$ such that $f(u + \phi) = u$. Applying $f$ to Fishburn's representation $u + \phi$ yields the expected utility representation for gambles, and, for outcomes $x$, $f$ transforms $(u + \phi)$ into what we call $v$.
4. For example, they use joint receipts, meaning that more than one act is received simultaneously; they consider multistage gambles; and they do not identify acts with mappings from states to outcomes.

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