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Dutch books: avoiding strategic and dynamic complications, and a comonotonic extension

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Abstract

This paper formalizes de Finetti's book-making principle as a static individual preference condition. It thus avoids the confounding strategic and dynamic effects of modern formulations that consider games with sequential moves between a bookmaker and a bettor. This paper next shows that the book-making principle, commonly used to justify additive subjective probabilities, can be modified to agree with nonadditive probabilities. The principle is simply restricted to comonotonic subsets which, as usual, leads to an axiomatization of rank-dependent utility theory. Typical features of rank-dependence such as hedging, ambiguity aversion, and pessimism and optimism can be accommodated. The model leads to suggestions for a simplified empirical measurement of nonadditive probabilities.

Keywords: Book making; Dutch book; Comonotonic; Choquet expected utility; Ambiguity aversion; Ordered vector space

JEL classification: D81; C60

1. Introduction

De Finetti's book-making principle entails that a gambler should not have preferences that can be linearly combined into a sure loss. A surprising implication is that all uncertainties have to be quantifiable by means of additive probabilities, possibly subjective (de Finetti, 1931, 1937, 1974). The principle has, since its discovery, served

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as a justification of Bayesianism. The main restriction of the book-making principle is that it requires outcomes to be expressed in utils, in other words, utility must be linear. This requirement is reasonable for small stakes (Rabin, 2000).

Linear combinations of gambles naturally arise in financial markets, where assets can be bought and sold at fixed rates. The book-making principle then amounts to a no-arbitrage requirement, which is commonly considered normative in finance (Nau and McCardle, 1991; Varian, 1987).

Section 2 presents a formalization of de Finetti's book-making principle that deviates from other presentations. First, we formulate the principle for a static individual preference system, thus eschewing all dynamic and game-theoretic complications. Second, our principle is completely formalized, whereas in the literature it is commonly used in a broad and informal sense. Our formalization is closely related to an additivity condition for preferences that is well known in decision theory and that has been studied extensively in the mathematics literature.

There are many descriptive reasons and, according to some authors, also normative reasons for deviations from Bayesianism. This insight has resulted from the Allais (1953) and Ellsberg (1961) paradoxes and has led to a rich literature (Camerer and Weber, 1992; Schmidt, 1998; Starmer, 2000). The most popular models today are the rank-dependent models (Quiggin, 1981; Schmeidler, 1989; Tversky and Kahneman, 1992; Yaari, 1987). They allow for a nonlinear weighting of uncertainty, modeled through nonadditive measures (capacities). Decision weights of events depend on how favorable the outcomes of the events are in comparison to the alternative outcomes of the gamble under consideration (rank-dependence). Basic rationality requirements such as transitivity and monotonicity are maintained but several other deviations from Bayesianism can be accommodated. Examples are pessimism (aversion to uncertainty; convex capacities), optimism (concave capacities), and insufficient sensitivity towards varying degrees of uncertainty (inverse-S capacities, overweighting unlikely events and underweighting likely events, see Tversky and Kahneman, 1992).

In financial portfolios, investing in negatively correlated assets (hedging) is desirable. This phenomenon can be modeled by pessimism and convex capacities. The nonlinear weighting of uncertainty is an important factor in insurance. Wakker et al. (1997) found that the common aversion to incomplete insurance cannot be explained by curvature of utility but can be explained by nonlinear probabilities.

Section 3 extends the book-making principle to the rank-dependent models. We maintain the hypothesis that outcomes are utils and then describe the books that can be made[2] against the rank-dependent models. Examples will demonstrate that books can be made because of hedging, optimism, or other phenomena related to noncomonotonic gambles. In situations where such phenomena, typical of rank-dependent utility, are descriptively or even normatively desirable, the exclusion of books is unwarranted. Therefore, the existence of books may be reasonable under rank-dependent utility when the gambles are not comonotonic, and books are only to be excluded when all gambles are comonotonic. This condition is called the comonotonic (Dutch) book principle. It is

[2] Making a book (also called Dutch book) means a violation of the book-making principle.

not only necessary but also sufficient for the rank-dependent models, given payment in utils. In a mathematical sense, our result extends Yaari's (1987) theorem (and the, almost identical, Theorem 3 of Weymark, 1981) from risk to uncertainty. Such an extension was obtained before by Chateauneuf (1991). The novelty of this paper lies in the new version of the book-making principle and its adaptation to nonadditive probabilities.

2. De Finetti's book-making principle

This section gives a new formalization of de Finetti's book-making principle. $S = \{s_1, \ldots, s_n\}$ is a finite *state space*, with subsets called *events*. One of the states is true and the others are not true. A decision maker is uncertain about which state is true. *Outcomes* are real numbers designating money. A gamble is a state-contingent payoff, e.g., a financial asset. Formally, a *gamble* f is a function from the state space to the outcomes. Gamble f will generate outcome $f(s)$ if s is the true state of nature. Gambles are often identified with n-tuples and, hence, the set of gambles is identified with IR^n. Sometimes probabilities of the states are given. Then the state space is a probability space and gambles are random variables. In general, probabilities need not be given.

By $\succcurlyeq$ we denote the preference relation of the decision maker over the gambles. It is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all gambles f, g) and transitive. The notation $\succ$ and $\sim$ is as usual. *Strict monotonicity* holds if $f \succ g$ whenever $f > g$ ($f > g$ means that $f(s) > g(s)$ for all states s). For a gamble f, a *fair price* is an outcome x such that $x \sim f$. As usual, outcomes are identified with constant gambles. A function V *represents* a preference relation $\succcurlyeq$ if $V(f) \geq V(g)$ if and only if $f \succcurlyeq g$, for all gambles f, g.

The book-making principle, also called coherence by de Finetti, is based on the idea that a number of good decisions, when taken together, should still be good. 'Taken together' is interpreted as state-wise addition of outcomes. A book, defined formally hereafter, consists of a number of preferences that, when taken together, yield a loss for each state of nature. Obviously, such a result is not good and therefore the *book-making principle* requires that no book exists.

Definition 1. A *book* consists of a number of preferences as depicted in Fig. 1. □

In words, if replacing g^j by f^j is good for each j, then the joint result of these replacements should not be a sure loss. Our presentation differs from de Finetti's in a number of respects. First, de Finetti also incorporated multiplication by positive scalars, replacing the final inequality in Fig. 1 by the condition that there exist positive λ^js such that $\sum_{j=1}^m \lambda^j f^j(s) < \sum_{j=1}^m \lambda^j g^j(s)$ for all s. We have dropped such scalar multiplication because, first, we find addition a more appealing way of combining gambles, and, second, the book principle thus becomes less restrictive so that we obtain more general theorems.

A second difference is that de Finetti considered a game situation where an outside person can take the decision maker up on any of his preferences. This generates distortions due to strategic considerations (Border and Segal, 1994, 2001; de Finetti,

$$f^1 \succcurlyeq g^1$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$f^m \succcurlyeq g^m \quad \text{but}$$

$$\sum_{j=1}^{m} f^j(s) < \sum_{j=1}^{m} g^j(s) \text{ for all states } s$$

Fig. 1. A book.

1937, footnote (a) in the 1964 translation; de Finetti, 1974, p. 93) and the state of information of the outside person. Our single-person condition avoids such distortions.

Third, as Theorem 2 will demonstrate, the book-making principle is based on two conditions, strict monotonicity and *additivity* ($f \succcurlyeq g$ implies $f + h \succcurlyeq g + h$ for all gambles f, g, h; absence of any ''income effect''). In his discussions, de Finetti emphasized monotonicity but we, like many other authors, think that the essence of the book-making principle lies in additivity (Camerer and Weber, 1992, p. 359, second full paragraph; Schick, 1986). For moderate stakes, additivity seems to be a reasonable condition. The receipt of gamble h does not change the situation or needs of the decision maker much and therefore it seems reasonable that the preference between f and g is not affected.

Fourth, de Finetti did not impose the completeness requirement on all gambles but, instead, he took an arbitrary set of gambles and their fair prices as the initial domain of preference. Because all linear combinations were also incorporated, his domain was a linear subspace on which, through the fair prices, a weak order was obtained. The extension of the following theorem to linear subspaces is omitted for simplicity.

One case of a linear subspace is of special interest. It results when the book-making principle is restricted to judgements of acceptability of single gambles f^j. Gamble f^j is called *acceptable* if $f^j \succcurlyeq (0, \ldots, 0)$. In other words, $g^j = (0, \ldots, 0)$ for all j in Fig. 1. The acceptable gambles are those that are evaluated nonnegatively. An alternative interpretation of the favorableness of gambles has sometimes been used (Camerer and Weber, 1992, p. 359) that relates a general preference $f^j \succcurlyeq g^j$ to a favorableness judgment $f^j - g^j \succcurlyeq 0$. In this interpretation, prior endowments of the g^js to the agent are assumed and exchanges for f^j are considered, for each j, leading to a sure net loss. Relating preferences $f^j \succcurlyeq g^j$ to favorableness judgments $f^j - g^j \succcurlyeq 0$ is not as innocuous as may seem at first sight, though. It entails additivity, which in itself already implies most of the book-making principle (Theorem 2).

Several modern papers have used the term (Dutch) book-making for dynamic decision principles. These descriptions can sometimes lead to confusion if some of the dynamic decision principles assumed are left implicit, or if the domain of choice options changes in the course of the example in ways that essentially change the strategic situation (criticized by Machina, 1989, as hidden nodes). We do not assume dynamic or sequential choices, and all preferences in Fig. 1 and elsewhere are assumed to be part of one static preference system. By eschewing dynamic and strategic aspects and, thereby, their distortions, we hope to obtain an unambiguous condition that contains the essence of de Finetti's book-making principle. The following theorem shows that our condition does indeed achieve de Finetti's main objective, i.e., it implies the existence of subjective probabilities.

Theorem 2. *The following three statements are equivalent for* $\succcurlyeq$ *on* IR^n.

(i) *There exist probabilities* $p_1, \ldots, p_n$ *such that preferences maximize expected value* $f \mapsto p_1 f(s_1) + \ldots + p_n f(s_n)$.
(ii) $\succcurlyeq$ *is a weak order, for each gamble there exists a fair price, and no book can be made*.
(iii) $\succcurlyeq$ *is a weak order, for each gamble there exists a fair price, and additivity and strict monotonicity are satisfied.*

Furthermore, the probabilities in (*i*) *are uniquely determined.* □

We end this section with some comments on related mathematical results. There are many results similar to the equivalence of (i) and (iii) with continuity instead of the fair price condition and with an invariance condition for scalar multiplication (homotheticity) added (Nau, 1992; Regazzini, 1987; Schervish et al., 2000; Weibull, 1985). Additivity of preference amounts to commutativity of an ordering and an addition operation, which has been extensively studied in the mathematics literature (Birkhoff, 1967, Chapter 15; Fuchssteiner and Lusky, 1981; Krantz et al., 1971, Section 2.2.5). These studies often considered more general state spaces and outcome spaces. Blackwell and Girshick (1954, Theorem 4.3.1 and Problem 4.3.1) and Wakker (1989, Theorem A2.1) presented related results that did not use scalar multiplication either but instead a stronger monotonicity condition plus continuity. Candeal and Induráin (1995) and Neuefeind and Trockel (1995) presented results without monotonicity for the preference relation or the representing linear functional.

The mathematics of our theorem is related to invariance conditions for preferences with respect to mixing operations (Fishburn, 1982; von Neumann and Morgenstern, 1944), which similarly lead to linear representations. In Theorem 2, we did not seek for maximal mathematical generality. The purpose of the theorem was to present, in a manner as accessible as possible, de Finetti's book-making principle for deriving subjective probabilities while avoiding game-theoretic and dynamic complications. Our derivation of subjective probabilities is cleaner, but admittedly less vivid, than de Finetti's. Our version of a book only entails an internal inconsistency, not a sure ruin to the benefit of an eager opponent.

3. Hedging, uncertainty aversion, and comonotonic books

This section presents three violations of the book-making principle as formalized in Definition 1. The first illustrates how books can help uncover what we consider to be irrationalities.[3] The second is based on hedging, which was put forward as a rationale for the rank-dependent models by Yaari (1987, p. 104). The third example, the Ellsberg paradox, shows how aversion to unknown probabilities leads to a book, illustrating once more that additive probabilities cannot describe this paradox. Camerer and Weber (1992, Section 5.8) described a similar example, formulated as a dynamic game.

Example 3. [Roulette] Consider gambles on a roulette wheel. There are 37 states of nature, corresponding to one of the numbers $0, \ldots, 36$ being selected. A bet of \$1 on a single number yields a net profit of $\$36 - \$1 = \$35$ if the number shows up and $-\$1$ otherwise. A gambler may be indifferent to the choice of number but prefer any of these bets to not betting. The resulting preferences constitute a book, depicted in Fig. 2. □

Example 4. [Hedging] Assume that a coin is tossed once and the state space is {heads, tails}. (20, 0) denotes the gamble yielding \$20 for heads and \$0 for tails. The other gambles are defined similarly and relate to the same toss of the coin. The preferences in Fig. 3 are natural but generate a book.

The preferences in this example are traditionally explained by expected utility with concave utility. For moderate stakes, however, utility is close to linear and an alternative explanation for the observed risk aversion seems to be more plausible. Such an alternative explanation, based on a nonlinear weighting of uncertainty, will be provided later. Note that when the gambles (20, 0) and (0, 20) are taken together, one gamble serves as a hedge for the other. □

$$(35, -1, -1, \ldots, -1) \succcurlyeq (0, \ldots, 0)$$

$$(-1, 35, -1, \ldots, -1) \succcurlyeq (0, \ldots, 0)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$(-1, -1, \ldots, -1, 35) \succcurlyeq (0, \ldots, 0) \text{ but}$$

$$(-1, -1, -1, \ldots, -1) < (0, \ldots, 0).$$

Fig. 2. Making book against roulette players.

[3] As a matter of personal opinion, we consider rank-dependent utility, and thereby our comonotonic weakening of the book-making principle, to be primarily of descriptive, not normative, interest.

$$(9,9) \succcurlyeq (20,0)$$

$$(9,9) \succcurlyeq (0,20) \quad \text{but}$$

$$(18,18) < (20,20).$$

Fig. 3. Making book against risk aversion.

Example 5. [Ellsberg Example] Assume an urn K (known) containing red and black balls in equal proportions and an urn A (unknown or ambiguous) containing red and black balls in an unknown proportion. A ball will be drawn at random from each urn, and its color will be inspected. The state space is $\{B_kB_a, B_kR_a, R_kB_a, R_kR_a\}$, where B_kR_a refers to a black ball from urn K and a red ball from urn A, and the other states are defined similarly. Gamble (1,0,1,0) yields \$1 if the ball from A is black and nothing otherwise; other gambles are defined similarly. The two preferences in the first two lines of Fig. 4 are commonly observed, even as strict preferences, for $\epsilon = 0$. For $\epsilon > 0$ sufficiently small, they will still hold and, when taken together as in the inequality in Fig. 4, yield a book.

The left gambles in the figure provide a hedge for each other, as in the preceding example, because taking them together changes risk into certainty. This same hedging takes place when the right gambles are taken together but, in addition, the uncertainty about the unknown probabilities is removed there also. It is well known that these preferences cannot be explained by expected utility or any other model using additive probabilities. □

The preceding examples have something in common. In each of them, the best-ranked outcomes of one gamble are combined with the worst-ranked outcomes of other gambles. In this manner, the outcomes neutralize each other, leading to a gamble of lower variance. Let us consider Fig. 2 of Example 3 in some detail. For each gamble, the good outcome \$35 is neutralized by the bad outcomes −\$1 of the other gambles when they are taken together. In this example, taking the gambles together may lead to an overall loss of value: if the gambler takes up a single gamble, then the probability of gaining \$35 changes from 0 to 1/37, i.e., from impossible to possible. It is well known that people especially appreciate such a change that brings hope. If the gambler already received one or more gambles, then he already has a possibility of receiving \$35. Receiving an extra gamble still increases the probability of winning \$35 by 1/37, but no more changes this event from impossible to possible. A sort of psychological

$$(1,1,0,0) \succcurlyeq (1+\epsilon, 0, 1+\epsilon, 0)$$

$$(0,0,1,1) \succcurlyeq (0, 1+\epsilon, 0, 1+\epsilon) \quad \text{but}$$

$$(1,1,1,1) < (1+\epsilon, 1+\epsilon, 1+\epsilon, 1+\epsilon).$$

Fig. 4. Making book against Ellsberg preferences.

substitutability occurs between the various gambles, where the positive value of hoping for \$35, provided by a single gamble, is reduced when such hope was already provided by another gamble. This psychological effect can explain why the gambler likes each gamble in isolation but not all gambles when taken together, and why it may be worthwhile to allow for such phenomena in a descriptive model.

In Example 4, the unfavorableness of the event of gaining nothing is especially salient, explaining the preference against (20, 0) or (0, 20) in isolation. If these gambles are taken together, however, then the favorable \$20 outcome of each gamble neutralizes the possibility of gaining nothing of the other gamble, and a sure gain results. Here a complementarity effect occurs in the taking together of the gambles. Besides this complementarity effect, another complementarity effect occurs when taking $(1+\epsilon, 0, 1+\epsilon, 0)$ and $(0, 1+\epsilon, 0, 1+\epsilon)$ together in Fig. 4 in Example 5, because the uncertainty about the unknown probabilities of the outcomes is removed.

In each example, the variability of one gamble is tempered by the counter-variability of the other(s). The above-mentioned interaction effects do not occur when the gambles taken together are comonotonic. A set of gambles is *comonotonic* if for each pair of elements f,g there do not exist states s,t such that $f(s) > f(t)$ and $g(s) < g(t)$.

The preceding considerations suggest a generalization of the book-making principle. A *comonotonic book* is a book as in Fig. 1 but with the extra restriction that the set of gambles considered, $\{f^1, \ldots, f^m, g^1, \ldots, g^m\}$, is comonotonic. As illustrated by the examples, the existence of books seems reasonable if the gambles are not comonotonic. Only if the gambles are comonotonic, are books problematic. The *comonotonic book-making principle* requires that no comonotonic book exists. Similarly, *comonotonic additivity* means that $f \succcurlyeq g$ implies $f+h \succcurlyeq g+h$ for all comonotonic gambles f, g, h.

Choquet expected value is the model characterized by the comonotonic book-making principle. It is the rank-dependent model for decision under uncertainty, i.e., the context where no probabilities are given. Because payment is in utils, no utility function need to be defined; utility is assumed to be linear. We therefore use the term Choquet expected value instead of Choquet expected utility. A *capacity* W is a function $W{:}2^S \to [0, 1]$ satisfying: (a) $W(\emptyset)=0$, (b) $W(S)=1$, and (c) W is nondecreasing with respect to set inclusion. *Choquet expected value* holds if there exists a capacity W such that

$$f \mapsto \sum_{j=1}^{n} \pi_j f(s_j)$$

represents $\succcurlyeq$, where the *decision weights* π_j are defined as follows. First, a permutation ρ is chosen such that $f(s_{\rho(1)}) \geqslant \cdots \geqslant f(s_{\rho(n)})$. Next, $\pi_{\rho(i)} = W(\{s_{\rho(1)}, \ldots, s_{\rho(i)}\} - :\}) - W(\{s_{\rho(1)}, \ldots, s_{\rho(i-1)}\};:\})$; in particular, $\pi_{\rho(1)} = W(s_{\rho(1)})$. The decision weights are nonnegative and sum to one. Many theorems in the literature have demonstrated that rank-dependent forms can be characterized by means of comonotonic restrictions of expected-utility axioms. The following theorem states such a result for the book-making principle, leading to a characterization of Choquet expected value.

Theorem 6. *The following three statements are equivalent for the preference relation* $\succcurlyeq$ *on* IR^n.

(i) *There exists a capacity W such that preferences maximize Choquet expected value.*
(ii) *The binary relation $\succcurlyeq$ is a weak order, for each gamble there exists a fair price, and no comonotonic book can be made.*
(iii) *The binary relation $\succcurlyeq$ is a weak order, for each gamble there exists a fair price, and comonotonic additivity and strict monotonicity are satisfied.*

Furthermore, the capacity W in (i) is uniquely determined. $\square$

Choquet expected value can accommodate the phenomena in the examples. For example, a capacity W that assigns a weight exceeding $1/36$ to each number can explain the risk seeking in Fig. 2 of Example 3. This capacity implies an overweighting of unlikely events and risk seeking for long-shot options. In Fig. 3, hedging can be explained by a capacity W with $W(Heads) = W(Tails) < 0.45$. This choice yields a decision weight of less than 0.45 for the outcome 20 and a decision weight exceeding 0.55 for the outcome zero. Consequently, the observed risk aversion is not ascribed to diminishing marginal utility as this was traditionally done, but to the extra attention paid to the zero outcome. The aversion to unknown probabilities in Fig. 4 of Example 5 can be explained by any capacity W assigning a greater value to the events $\{B_k B_a, B_k R_a\}$ and $\{R_k B_a, R_k R_a\}$, which describe the colors from the known urn K, than to the events $\{B_k B_a, R_k B_a\}$ and $\{B_k R_a, R_k R_a\}$, which describe the colors from the unknown urn A.

We end this section with some comments on related mathematical results. Several papers have considered variations of Statement (iii). De Waegenaere and Wakker (2001) used comonotonic additivity together with continuity but without any monotonicity to characterize a nonmonotonic generalization. Schmeidler (1986) used a comonotonic additivity condition for functionals, in combination with continuity, to characterize noncomonotonic functionals; he also characterized the monotonic case. Schmeidler's (1989) comonotonic mixture-invariance condition for preferences is famous. It was used to obtain linearity with respect to second-stage probabilities. Chateauneuf (1991, Theorem 1) is closest to our result. He used a weakened version of comonotonic independence (similar to Anger's, 1977, Theorem 3, which is more general than Schmeidler's 1986 result), considered mixtures of outcomes rather than of probabilities, and used continuity instead of certainty equivalence. Thus, he characterized the same form as our Theorem 6.

4. Discussion

The book-making principle relies on linear utility. Utility is approximately linear for moderate amounts of money (Edwards, 1955; Fox et al., 1996; Lopes and Oden, 1999, p. 290; Luce, 2000, p. 86; Ramsey, 1931, p. 176; Rabin, 2000; Savage, 1954, p. 91). The rank-dependent model suggests that a considerable part of the deviations from expected value observed for moderate amounts of money, traditionally ascribed to curvature of utility, is due to a nonlinear weighting of probability. This suggestion is supported empirically by Selten et al. (1999). They compared the effects of nonlinear utility with those of nonlinear probability weighting. For the small outcomes considered (ranging between $-\$1$ and \$3), the nonlinearity of probability weighting was more pronounced. Yaari (1987) also assumed linear utility in his derivation of rank-dependent utility for risk, and our model can be considered the generalization of Yaari's model to uncertainty.

We next consider some empirical implications of our work. Many studies into the nature of nonadditive probabilities are going on today. If both utilities and probability weights are unknown, complex measurement methods have to be used (Abdellaoui, 2000; Bleichrodt and Pinto, 2000; Gonzalez and Wu, 1999; Loehman, 1998; Tversky and Kahneman, 1992). With linear utility, axiomatized in the present paper, as an approximation for moderate stakes, gambles with such stakes provide a convenient tool for measuring nonlinear probability weighting (Kilka and Weber, 2000; Diecidue, Wakker, and Zeelenberg, in preparation).

Our model can be interpreted as a return to Preston and Baratta (1948). This paper, one of the earliest empirical studies of risk attitude, used nonlinear probabilities rather than nonlinear utilities to explain deviations from expected value. In the following decades, expected utility was the dominant model and Preston and Baratta's study was usually criticized for its way of modeling risk attitude. From the current perspective of rank-dependent utility and prospect theory, however, nonlinear probabilities are useful concepts. If the plausible assumption of linear utility for small stakes is added, then the analysis of Preston and Baratta seems to be appropriate again, and our paper has provided a preference axiomatization for their approach.

5. Conclusion

This paper has demonstrated how de Finetti's book-making principle can be formulated without strategic or dynamic complications. A comonotonic restriction of the principle characterizes rank-dependent utility. It is remarkable that de Finetti's book-making principle, usually considered to be inextricably associated with additive probabilities, can so easily be adapted to nonadditive probabilities.

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Robert Kast and an anonymous referee made helpful comments.

Appendix A. Proofs

Proof of Theorem 2. The implication (i)$\Rightarrow$(ii) follows from substitution. Next we assume (ii) and derive (iii). For strict monotonicity, assume that $f \succcurlyeq g$ and $f(s) < g(s)$ for all s. This preference and these inequalities constitute a book (with $m = 1$ in Fig. 1) and, hence, a contradiction. Strict monotonicity must hold. For each gamble f, define $FP(f)$ as the fair price of gamble f. FP is uniquely determined and represents preference ($f \succcurlyeq g$ if and only if $FP(f) \geqslant FP(g)$; note that $x > y$ implies $x \succ y$ because of strict monotonici-

$$f \succcurlyeq FP(f)$$

$$g \succcurlyeq FP(g)$$

$$FP(f+g) \succcurlyeq f+g, \text{ but}$$

$$f+g+FP(f+g) < f+g+FP(f)+FP(g)$$

Fig. 5. Making book against violations of additivity.

ty). We claim that FP satisfies *additivity* ($FP(f+g)=FP(f)+FP(g)$, also known as Cauchy's functional equation). To clarify this point, if $FP(f+g)<FP(f)+FP(g)$ then the book depicted in Fig. 5 results and, hence, a contradiction. If $FP(f+g)>FP(f)+FP(g)$ then the reversed preferences result in a book. Additivity of FP follows. This implies additivity of $\succcurlyeq$; hence, Statement (iii) follows.

We finally assume (iii) and derive (i) and the uniqueness result. FP is defined as above and represents preferences. We again derive additivity of FP. $f \sim FP(f)$ implies, by 2-fold application of additivity (with $\succcurlyeq$ and with $\preccurlyeq$), that $f+g \sim FP(f)+g$. Additivity and $g \sim FP(g)$ imply that $g+FP(f) \sim FP(g)+FP(f)$. Transitivity implies that $f+g \sim FP(f)+FP(g)$; hence, $FP(f+g)=FP(f)+FP(g)$. We conclude that FP is additive.

Additivity means that Cauchy's functional equation holds which, together with strict monotonicity, implies that FP is a linear functional (Aczél, 1966, Theorem 5.1.1.1; our strict monotonicity implies the existence of a measurable majorant on a set of positive measure, e.g., $FP(1,\ldots,1)$ is the majorant on the set of gambles dominated by $(1,\ldots,1)$). $FP(f)=\Sigma_{j=1}^{n} p_j f(s_j)$ for real numbers p_j. The p_js are nonnegative for if one, say p_1, were negative then we could find a gamble $(M,1,\ldots,1)$ with M so large that the FP of the gamble would be negative, implying that it is less preferred than the 0 gamble, thus violating strict monotonicity. $FP(1)=1$ implies that the p_js sum to one. Statement (i) has been proved.

For uniqueness, $FP(1,0,\ldots,0)=p_1$ determines p_1 in a unique manner because of strict monotonicity. Similarly, every p_j is uniquely determined. □

Proof of Theorem 6. The implication (i)⇒(ii) follows from substitution. Next assume that (ii) holds. We derive (iii). For strict monotonicity, assume that $f \succcurlyeq g$ and $f(s)<g(s)$ for all s. Also assume, first, that f and g are comonotonic. Then the preference and inequalities constitute a comonotonic book and, hence, a contradiction. Therefore, *comonotonic strict monotonicity* holds, i.e., strict monotonicity holds within sets of comonotonic gambles. Lemma 7 will demonstrate that strict monotonicity holds in full force.

Comonotonic additivity is derived as in the proof of (ii)⇒(iii) in Theorem 2, with the appropriate comonotonicity requirements added. These do not complicate the reasoning. Note that constant gambles are comonotonic with all other gambles.

We finally assume (iii) and derive (i) and uniqueness. That FP represents preferences and satisfies *comonotonic additivity* ($FP(f+g)=FP(f)+FP(g)$ holds whenever f and

g are comonotonic) is demonstrated exactly as in the proof of Theorem 2, again with all appropriate comonotonicity requirements added. We show that FP is a Choquet integral.

For any event E and real λ, λE denotes λ times the indicator function of E. For any fixed E, $\lambda \mapsto FP(\lambda E)$ satisfies Cauchy's equation on the nonnegative reals. On this set, the mapping is bounded on a nondegenerate interval, i.e., it is bounded above on [0, 1] by $FP(2,\ldots,2)$. Hence, FP is linear on this set (Aczél, 1966, Theorem 2.1.1.1) and $FP(\lambda E) = \lambda W(E)$ for the real number $W(E) = FP(1E)$. $W(\emptyset) = 0$ and $W(S) = 1$ follow because FP assigns fair prices. W is monotonic with respect to set inclusion: If $A \supset B$ but $W(A) < W(B)$, then we can find a λ sufficiently large to imply $FP(\lambda A + (1,\ldots,1)) = FP(\lambda A) + FP(1,\ldots,1)) = \lambda W(A) + 1 < \lambda W(B) = FP(\lambda B)$, contradicting strict monotonicity. Hence, W is monotonic with respect to set inclusion, which implies that W is nonnegative.

Every gamble can be written as a sum $\Sigma_{j=1}^{k} \lambda_j E_j - (M,\ldots,M)$ for nonnegative λ_j, nonnegative M, and decreasing sets $E_1 \supset \cdots \supset E_k$. To see this point, in $E_1 - E_2$ the gamble is minimal, its second-smallest value is taken in $E_2 - E_3$, etc. If the minimal value is negative then M is taken positive and large enough so as to have λ_1 nonnegative. By comonotonic additivity, $FP(\Sigma_{j=1}^{k} \lambda_j E_j - (M,\ldots,M)) = \Sigma_{j=1}^{k} FP(\lambda_j E_j) - FP(M,\ldots,M)) = \Sigma_{j=1}^{k} \lambda_j W(E_j) - M$, which is the Choquet expected value of the gamble with respect to the capacity W. Statement (i) has been proved.

Uniqueness of W follows because the sure amount of money $W(E)$ is the certainty equivalent of the indicator function $1E$ and it is uniquely determined because of strict monotonicity. $\square$

Lemma 7. *Let $\succcurlyeq$ be a weak order on the set of gambles that satisfies comonotonic strict monotonicity. Then it satisfies strict monotonicity.*

Proof. Throughout this proof, we write h_j for $h(s_j)$, for all h,j. Assume that $g_j > f_j$ for all j. If f and g are comonotonic then we are done, so assume they are not. The plan is to change gamble f, step by step, into a gamble that is weakly preferred to f, that is strictly dominated by g statewise, and that is also comonotonic with g. Then g is strictly preferred to that gamble and, by transitivity, the desired preference $g \succ f$ follows. In each step, the new gamble is comonotonic with, and weakly preferred to, the one constructed before. We will assume, without loss of generality, that

$$g_1 \geq \ldots \geq g_m. \tag{A.1}$$

Take any permutation $\rho_1,\ldots,\rho_m$ of $1,\ldots,m$ such that $f_{\rho_1} \geq \ldots \geq f_{\rho_m}$. Because f is not comonotonic with g, ρ cannot be the identity, and there is an i such that $\rho_i > \rho_{i+1}$. The squared Euclidean distance between $(\rho_1,\ldots,\rho_m)$ and $(1,\ldots,m)$, i.e., $\Sigma_{j=1}^{m}(\rho_j - j)^2$, is positive and is a natural number. It will be reduced in each step until the newly constructed gamble is comonotonic with g and the distance is zero.

We change the pair (f, ρ) into a pair (f', ρ') with again $f'_{\rho'_1} \geq \ldots \geq f'_{\rho'_m}$, as follows. First, with i as above, $\rho'_{i+1} = \rho_i$ and $\rho'_i = \rho_{i+1}$, $\rho'_k = \rho_k$ for all other k. Further

Case 1. $f_{\rho_i} = f_{\rho_{i+1}}$. Set $f = f'$.

Case 2. $f_{\rho_i} > f_{\rho_{i+1}}$.

We will increase the ρ_{i+1}th coordinate to become equal to the ρ_ith. Because we are only given comonotonic monotonicity in a strict sense, we next increase all coordinates by a small positive ϵ so as to guarantee that the new gamble is weakly (even strictly) preferred to the previous one. Formally, define $f'_{\rho_k} = f_{\rho_k} + \epsilon$ for all $k \neq i+1$ with the positive ϵ so small that still $f'_{\rho_k} < g_{\rho_k}$ for all $k \neq i+1$. Define $f'_{\rho_{i+1}} = f_{\rho_i} + \epsilon$. We have $f'_{\rho_{i+1}} = f'_{\rho_i} < g_{\rho_i} \leq g_{\rho_{i+1}}$, the latter inequality following from $\rho_i > \rho_{i+1}$ and Eq. (A.1), so that f' is still dominated by g statewise.

We have, in each case: f' is comonotonic with f; f' is still dominated by g statewise, $f' \succcurlyeq f$ (by reflexivity in the first case and by comonotonic strict monotonicity in the second). The squared Euclidean distance between $(\rho'_1, \ldots, \rho'_m)$ and $(1, \ldots, m)$ has decreased compared to that between $(\rho_1, \ldots, \rho_m)$ and $(1, \ldots, m)$. (The square function being convex, the sum of squares is minimized when 1 is subtracted from the larger of $\rho_i - i$, $\rho_{i+1} - 1$.)

Each time when the newly constructed gamble is not comonotonic with g, we apply the same procedure, each time decreasing the Euclidean distance between the new permutation and the identity permutation. Because the Euclidean distances are natural numbers, the process must stop at some stage. It can (but need not) be seen that the number of steps is identical to the number of pairs i,j such that $i > j$ but $\rho_i < \rho_j$, with ρ as defined below Eq. (A.1).

A gamble has resulted that is strictly dominated by g statewise, is also comonotonic with g, and is weakly preferred to f. $g \succ f$ follows. $\square$

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