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Nonmonotonic Choquet integrals

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Abstract

This paper shows how the signed Choquet integral, a generalization of the regular Choquet integral, can model violations of separability and monotonicity. Applications to intertemporal preference, asset pricing, and welfare evaluations are discussed. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper deals with signed Choquet integrals. These integrals relax the monotonicity condition of regular Choquet integrals. The belonging capacities (nonadditive measures) are nonmonotonic and may even take negative values. We give preference axioms for the signed Choquet integral and show that results on convexity, well known for monotonic Choquet integrals, naturally extend to signed Choquet integrals. In particular, a concave signed Choquet integral still is the minimum of Core integrals (Theorem 3).

Several papers have considered signed Choquet integrals. Schmeidler (1986) characterized these integrals through functional equations. Gilboa and Schmeidler (1994), Gilboa and Schmeidler (1995), and Denneberg (1997) used signed Choquet integrals in studies of Möbius inverses and Jordan decompositions. Murofushi et al. (1994) used them in a study of fuzzy measures of bounded variation. We demonstrate how the signed Choquet integral can be applied to intertemporal preferences, modeling interactions between separate periods that can be so strong that monotonicity is violated. For intertemporal preferences, the signed Choquet integral provides a common generalization of the functionals of Gilboa (1989) and

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Shalev (1997). Applications of signed Choquet integrals to other areas are mentioned, in particular to asset pricing in a context with uncertainty. Proofs are given in Appendix A.

2. Notation, definitions, and preparatory results

Let $\{1, \ldots, n\}$ be a finite set of *time points*. *Profiles* are n-tuples $(x_1, \ldots, x_n)$, also denoted by x. They can be interpreted as functions from $\{1, \ldots, n\}$ to $\mathbb{R}$, describing consumption or income at each time point. For simplicity, we consider only a finite number of time points. The extension to infinitely many time points can be obtained through Wakker's (1993) techniques and is a topic for future research.

For decision under uncertainty, time points are reinterpreted as states of nature and profiles as acts, for welfare theory time points are persons and profiles are welfare allocations. Other interpretations are possible. We will use the terminology of the main field of application in this paper, i.e. intertemporal preference.

Profiles x and y are *comonotonic* if there are no time points i, j such that $x_i > x_j$ and $y_i < y_j$. A subset of $\mathbb{R}^n$ is *comonotonic* if every pair of profiles in the subset is comonotonic. We consider general set functions $\nu : 2^{\{1,\ldots,n\}} \to \mathbb{R}$; ν is permitted to take negative values. Here ν is a *capacity* if $\nu(\emptyset) = 0$, $\nu(\{1, \ldots, n\}) = 1$, and ν satisfies *monotonicity* (with respect to set inclusion), i.e. $A \supset B \Rightarrow \nu(A) \geq \nu(B)$. A capacity cannot take negative values. For a *signed capacity* ν, the monotonicity requirement is dropped, and $\nu(\emptyset) = 0$ and $\nu(\{1, \ldots, n\}) = 1$ are the only requirements. A signed capacity can therefore take negative values.

Let ν be an arbitrary set function. For any $x \in \mathbb{R}^n$, we define the *signed Choquet integral* $\int x \, \mathrm{d}\nu$, analogously to the Choquet integral, as

$$\int_0^{\infty} [\nu(\{j : x_j \geq t\}) - \nu(\emptyset)] \, \mathrm{d}t + \int_{-\infty}^{0} [\nu(\{j : x_j \geq t\}) - \nu(\{1, \ldots, n\})] \, \mathrm{d}t. \tag{1}$$

In this paper, the following method for calculating the signed Choquet integral is useful. It follows from the preceding equation by integration by parts.

(i) Take a permutation ρ on $\{1, \ldots, n\}$ that is *compatible* with x, i.e. $x_{\rho(1)} \geq \cdots \geq x_{\rho(n)}$.

(ii) Define

$$\pi_{\rho(j)} := \nu(\{\rho(1), \ldots, \rho(j)\}) - \nu(\{\rho(1), \ldots, \rho(j-1)\}) \tag{2}$$

for all j (thus, $\pi_{\rho(1)} = \nu(\{\rho(1)\}) - \nu(\emptyset)$).

(iii) $\int x \, \mathrm{d}\nu = \sum_{j=1}^{n} \pi_j x_j$ is the *signed Choquet integral* of x with respect to ν.

The integral remains the same if ν is replaced by $\nu' = \nu - c$ for any constant c. In particular, we can take $\nu' = \nu - \nu(\emptyset)$ and thus restrict attention to set functions assigning 0 to the empty set. The numbers π_j are called *decision weights*. In general, they can well be negative. They are all nonnegative if and only if ν is monotonic. The signed Choquet integral of $(\alpha, \ldots, \alpha)$ with respect to a signed capacity ν is α for all $\alpha \in \mathbb{R}$, because $\nu(\{1, \ldots, n\}) = 1$.

Schmeidler (1986) gave the following functional characterization, and extended the result to infinitely many time points and bounded profiles. The characteristic property of the signed

Choquet integral is *comonotonic additivity*, which for a general functional V means that $V(x+y) = V(x) + V(y)$ whenever x and y are comonotonic. If the equality also holds for all noncomonotonic x, y, then V is *additive*.

Theorem 1. *$V : \mathbb{R}^n \to \mathbb{R}$ is a signed Choquet integral if and only if it is continuous and satisfies comonotonic additivity.*

We next characterize the preference relations that can be represented by signed Choquet integrals with respect to signed capacities. The characterization follows from Schmeidler's (1986) Proposition 2 (our Theorem 1), in the same way as Schmeidler's (1989) famous preference characterization of Choquet expected utility follows from the functional characterization of the Choquet integral in Schmeidler's (1986) Corollary. Schmeidler (1989) assumed that the outcome set is a convex set of probability distributions and that linearity refers to probabilistic mixing. We follow Chateauneuf's (1991) approach where the outcome set is $\mathbb{R}$ and linearity refers to outcome-mixing. This difference in interpretation does not affect the mathematics. The generalization of our results to nonlinear utility, and thus of Gilboa (1987), Wakker (1989a), Oginuma (1994), and Chateauneuf (1999) to the nonmonotonic case, is a topic for future research.

A binary relation $\succeq$ on $\mathbb{R}^n$ is a *weak order* if it is *complete* ($x \succeq y$ or $y \succeq x$ for all x, y) and transitive. It is *continuous* if $\{x \in \mathbb{R}^n : x \succeq y\}$ and $\{x \in \mathbb{R}^n : x \preceq y\}$ are closed for all $y \in \mathbb{R}^n$. It satisfies *comonotonic additivity* if $x \succeq y$ implies $x + z \succeq y + z$ for all comonotonic x, y, z. $(\alpha, \dots, \alpha)$ is a *constant equivalent* of x if it is equal in preference to x. A binary relation $\succeq$ is *constant–monotonic* if $\alpha > \beta \Rightarrow (\alpha, \dots, \alpha) \succ (\beta, \dots, \beta)$ for all real α, β. Constant monotonicity implies that the set function of a representing signed Choquet integral assigns a positive value to the set of all time points. A function V *represents* $\succeq$ if $x \succeq y \Leftrightarrow V(x) \geq V(y)$.

Corollary 1. *Let $\succeq$ be a binary relation on $\mathbb{R}^n$. Then $\succeq$ can be represented by a signed Choquet integral with respect to a signed capacity if and only if*

(i) $\succeq$ *is a weak order*;
(ii) $\succeq$ *is continuous*;
(iii) $\succeq$ *is constant–monotonic*;
(iv) $\succeq$ *satisfies comonotonic additivity*.

Further, the representing signed Choquet integral is uniquely determined.

We next show that convexity and concavity results, well known in the literature for Choquet integrals, naturally extend to signed Choquet integrals. Although these extensions are elementary and may have been known before, we have not been able to find references where they are stated. Throughout the rest of this section, V is a signed Choquet integral with respect to a set function v. We consider V as a function on $\mathbb{R}^n$ and convexity and concavity properties of V refer to mixtures of elements of $\mathbb{R}^n$ (and not to mixtures of set functions). A set function v is *convex* if

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B) \tag{3}$$

for all A, B, and it is *concave* if the reversed inequality holds.

It is elementarily verified that V satisfies *positive homogeneity*, i.e. $V(\lambda x) = \lambda V(x)$ for all $\lambda > 0$. V is *superadditive* if $V(x+y) \geq V(x) + V(y)$ for all x, y, and V is *subadditive* if the reversed inequality holds. For a set function ν with $\nu(\emptyset) = 0$, *Core*(ν) is the set of additive set functions μ that (weakly) exceed ν everywhere and satisfy $\mu(\{1, \ldots, n\}) = \nu(\{1, \ldots, n\})$. Then also $\mu(\emptyset) = 0$, because of additivity. If ν is a capacity (hence, nonnegative) then all elements of Core(ν) are nonnegative and therefore are probability measures. The following lemma gives the main tool for extending results from monotonic Choquet integrals to signed Choquet integrals. It was pointed out to us by Denneberg (1998).

Lemma 1 (Denneberg). *For each set function ν and additive measure P,*

(i) *ν is convex if and only if $\nu + P$ is.*

(ii) *μ is a Core member of ν if and only if $\mu + P$ is of $\nu + P$.*

Further:

(iii) *For each set function ν with $\nu(\emptyset) \geq 0$ there exists an additive measure P such that $\nu + P$ is nonnegative.*

(iv) *If $(\nu + P)(\emptyset) = 0$, $\nu + P$ is nonnegative and $\nu + P$ is convex, then $\nu + P$ is also monotonic.*

The following two theorems extend Schmeidler's (1986) Proposition 3 to signed Choquet integrals. For Choquet integrals, many alternative results are given by Schmeidler (1989, the "Proposition"). In this paper, we use linear as equivalent to affine. Hence, a linear function need not be zero at the origin.

Theorem 2. *The following four statements are equivalent for a signed Choquet integral V.*

(i) *V is concave.*

(ii) *V is superadditive.*

(iii) *V is the minimum of dominating linear functionals.*

(iv) *ν is convex.*

The following theorem, demonstrating that a set function is convex if and only if its signed Choquet integral is the minimum of Core integrals, has often been stated for monotone capacities (Anger, 1977; Huber, 1981). The result turns out to hold for all set functions that vanish on the empty set.

Theorem 3. *Assume that $\nu(\emptyset) = 0$. Then the set function ν is convex if and only if* Core(ν) *is nonempty and* $V(x) = \min\{\int x \, d\mu | \mu \in \text{Core}(\nu)\}$.

For capacities, Shapley (1965) first proved that convexity implies a nonempty Core. We next turn to dual versions of the preceding results.

Observation 1. Theorems 2 and 3 *can be applied to $-V$ and $-\nu$, resulting in substitution of the terms convex for concave, subadditive for superadditive, maximum for minimum, dominated for dominating, anti Core(ν) (set of additive set functions μ that lie below ν everywhere and satisfy $\mu(\{1, \ldots, n\}) = \nu(\{1, \ldots, n\})$) for Core$(\nu)$, below for above, and vice versa.*

3. Nonmonotonicity in multi-period decisions

Classical models for intertemporal preference have usually assumed separability over disjoint periods. Such separability underlies, for instance, discounted utility (Koopmans, 1972) as well as the Quality-Adjusted-Life-Years model commonly used in health economics (Drummond et al., 1987). Interaction between different periods is, however, central in human preference. The utility of seeing a movie, listening to music, buying a brand of coffee, the salary of a new job, or a year in Australia, crucially depends on past experience and consumption. Such dependency has been amply demonstrated (Kreps and Porteus, 1978; Loewenstein and Thaler, 1989; Loewenstein and Elster, 1992). Dependence on past consumption underlies many phenomena, such as habit formation and the equity premium puzzle (Pollak, 1970; Constantinides, 1990), fashion and addiction (Becker and Murphy, 1988), and changing tastes (Strotz, 1956; Hammond, 1976; Karni and Schmeidler, 1990).

Gilboa (1989) demonstrated how the Choquet integral can be used in intertemporal preference to model interactions between different periods. The integral can model dependency on orderings of outcomes and thus it can incorporate sensitivity towards increases and decreases of income. Many studies have demonstrated that people are especially sensitive to such *changes* in income (Scitovsky, 1965; Frank, 1989; Constantinides, 1990; Loewenstein and Prelec, 1991; Hsee and Abelson, 1991; Erev et al., 1997). This special sensitivity can be modeled by means of Gilboa's (1989) special, Markovian, version of the Choquet integral. Gilboa retained the classical monotonicity condition of Choquet integrals, thus did not permit that increases in income can lead to decreases in utility.

In extreme cases, the interaction between different periods can be so strong that it dominates the intrinsic values of outcomes. Then violations of monotonicity can result. For example, the dislike of decreases in income can be so extreme that people prefer an increasing wage profile to a decreasing one even if the latter at every time point yields at least as much money, and thereby larger budget sets to choose from (Benzion et al., 1989; Loewenstein and Sicherman, 1991; Hsee and Abelson, 1991). One reason can be that increases in salary are interpreted as a signal of success, and that signal may be more important than the absolute amount of money involved. Another reason can be that people find a reduction of standard of living especially hard to adapt to. A different kind of violation of monotonicity was found in experiments by Kahneman. He found that subjects violate temporal monotonicity when choosing between aversive episodes such as immersing hands in cold water (Kahneman et al., 1993). Shalev (1997) introduced a new functional for intertemporal preference that accommodates violations of monotonicity. He characterized the functional in terms of preference conditions.

Our model generalizes the models of Gilboa (1989) and Shalev (1997). A difference between our model and theirs is that we assume a one-stage model with linearity directly in outcomes, similar to Chateauneuf (1991). Gilboa and Shalev assumed a two-stage Anscombe–Aumann (1963) model where outcomes are probability distributions over prizes and expected utility applies to those. That is, Gilboa and Shalev use linearity with respect to probabilistic mixtures rather than with respect to quantitative outcomes. This is a difference in interpretation that does not essentially affect the mathematics, which is based solely upon linearity in outcomes.

Shalev introduced the following, basic, condition. Two profiles x, y are *sequentially comonotonic* if there are no two adjacent periods s, t such that $x_s > x_t$ and $y_s < y_t$. Note that the natural ordering of the time points $\{1, \dots, n\}$ is essential. A set of profiles is *sequentially comonotonic* if every contained pair of profiles is sequentially comonotonic. The preference relation $\succeq$ satisfies *sequential additivity* if $x \succeq y$ implies $x + z \succeq y + z$ whenever $\{x, y, z\}$ is sequentially comonotonic. Obviously, if profiles are comonotonic then they are also sequentially comonotonic. Hence, sequential additivity implies comonotonic additivity.

The functional used to value profiles in this section is of the form

$$\lambda_1 x_1 + \sum_{j=2}^{n} [\lambda_j x_j - \tau_j (x_{j-1} - x_j)^+] \tag{4}$$

where the λ_j's sum to one and for any real number β we define $\beta^+ = \max\{0, \beta\}$. We call such a functional a *sequential Choquet integral*.

Under Eq. (4), a regular weighting scheme $\sum_{j=1}^{n} \lambda_j x_j$ applies if the sequence is nondecreasing, in other words, if $x_{j-1} \leq x_j$ for all j. For decreases there is a "decision weight penalty", that is, if $x_{j-1} > x_j$, then decision weight τ_j is shifted from x_{j-1} to the lower outcome x_j, leading to subtraction of a term $\tau_j(x_{j-1} - x_j)$. In some cases, the τ_j's may be so large that violations of monotonicity result. The terminology in this interpretation is adapted from the example of income valuation, where preference is monotonic if income is increasing ($\lambda_j > 0$ for all j), but for decreases in salary a penalty is subtracted, i.e. $\tau_j > 0$ for all j. The following analysis also considers the general case in which some λ_j's and τ_j's may be negative. Observation 2 could be proved by deriving comonotonic additivity and then invoking Theorem 1. We present an independent proof in the main text because this proof clarifies the role of the parameters in the sequential Choquet integral.

Observation 2. *The functional in Eq. (4) is a signed Choquet integral.*

Proof. The "Choquet-integral" decision weights π_j (Eq. (2)) can be obtained as follows for a profile x, where we use the notational convention that $x_0 = -\infty$ and $x_{n+1} = \infty$. At first the decision weight, still to be modified, of each x_j is λ_j. If $x_{j-1} > x_j$, then τ_j is added to the decision weight of x_j (and similarly subtracted from the decision weight of x_{j-1}). If $x_j > x_{j+1}$, then τ_{j+1} is subtracted from the decision weight of x_j (and similarly added to the decision weight of x_{j+1}). The following decision weights result:

(i) $\pi_j = \lambda_j$ if $x_{j-1} \leq x_j \leq x_{j+1}$.

(ii) $\pi_j = \lambda_j + \tau_j$ if $x_{j-1} > x_j$ and $x_j \leq x_{j+1}$.

(iii) $\pi_j = \lambda_j - \tau_{j+1}$ if $x_{j-1} \leq x_j$ and $x_j > x_{j+1}$.

(iv) $\pi_j = \lambda_j + \tau_j - \tau_{j+1}$ if $x_{j-1} > x_j > x_{j+1}$.

Setting $\tau_{n+1} = 0$, the signed capacity satisfies $\nu(i) = \lambda_i - \tau_{i+1}$ for all singletons $\{i\}$. For a general set E, the signed capacity is given by

$$\nu(E) = \sum_{j \in E} \lambda_j - \sum_{j \in E: j+1 \notin E} \tau_{j+1}.$$

ν assigns value 0 to the empty set and value 1 to $\{1, \ldots, n\}$ and, hence, it is a signed capacity indeed. The verification that this capacity generates the appropriate decision weights is left to the reader. □

The sequential Choquet integral can also be used if subjects are especially sensitive to *changes* of outcomes in a profile, positive as well as negative. To illustrate this point, we rewrite the sequential Choquet integral as a weighted sum, adjusted for variations of any kind between successive terms. The following equation is equivalent to Eq. (4) after appropriate substitutions, described next.

$$p(s_1)x_1 + \sum_{j=2}^{n}(p(s_j)x_j + \delta_j|x_j - x_{j-1}|). \tag{5}$$

To establish equivalence with Eq. (4), rewrite Eq. (4) as

$$\lambda_1 x_1 + \sum_{j=2}^{n}\left[\lambda_j x_j - \frac{\tau_j}{2}|x_j - x_{j-1}| + \frac{\tau_j}{2}(x_j - x_{j-1})\right].$$

Then use the following substitutions: $\delta_j = -\tau_j/2$ for $j = 2, \ldots, n$, $p(s_1) = \lambda_1 - \tau_2/2$, $p(s_j) = \lambda_j + \tau_j/2 - \tau_{j+1}/2$ for $j = 2, \ldots, n-1$, and $p(s_n) = \lambda_n + \tau_n/2$. Gilboa (1989) used Eq. (5) for monotonic sequential Choquet integrals. Yet another, equivalent, formula was used by Shalev (1997, Theorem 1). The following theorem characterizes the sequential Choquet integral.

Theorem 4. *Let $\succeq$ be a binary relation on $\mathbb{R}^n$. It can be represented by a sequential Choquet integral with respect to a signed capacity if and only if*

(i) $\succeq$ *is a weak order*;
(ii) $\succeq$ *is continuous*;
(iii) $\succeq$ *is constant–monotonic*;
(iv) $\succeq$ *satisfies sequential additivity*.

The sequential Choquet integral is uniquely determined.

The functional, characterized in Theorem 4, reduces to Gilboa's (1989) functional if monotonicity is added; $x \succeq y$ whenever $x_j \geq y_j$ for all j. Gilboa used a somewhat different axiom than our sequential additivity or Shalev's sequential comonotonicity. He assumed, first, comonotonic independence as in Schmeidler (1989), which plays the same role as comonotonic additivity in our approach and implies a Choquet integral representation. Then a variation-preserving sure-thing principle was added. It requires, loosely speaking, that preference be unaffected if common outcomes on a connected set of time points are replaced by other common outcomes in such a manner that the variations between successive noncommon outcomes remain unaffected. The latter does permit changes in rank ordering, thus imposing the additional restrictions on the Choquet integral that reduce it to a monotonic sequential Choquet integral.

The case in which all λ_j's and τ_j's are nonnegative is most interesting for the application to income evaluation. This case is characterized as follows. The λ_j's are nonnegative if and only if monotonicity holds for all profiles x with $x_1 \leq \cdots \leq x_n$. The τ_j's (for $j \geq 2$) are nonnegative if and only if

$$\begin{aligned}
&(x-a,\ldots,x-a,\boldsymbol{x-a},\boldsymbol{x+b},x+b,\ldots,x+b)\\
&\quad\sim(x-a,\ldots,x-a,\boldsymbol{x},\boldsymbol{x},x+b,\ldots,x+b)\Rightarrow(x+b,\ldots,x+b)\\
&\quad\succeq(x+b,\ldots,x+b,\boldsymbol{x+a+b},\boldsymbol{x},x+b,\ldots,x+b),
\end{aligned}$$

where the $(j-1)$th and jth coordinate have been boldprinted and a and b are nonnegative. In this preference condition, the second preference has been generated by raising the first $j-1$ incomes by a term $a+b$. This implies a decrease in income in period j, which generates a lower valuation of the last profile. A preference condition characterizing negative δ_j's in Eq. (5), called variation aversion, is available in Gilboa (1989).

This section has presented a special case of the signed Choquet integral that is useful in intertemporal preference. The signed Choquet integral can be of use in other areas, such as welfare evaluations where an allocation (10, 10, 10, 10) (US\$ 10 for person $1,\ldots,4$) should sometimes be preferred to an allocation (11, 12, 13, 15) if the latter can arise envy and conflict (Crosby, 1976; Tversky and Griffin, 1991; Ng, 1997, p. 1849). The next section considers an application to asset pricing.

4. Nonmonotonicity in asset pricing

This section elaborates on an application in asset pricing. Here, the elements of $\{1,\ldots,n\}$ are states of nature. At date zero, assets can be traded. At a later date, date one, exactly one of the states of nature will turn out to be true, and the assets yield a pay off depending on the true state.

Asset trading often occurs through brokers or dealers charging a price for their intermediation. This implies that the functional that assigns a price to any asset portfolio is nonlinear. Indeed, the price to be paid to the broker for buying an asset is strictly larger than the price received from the broker when selling the same asset, the difference being the *bid-ask spread*. Chateauneuf et al. (1996) considered asset markets with a dealer charging bid-ask spreads, and showed how the asset prices can be represented by a Choquet integral with respect to a concave capacity. In the related area of insurance pricing, the development of premium systems based on Choquet integrals has recently gained considerable interest. Wang (1996) argued that when there is ambiguity regarding the loss distribution, or when there is considerable correlation between the individual risks, the traditional pricing principles may be inadequate to determine premiums that accurately cover the risk. Moreover, insurance price functionals should be subadditive in order to capture the pooling effect. Wang (1996) therefore proposed to use the *proportional hazard mean* as an alternative pricing rule and shows that it is equivalent to taking the Choquet integral of the insured risk with respect to a concave measure. Wang et al. (1997) gave an axiomatic characterization of insurance prices implying that the price of an insurance risk has a Choquet integral representation. The use of Choquet integrals in insurance pricing has been further developed by Wang and Dhaene (1998), Wang and Young (1998), and Hürlimann (1998).

The trade on asset markets or insurance markets is often subject to exogenous constraints on portfolio holdings such as leverage constraints or no-overinsurance constraints (i.e. an insurance contract for a certain risk can only be bought by agents who bear the risk). The following example shows that in the presence of exogenous trading constraints, the concept of Choquet pricing needs to be extended to allow for signed Choquet integrals in order to assure existence of equilibrium Choquet pricing rules. Indeed, the example shows that on markets with constraints on portfolio holdings and an intermediary or insurer charging subadditive, comonotonically additive premiums, an equilibrium price functional can be nonmonotonic. The equilibrium price of a portfolio then equals the signed Choquet integral of its pay off with respect to a concave set function. The standard general equilibrium model on incomplete asset markets on which this example is based can be found in Magill and Shafer (1991).

Example 1. We consider a two-period financial market model where trade takes place through an intermediary, and portfolio holdings are constrained by the prohibition of overinsurance.

There are three assets that can be traded in the first period (date zero) and pay off in the second period (date one). There are three states of nature in this second period, i.e. $n = 3$. The pay off of an asset can therefore be represented by a vector in $\mathbb{R}^3$. The pay off vectors for the three assets are given by $A^1 = (1, 1, 1)$, $A^2 = (1, 1, 0)$, and $A^3 = (0, 1, 1)$, respectively. Hence, asset 1 is a riskless bond that pays off one regardless of the true state at date one, asset 2 is an insurance contract that pays off one if state 1 or state 2 is the true state at date one and nothing otherwise, and asset 3 is an insurance contract that pays off one if state 2 or state 3 is the true state at date one and nothing otherwise.

For trading a portfolio $z = (z_1, z_2, z_3) \in \mathbb{R}^3$, i.e. buying z_1 units of asset 1 (or selling $-z_1$ if z_1 is negative), buying z_2 units of asset 2, and buying z_3 units of asset 3, the intermediary charges an amount $q(z) = \pi Az + \gamma((Az)_{\rho(1)} - (Az)_{\rho(2)})/3 + \gamma((Az)_{\rho(1)} - (Az)_{\rho(3)})/3$, where ρ is a permutation that is compatible with Az, $\pi \in \mathbb{R}^3$, and $\gamma \geq 0$. Thus, $q(z)$ consists of a linear part πAz, which is the "price", augmented by a subadditive, positive part representing the "risk-premium" charged by the intermediary. Because then $q(z) > -q(-z)$ for all portfolios with a risky pay off (i.e. with $(Az)_i \neq (Az)_j$ for some i, j), the intermediary makes a profit by buying this portfolio from one agent and selling it to the other. The two agents each have a 50% share in the intermediary's firm. Hence, the intermediary's profit, denoted π^{d}, is divided equally between the two agents after trade.

The no-overinsurance constraint implies that the amount of assets 2 and 3 that an agent is allowed to buy is bounded above, with the upperbound depending on the risk he bears. In this case, the agents are only allowed to buy portfolios satisfying $z_2 \leq 0.5$ and $z_3 \leq 0.5$. There is no exogenous constraint on trading the riskless asset. Hence, the *portfolio choice set* of each agent equals $Z^i = \mathbb{R} \times]-\infty, 0.5]^2$.

Since there are three states at date one, a consumption bundle of an agent consists of a vector $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_+$, where x_0 denotes the amount of money the agent owns at date zero, and x_j denotes the amount of money he will own at date one if state j occurs, $j = 1, 2, 3$. The endowment (before trading) of the agents is given by $w^1 = (5.5, 5, 4.5, 4)$ and $w^2 = (11.5, 12, 12.5, 13)$. The trade of portfolio $z^i \in Z^i$ by agent i has the following effect on his resources:

$$x_0^i = w_0^i - q(z^i) + \xi^i \pi^{\mathrm{d}}$$
$$x_1^i = w_1^i + z_1^i + z_2^i$$
$$x_2^i = w_2^i + z_1^i + z_2^i + z_3^i$$
$$x_3^i = w_1^i + z_1^i + z_3^i,$$

where $\xi^1 = 0.5$ and $\xi^2 = 0.5$ denote the shares of the respective agents in the intermediary's profit.

Now, using his initial resources w^i, the agent can trade asset portfolios $z = (z_1, z_2, z_3) \in Z^i$ in order to maximize his utility. Agent 1 has utility function $u^1(x_0, x_1, x_2, x_3) = 60\sqrt{x_0} + 7(\sqrt{x_1} + \sqrt{x_2}) + 31\sqrt{x_3}$. Agent 2 has utility function $u^2(x_0, x_1, x_2, x_3) = 15\sqrt{x_0} + 124\sqrt{x_1} + 28(\sqrt{x_2} + \sqrt{x_3})$. Let $x^i \in \mathbb{R}_+^4$ and $z^i \in Z^i$ denote the consumption bundle and the asset portfolio that agent i will choose as a result of his utility maximizing problem for a given π^{d}, for $i = 1, 2$. It can be shown that, for

$$\pi^{\mathrm{d}} = 0.1$$
$$\pi = (2, -1, 2)$$
$$\gamma = 0.1$$
$$x^1 = (16, 1, 1, 1); \qquad x^2 = (1, 16, 16, 16)$$
$$z^1 = (-3.5, -0.5, 0.5); \qquad z^2 = (3.5, 0.5, -0.5),$$

the market is in equilibrium. That is, both agents have maximized their utility, the market in assets and money clears (i.e. there is no excess demand or excess supply), and the intermediary's profit equals $\pi^{\mathrm{d}} = q(z^1) + q(z^2)$.

Define the set function ν on $2^{\{1,2,3\}}$ as follows: $\nu(\emptyset) = 0$, $\nu(\{1\}) = \nu(\{3\}) = 31/15$, $\nu(\{2\}) = -14/15$, $\nu\{(i, j)\} = \nu(\{i\}) + \nu(\{j\}) - 0.1$, for all $i \neq j$, and $\nu(\{1, 2, 3\}) = 3$. Then it immediately follows that $q(z) = \int (Az)\,\mathrm{d}\nu$ for all $z \in \mathbb{R}^3$, i.e. the equilibrium price of a portfolio equals the signed Choquet integral of its pay off. Moreover, it is clear that because $\nu(\{2\})$ is negative, this equilibrium price functional cannot be represented by a Choquet integral with respect to a monotone set function.

Notice that in the above example the price for insurance in state 2 (i.e. $\int (0, 1, 0)\,\mathrm{d}\nu = \nu(\{2\})$) is negative. Such a "free lunch" would clearly be incompatible with equilibrium in frictionless markets. The explanation is that the no-overinsurance constraints imply that the amount of insurance bought for state 2 is bounded above by the amount of insurance bought for state 1 plus 0.5, and by the amount of insurance bought for state 3 plus 0.5. Once one of these constraints becomes binding, purchasing extra insurance for state 2 is no longer costless. Consequently, the arbitrage possibility only yields bounded profits. This implies that, because agent 1 has a higher utility for money at date zero than for insurance at date one, it is optimal for him to sell insurance in state 2, rather than to buy it. Because the opposite holds for agent 1, equilibrium results.

5. Conclusion

Many properties of the Choquet integral, such as preference axiomatizations and concavity and convexity, naturally extend to signed Choquet integrals. Signed Choquet integrals can be used to model interactions in intertemporal preference that are so strong that monotonicity is violated. Signed Choquet integrals can also be useful in other areas, such as asset pricing. In general, they are applicable to aggregations where rank dependence is relevant but violations of monotonicity should be permitted.

Appendix A. Proofs

Proof of Corollary 1. There are many derivations of linear representations under an additivity axiom available in the literature. One deviation of our result from most results is that we do not have monotonicity. Further, we have to deal with comonotonicity restrictions.

Necessity of conditions (i)–(iii) is obvious; (iv), comonotonic additivity of $\succeq$, follows from comonotonic additivity of the signed Choquet integral. We therefore assume conditions (i)–(iv) and derive the representation. Real numbers (outcomes) are identified with constant profiles, that is, constant n-tuples, throughout. □

Lemma 2. *Each profile has a unique constant equivalent.*

Proof. Suppose, for contradiction, that there exists a profile x such that $x \succ \alpha$ for each constant profile α. For any natural number n, $x/n \succ \alpha/n$ (to wit, $x/n \preceq \alpha/n$ would imply, because of comonotonicity and comonotonic additivity, $2x/n \preceq \alpha/n + x/n \preceq 2\alpha/n$, and then, by induction, $m \times x/n \preceq m \times \alpha/n$ for all natural numbers m; for $m = n$ a contradiction would result). Because this holds for all real numbers α, we have $x/n \succ \beta$ for all real numbers β. Limit taking for $n \to \infty$ and continuity of $\succeq$ then imply $0 \succeq \beta$ for each real number β, contradicting constant monotonicity. A contradiction similarly results if $x \prec \alpha$ for all real α. Hence, for each profile x there exist real numbers α, β such that $\alpha \succeq x \succeq \beta$. By continuity, $\{\gamma \in \mathbb{R} : \gamma \succeq x\}$ and $\{\gamma \in \mathbb{R} : \gamma \preceq x\}$ are closed, we have already seen that both sets are nonempty, and, hence, because of connectedness of $\mathbb{R}$, they must intersect. The intersection contains the constant equivalent of x. It is unique because of constant monotonicity. *QED* □

Define, for each profile x, $V(x)$ as its constant equivalent. By constant–monotonicity, this function represents $\succeq$. Consider $V(x)$ and $V(y)$, for comonotonic x, y. Then $x \sim V(x)$ implies $x + y \sim V(x) + y$. (Note here that each constant profile is comonotonic with each other profile.) Now $y \sim V(y)$ implies $V(x) + y \sim V(x) + V(y)$. Transitivity implies $x + y \sim V(x) + V(y)$. Hence, $V(x + y) = V(x) + V(y)$. That is, V satisfies comonotonic additivity. V is continuous because $\{x : V(x) \geq \alpha\} = \{x : x \succeq (\alpha, \dots, \alpha)\}$ and $\{x : V(x) \leq \alpha\} = \{x : x \preceq (\alpha, \dots, \alpha)\}$ are closed for all α, because of continuity of $\succeq$. By Theorem 1, V is a signed Choquet integral.

For uniqueness of the representation, note that the signed capacity ν assigns value 1 to the set of all time points, implying that the signed Choquet integral assigns value α to each constant profile α. This uniquely defines the representing signed Choquet integral as the constant equivalent of each profile.

Proof of Lemma 1. (i) and (ii) follow from linearity of the signed Choquet integral with respect to the signed capacity, and substitution. For (iii), take any additive P' that is positive for all singletons and multiply it by a sufficiently large scalar to obtain P. (iv) follows from Eq. (3) when applied to disjoint sets A, B. □

Proof of Theorem 2. We first derive the equivalence of (i) and (ii). Because V is continuous, concavity holds if and only if midpoint concavity holds, we therefore consider midpoint concavity. The equivalence follows from:

$$V(x+y) \geq V(x) + V(y) \Leftrightarrow \frac{V(x+y)}{2} \geq \frac{V(x)+V(y)}{2} \Leftrightarrow V\left(\frac{x+y}{2}\right) \geq \frac{V(x)+V(y)}{2}$$

where the last step applies positive homogeneity to the left-hand side.

Equivalence of (i) and (iii) is well known, even holding for general functions V; it will not be proved here. We finally turn to the equivalence of (i) and (iv). It is first shown that concavity of V implies convexity of ν. Consider any two subsets A, B of $\{1, \ldots, n\}$. Then $\nu(A \cup B) + \nu(A \cap B) = V(1_{A\cup B}) + V(1_{A\cap B}) =$ (because $1_{A\cup B}$ and $1_{A\cap B}$ are comonotonic) $V(1_{A\cup B} + 1_{A\cap B}) =$ (because $1_{A\cup B} + 1_{A\cap B} = 1_A + 1_B$) $V(1_A + 1_B) \geq$ (because V is concave and, hence, as shown before, superadditive) $V(1_A)+V(1_B) = \nu(A)+\nu(B)$.

The implication (iv) $\Rightarrow$ (i) follows from Lemma 1: neither V nor convexity of ν are affected if ν is replaced by $\nu - \nu(\emptyset)$, and, hence, we may assume $\nu(\emptyset) = 0$. First take P as in (iii) of the lemma. Then convexity of ν implies convexity of $\nu + P$ by (i) of the lemma and, hence, monotonicity by (iv) of the lemma. The Choquet integral of $\nu + P$ is concave by Schmeidler's (1986) Proposition 3. Because of linearity of the signed Choquet integral with respect to the set-function, the Choquet integral of ν must also be concave. □

Proof of Theorem 3. If V is the minimum as described in the theorem, then V is a minimum of linear functionals and therefore is concave. By the implication (i) $\Rightarrow$ (iv) in Theorem 2, ν is convex.

Conversely, assume that ν is convex. Then V is the minimum of dominating linear functionals (Theorem 2). We have to prove that V is also the minimum of a subset of these dominating functions, i.e. of the Core integrals. This follows from Lemma 1:

Take P as in (iii) there; $\nu + P$ is convex by (i) and monotonic by (iv). Therefore, by Theorem 3 when applied to (monotonic) capacities (Anger, 1977; Huber, 1981), the Choquet integral of $\nu + P$ is the minimum of Core integrals of $\nu + P$. By subtracting P from the Core elements of $\nu + P$ and the P integral from their Choquet integrals and invoking (ii), we find that the signed Choquet integral of ν is the minimum of its Core integrals. □

Proof of Theorem 4. Throughout this proof, we use the notational convention that, for all $x \in \mathbb{R}^n$, $x_0 = -\infty$ and $x_{n+1} = \infty$. For any permutation ρ on $\{1, \ldots, n\}$, the *comonotonic cone* C_ρ associated with ρ is defined as $\{x \in \mathbb{R}^n : x_{\rho(1)} \geq \cdots \geq x_{\rho(n)}\}$. Hence, to each rank number j, ρ assigns the time point that has the jth place in the rank ordering with respect to outcomes. A set is comonotonic if and only if there is a permutation ρ such that the subset is contained in the comonotonic cone C_ρ (Wakker, 1989b, Lemma VI.3.3).

As a tool in the proof, a *change vector* c is defined as an $n+1$-tuple of plusses and minuses, with $c_1 = + = c_{n+1}$, and for each j either c_j is $+$ or c_j is $-$. For every change vector c, C^c is defined as the set containing all x such that for all $j \geq 2$, $x_j \geq x_{j-1}$ if $c_j = +$ and $x_{j-1} \geq x_j$ if $c_j = -$. C^c is sequentially comonotonic. Note that x is an element of several C^c's if $x_j = x_{j-1}$ for some j. The constant profiles are contained in all sets C^c. C^c is the union of all rank ordered cones whose change vector agrees with c. C^c is a convex cone because weak inequalities are preserved under convex combinations. □

Lemma 3. *A set $E \subset \mathbb{R}^n$ is sequentially comonotonic if and only if it is contained in one set C^c.*

Proof. For any change vector c, the set C^c, and thus any of its subsets, is sequentially comonotonic. Next assume that E is any sequentially comonotonic set. We define the change vector c. Consider three, exclusive and exhaustive, cases as follows.

(i) There is $x \in E$ with $x_i > x_{i-1}$. Then, by sequential consistency, $y_i \geq y_{i-1}$ for all $y \in E$. Define $c_i = +$.

(ii) There is $x \in E$ with $x_i < x_{i-1}$. Then, by sequential consistency, $y_i \leq y_{i-1}$ for all $y \in E$. Define $c_i = -$ in this case.

(iii) $x_i = x_{i-1}$ for all i. Then c_i can be chosen arbitrarily.

Now $E \subset C^c$. *QED* □

Lemma 4. *A sequential Choquet integral V is additive on any set C^c.*

Proof. Within one set C^c, we can use the same decision weights π_j, defined before Observation 2, for all Choquet integral calculations of V. Hence, V is additive there. *QED* □

After these preparations, we turn to the proof of the theorem. First assume the representation. Necessity of preference conditions (i), (ii), and (iii) is obvious, and (iv) follows from Lemmas 3 and 4.

We next assume conditions (i)–(iv) and derive the sequential Choquet integral representation. By the proof of Corollary 1, there exists a constant equivalent $V(x)$ for each profile x, and V represents preference. In fact, by the Corollary, V is a signed Choquet integral and satisfies additivity within each comonotonic cone. We prove, in a number of steps, that the decision weights of V are as for a sequential Choquet integral.

STEP 1. The decision weights depend only on the change vector c.

We demonstrate that V satisfies additivity on larger domains than comonotonic subsets, i.e. on whole sets C^c. By sequential additivity, for $x, y \in C^c$, $x \sim (\alpha, \dots, \alpha)$ and $y \sim (\beta, \dots, \beta)$ implies $x + y \sim (\alpha, \dots, \alpha) + y \sim (\alpha, \dots, \alpha) + (\beta, \dots, \beta)$, which implies $V(x + y) = \alpha + \beta = V(x) + V(y)$. In other words, the continuous functional V satisfies additivity on the convex cone C^c. Hence, it is linear there. Linearity implies that there are "decision weights" π_j^c such that $V(x) = \sum_{j=1}^n \pi_j^c x_j$ on C^c. Because of uniqueness, the decision weights of the signed Choquet integral on all comonotonic cones contained within C^c must coincide with the π^c's. Therefore, the term decision weight is justified for these numbers. The reasoning shows that the decision weights in the signed Choquet integral are completely determined by the change vector of a profile.

STEP 2. The jth decision weight depends only on c_j and c_{j+1}.

It was demonstrated before that a decision weight π_j in signed Choquet integrals does not depend on all of the rank ordering, but only on the "dominating set" of time points i that are rank ordered before j. Step 2 in this proof is similar. For j, and a change vector c', consider the convex cone of profiles x such that

- $x_1 = \cdots = x_{j-1}$;
- x_{j-1} and x_j are ordered in agreement with c'_j (hence, $x_j \geq x_{j-1}$ if $c'_j = +$, $x_j \leq x_{j-1}$ if $c'_j = -$);
- x_j and x_{j+1} are ordered in agreement with c'_{j+1};
- $x_{j+1} = \cdots = x_n$.

This cone is at least two-dimensional (unless $n = 1$, but for this case the theorem is trivial). It is the intersection of all cones C^c for which $c_j = c'_j$ and $c_{j+1} = c'_{j+1}$. Hence, $V(x) = ax_{j-1} + bx_j + dx_{j+1}$ for uniquely determined weights a, b, d on this cone. It follows that $\pi_j^c = b$ for any change vector c with $c_j = c'_j$ and $c_{j+1} = c'_{j+1}$ (and $a = \pi_1^c + \cdots + \pi_{j-1}^c$, $d = \pi_{j+1}^c + \cdots + \pi_n^c$). Hence, the decision weight π_j^c does not depend on all of the change pattern c, but only on c_j and c_{j+1}.

STEP 3. If c and c' differ only on the jth coordinate, say $c_j = +$ and $c'_j = -$, then $\pi_j^{c'} - \pi_j^c$ is independent of c and c' otherwise. This difference is called the jth *decision weight penalty* and is denoted by τ_j.

We next prove that τ_j is independent of c and c' indeed, for c and c' as just described. τ_j can be interpreted as the decision weight, subtracted from π_{j-1} if outcome x_j falls below x_{j-1}. Because each decision weight π_i depends only on the ith and $(i+1)$th coordinate of the change vector, $\pi_i^c = \pi_i^{c'}$ for all $i < j-1$ and $i > j$. This implies that $\pi_{j-1}^c + \pi_j^c = \pi_{j-1}^{c'} + \pi_j^{c'}$ and, hence, $\pi_{j-1}^{c'} = \pi_{j-1}^c - \tau_j$. In other words, changing the change vector c only on its jth coordinate generates a decision weight shift from the $(j-1)$th coordinate to the jth coordinate and leaves all other decision weights unaffected.

$\pi_j^{c'}$ and π_j^c and therefore also their difference τ_j depend only on c_j and c_{j+1}, because the decision weights of the jth coordinate are independent of c's coordinates other than c_j and c_{j+1}. In particular, τ_j is independent of $c_1, \ldots, c_{j-1}$. Similarly, $\pi_{j-1}^{c'}$ and π_{j-1}^c, thus also their difference τ_j, depend only on c_{j-1} and c_j. In particular, τ_j is independent of $c_{j+1}, \ldots, c_n$. We conclude that τ_j is independent of $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n$, consequently it is independent of c and c'.

STEP 4. Definition of λ_j's.

Define, for $c' = (+, \dots, +)$, $\lambda_j = \pi_j^{c'}$ for all j. In the context of income evaluation, these are the decision weights for the empirically most favorable case in which income always increases. The decision weight π_j^c depends only on c_j and c_{j+1}, and is as follows.

(i) If $c_j = +$ and $c_{j+1} = +$, then $\pi_j^c = \lambda_j$, by the definition of λ_j.

(ii) If $c_j = -$ and $c_{j+1} = +$, then $\pi_j^c = \lambda_j + \tau_j$, by (i) and Step 3.

(iii) If $c_j = +$ and $c_{j+1} = -$, then $\pi_j^c = \lambda_j - \tau_{j+1}$, by (i) and Step 3.

(iv) If $c_j = -$ and $c_{j+1} = -$, then $\pi_j^c = \lambda_j + \tau_j - \tau_{j+1}$, by (ii) (or (iii)) and Step 3.

By substitution (see also after Observation 2) it now follows that V is as in Eq. (4), which means that V is a sequential Choquet integral. Uniqueness follows from Corollary 1.

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