TESTING AND CHARACTERIZING PROPERTIES OF NONADDITIVE MEASURES THROUGH VIOLATIONS OF THE SURE-THING PRINCIPLE

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In expected utility theory, risk attitudes are modeled entirely in terms of utility. In the rank-dependent theories, a new dimension is added: chance attitude, modeled in terms of nonadditive measures or nonlinear probability transformations that are independent of utility. Most empirical studies of chance attitude assume probabilities given and adopt parametric fitting for estimating the probability transformation. Only a few qualitative conditions have been proposed or tested as yet, usually quasi-concavity or quasi-convexity in the case of given probabilities. This paper presents a general method of studying qualitative properties of chance attitude such as optimism, pessimism, and the "inverse-S shape" pattern, both for risk and for uncertainty. These qualitative properties can be characterized by permitting appropriate, relatively simple, violations of the sure-thing principle. In particular, this paper solves a hitherto open problem: the preference axiomatization of convex ("pessimistic" or "uncertainty averse") nonadditive measures under uncertainty. The axioms of this paper preserve the central feature of rank-dependent theories, i.e. the separation of chance attitude and utility.

Keywords: Risk attitude, rank-dependent, comonotonic, Choquet expected utility, prospect theory, pessimism, optimism, sure-thing principle.

1. INTRODUCTION

Under expected utility, risk attitudes are modeled solely through outcome sensitivity, i.e. curvature of utility. Classical economics thus identifies risk aversion with concave utility (Arrow (1953)), and the Pratt-Arrow utility index is used as a measure of risk aversion. Dissatisfaction with such an entangling of risk attitude and marginal utility, and empirical paradoxes, have led to a number of new theories. For risk (i.e. given probabilities), rank-dependent utility was introduced by Quiggin (1981). Schmeidler (1989) introduced rank-dependent utility for uncertainty (i.e. no probabilities of events need to be given). The theory was generalized to allow for a different treatment of gains and losses in cumulative prospect theory (Starmer and Sugden (1989), Luce and Fishburn (1991), Tversky and Kahneman (1992)).

In the rank-dependent models, risk/uncertainty attitudes consist of two independent and clearly separate components: (a) sensitivity towards outcomes, modeled through utility, and (b) sensitivity towards uncertainty (chance attitude), modeled through a nonadditive measure (capacity) under uncertainty and through a probability transformation under risk. In the context of risk, chance attitude is sometimes called probabilistic risk attitude. Descriptively, chance attitude seems to be as important as utility. Recent work suggests, for instance,

that there is an important role for chance attitude in insurance (Viscusi (1995), Wakker, Thaler, and Tversky (1997)). Selten, Sadrieh, and Abbing (1999) and Bleichrodt, van Rijn, and Johannesson (1999, p. 253) suggest that there is more curvature for probability than for moderate monetary outcomes or life-duration outcomes, respectively.

Most studies of chance attitude have considered decision under risk and here a host of results is available (Appendix A). In particular, Segal (1987) and Wu and Gonzalez (1996) used special forms of the conditions of this paper to characterize various properties of weighting functions. In most applications, probabilities are not given. Testable preference axiomatizations for uncertainty are therefore warranted.

This paper presents a general method of axiomatizing and testing properties of nonadditive measures ("capacities") through observable choice behavior. It demonstrates that several conditions, studied before in the literature and tested in experiments, are special cases of the general method. A unifying framework is thus obtained for the study of capacities. In particular, a preference axiomatization of convex (pessimistic) capacities is presented. Even though convexity is the most-studied property of capacities, a general preference axiomatization has not been known before. As demonstrated in the formal analysis of Sections 3 to 5, convex capacities naturally follow from (the common consequence version of) the Allais paradox. That the Allais—rather than the Ellsberg—paradox is relevant here is contrary to what has sometimes been thought.

This paper is based on three observations. First, decision under risk is the special case of decision under uncertainty where probabilities of events are given. Second, the Allais paradox pertains to uncertainty as well as to risk (MacCrimmon and Larsson (1979, p. 364−365), Tversky and Kahneman (1992, Section 1.3)). Third, whereas the Ellsberg paradox exhibits a relative phenomenon, i.e. more pessimism for unknown probabilities than for known probabilities, the Allais paradox exhibits pessimism in an absolute (noncomparative) sense.

The axiomatizations of convex capacities known up to now (see Section 2) do not disentangle utility from chance attitude and require concave or linear utility. The conditions presented in this paper characterize capacities independently of utility. They thus provide tools for analyzing chance attitude, the new component of risk attitude, while preserving a central feature of rank-dependent utility, namely the separation of chance attitude and outcome sensitivity. For decision under risk, Abdellaoui (2001) presented an alternative method of nonparametric qualitative tests of probability transformation that are also independent of outcome sensitivity. Abdellaoui's method is more complex than the one presented here but in return gives more powerful results, allowing for quantitative measurements of probability transformation.

The central result of this paper is Theorem 4.1. This theorem describes the general method of analyzing qualitative properties of capacities through simple violations of the sure-thing principle. The paper up to Theorem 4.1 prepares for the general method, and the rest of the paper describes various applications.

Appendix A presents empirical evidence and Appendix B presents formal proofs.

2. CHOQUET EXPECTED UTILITY AND CONVEX CAPACITIES

This section presents basic concepts of decision under uncertainty, the context assumed in the major part of the paper. S denotes a *state space*, with elements called *states* and subsets called *events*. S is infinite in Section 5 and can be finite or infinite in the rest of the paper. The *outcome* set is $\mathbb{R}$, designating money. The set of *acts* contains all finite-valued functions from S to $\mathbb{R}$. $(E_1, x_1; \ldots; E_n, x_n)$ denotes the act assigning outcome x_j to all states in event E_j, $j = 1, \ldots, n$. It is implicitly understood in this notation that $E_1, \ldots, E_n$ are disjoint events partitioning S and that $x_1, \ldots, x_n$ are outcomes.

$\succcurlyeq$ is the *preference relation* on the set of acts. Preferences are denoted $f \succcurlyeq g$, etc.; $\succ$ (*strict preference*), $\sim$ (*indifference*), $\preccurlyeq$, and $\prec$ are as usual. A function V *represents* $\succcurlyeq$ if V maps the set of acts to $\mathbb{R}$ and $f \succcurlyeq g \Leftrightarrow V(f) \geq V(g)$.

We assume *Choquet expected utility* (*CEU*) throughout this paper:

(i) A function $U: \mathbb{R} \to \mathbb{R}$ is given, the *utility function*.

(ii) A *capacity* W is given on S, i.e. W is defined on the collection of subsets of S with $W(\varnothing) = 0$, $W(S) = 1$, and $C \supset D \Rightarrow W(C) \geq W(D)$.

(iii) $\succcurlyeq$ is represented by $f \mapsto \int_S U(f(s)) \, dW(s)$,

the *Choquet expected utility* (CEU) of f (the integral is defined next).

To define the integral in (iii), let $d(\mathbf{A}, H) = W(\mathbf{A} \cup H) - W(H)$. It is implicit in this notation that $\mathbf{A}$ and H are disjoint. $d(\mathbf{A}, H)$ will be the *decision weight* of event $\mathbf{A}$ in what follows. Consider $f = (E_1, x_1; \ldots; E_n, x_n)$. The integral in (iii) is

$$(2.1) \qquad \sum_{j=1}^{n} \pi_j U(x_j)$$

where the π_j's are defined as follows. Take a permutation ρ on $\{1, \ldots, n\}$ such that $x_{\rho(1)} \geq \cdots \geq x_{\rho(n)}$. The *decision weight* $\pi_{\rho(j)}$ of event $E_{\rho(j)}$ (or of outcome $x_{\rho(j)}$) is defined as $d(E_{\rho(j)}, E_{\rho(1)} \cup \cdots \cup E_{\rho(j-1)})$, i.e. the marginal capacity contribution of event $E_{\rho(j)}$ to the events yielding better outcomes. If two acts f and g are *comonotonic* ($f(s) > f(t)$ and $g(s) < g(t)$ for no s, t), then the same partition $(E_1, \ldots, E_n)$ and the same permutation ρ, and therefore also the same decision weights π_j, can be used for f and g (Wakker (1989, Section VI.3)).

W is *convex* if $W(C) + W(D) \leq W(C \cup D) + W(C \cap D)$ for all events C, D. W is *concave* if the reversed inequality holds. Convexity can be rewritten as $W(D) - W(C \cap D) \leq W(C \cup D) - W(C)$. Equivalently, by substitution of $H = C \cap D$, $\mathbf{A} = D - C$, $I = C - D$ (where $-$ denotes set difference), convexity means that for all disjoint events $\mathbf{A}, H, I$,

$$(2.2) \qquad W(\mathbf{A} \cup H) - W(H) \leq W(\mathbf{A} \cup H \cup I) - W(H \cup I).$$

Equation (2.2) is equivalent to $d(\mathbf{A}, H) \leq d(\mathbf{A}, H \cup I)$, i.e.

$$(2.3) \qquad d(A, H) \text{ is nondecreasing in } H.$$

These inequalities correspond to nondecreasing differences of a convex probability transformation w, where $w(\mathbf{q} + h) - w(h)$ is nondecreasing in h. Convexity of the capacity means that an event receives more decision weight as its associated outcome is ranked lower. Convexity reflects a pessimistic attitude where more attention is paid to unfavorable than to favorable outcomes. Similar observations hold for concavity, with reversed inequalities and an optimism interpretation.

Capacities are usually assumed to be convex in the current literature. Many papers assume an Anscombe-Aumann (1963) two-stage model with unknown probabilities ("ambiguity") and deviations from expected utility in the first stage, and known probabilities and expected utility in the second stage. Convexity of the capacity can then be interpreted as "ambiguity aversion" or "uncertainty aversion" (Schmeidler (1989)). Convex capacities are used in statistics (Kadane and Wasserman (1996, p. 1251)) and ensure the existence of equilibria in zero-sum games (Debreu (1952, Section 4)). In artificial intelligence, belief functions are important. They are a subclass of convex capacities. In empirical studies, the predominantly found pattern is not convexity but an inverse-S shape (Section 5).

Early preference axiomatizations of convex capacities applied only to linear utility (Schmeidler (1989), Chateauneuf (1991)). Chew, Karni, and Safra (1987) characterized joint concavity of utility and convexity of the capacity for risk; Chateauneuf and Tallon (1998) extended this result to uncertainty. These results do not disentangle chance attitude and utility. Tversky and Wakker (1995) gave preference axiomatizations for "bounded subadditivity" that are independent of utility. The mathematical principle underlying the preference conditions of this paper was described by Wakker (1986, Section VI.11). Special versions of these conditions were tested empirically by Wu and Gonzalez (1999). They showed that the conditions are necessary for convex capacities. This paper generalizes their conditions and shows that the resulting conditions are not only necessary, but also sufficient, so that axiomatizations are obtained.

3. CONVEX CAPACITIES AS A GENERALIZATION OF THE ALLAIS PARADOX

This section demonstrates how convex capacities naturally arise from the Allais paradox, starting with some notational conventions for the figures.[2] In Figure 1, H, $\mathbf{A}$, and I are three events. One and only one of these events will obtain and it is unknown to a decision maker which event that will be. It is always assumed without further mention that the events partition the universal event in such illustrations of acts. Figure 1a depicts an indifference between a left act, yielding m (i.e., $\$m$) if H obtains, $\mathbf{m}$ if $\mathbf{A}$ obtains, and 0 if I obtains, and a right act interpreted similarly. In Figure 1b, the common outcome 0 at I has been changed into the common outcome m, and a weak preference is depicted.

[2] A similar presentation, using the Allais paradox as a starting-point for deriving preference conditions, was used by Segal (1987) for decision under risk; see also Conlisk (1989, second test).
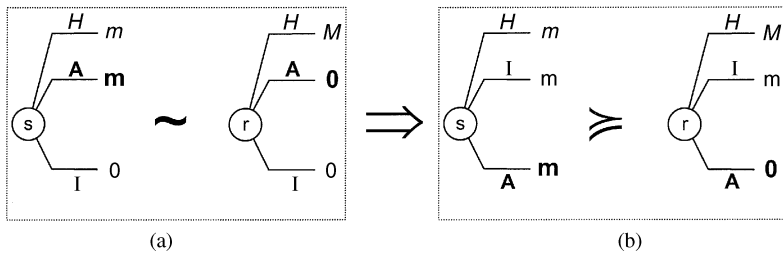
FIGURE 1: ($M \geq m \geq 0$). Variation on the Allais paradox: the certainty effect.

Let me explain why the I branch has been moved from the lowest to the middle position in Figure 1b. For a given gamble, events can be rank-ordered according to the desirability of their associated outcomes. Events are depicted higher as their outcomes are rank-ordered higher. Hence, the I branch has been moved from the lowest to the middle position in Figure 1b. Event H is always associated with <u>H</u>ighest outcomes in this paper. For the outcome resulting under event I it is always <u>I</u>rrelevant which gamble is chosen. We will study the change of event **A**'s decision weight induced by the change of event **A**'s ranking position. As a convention, the left gamble is the <u>S</u>afer one (Ⓢ) and the <u>R</u>ight gamble is more <u>R</u>isky (ⓡ).

Figure 1 has the structure of the Allais paradox. In the Allais paradox, events H, **A**, and I have probabilities *.10*, **.01**, and .89, respectively, M is five million dollars, and m is one million dollars. For these values, people typically have a strict preference $\prec$ in Figure 1a and a reversed strict preference $\succ$ in Figure 1b, demonstrating that the replacement of the common outcome leads to a stronger preference for the left act in Figure 1b. This change in preference can be ascribed to the certainty effect: In Figure 1b, the left act yields a certain outcome; the absence of risk adds to the preference value of the act. These preferences imply a violation of expected utility because they have been affected by a change of the common outcomes, thus violating Savage's (1954) *sure-thing principle*. After decreasing the payment M (five million dollars) to a lower amount such that indifference results in Figure 1a, the weak preference in Figure 1b is natural.

If the Allais paradox is not an isolated example but reflects a general principle of decision under uncertainty, then it may be conjectured that the implication in Figure 1 holds for all outcomes $M \geq m \geq 0$ and partitions $\{H, \mathbf{A}, I\}$ of the state space. This preference condition, formalizing the certainty effect, is called *upper subadditivity*. Many studies have tested and confirmed upper subadditivity for risk (Appendix A). For uncertainty, upper subadditivity was empirically found by Tversky and Kahneman (1992); see Example A.1. Several studies suggest that upper subadditivity is more pronounced for uncertainty than for risk (Hogarth and Kunreuther (1985), Kahn and Sarin (1988), Tversky and Fox (1995), Fox, Rogers, and Tversky (1996)).
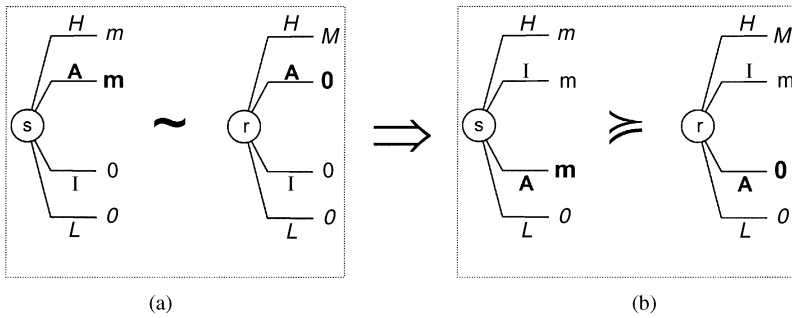
FIGURE 2: ($M \geq m \geq 0$). Generalization of the Allais paradox: pessimism.

The Allais paradox may be caused by a phenomenon more general than the certainty effect, i.e. it may be caused by a kind of pessimism. Pessimism entails more attention for events as their outcomes are ranked lower. This phenomenon can also explain why event **A** receives more attention in Figure 1b than in Figure 1a. Under the pessimism interpretation, the implication in Figure 2 becomes reasonable. In Figure 2, an event L has been added that yields a common outcome lower than all other outcomes. As a notational convention, event L is associated with Lowest outcomes throughout this paper. Event **A** is ranked lower in Figure 2b than in Figure 2a, although now, contrary to Figure 1, it is not ranked lowest and no certainty results in Figure 2b. The implication of Figure 2 is imposed for all $M \geq m \geq 0$ and partitions $\{H, \mathbf{A}, \mathrm{I}, L\}$ of the state space, and is formally called *pessimism*. Pessimism is more restrictive than upper subadditivity and includes Figure 1 as the special case where L is empty.

Example A.2 describes an empirical test of the general condition of Figure 2. We now proceed with a theoretical analysis of the choices in Figure 2. It is first demonstrated that convexity of the capacity implies pessimism. Then the reversed implication is explained informally. For the first implication, we substitute Choquet expected utility in Figure 2a:

$$(3.1) \qquad \pi_H U(m) + \boldsymbol{\pi_A} U(\mathbf{m}) + \pi_\mathrm{I} U(0) + \pi_L U(0)$$

$$= \pi_H U(M) + \boldsymbol{\pi_A} U(\mathbf{0}) + \pi_\mathrm{I} U(0) + \pi_L U(0),$$

where the π's, the decision weights, are depicted in Figure 3a. The rank-ordering of events is from left (high) to right (low) in Figures 3a and 3b. The same decision weights can be used for both acts in Figure 2a because the acts are comonotonic.

Substitution of Choquet expected utility in Figure 2b, where the acts are also comonotonic, implies

$$(3.2) \qquad \pi_H U(m) + \overline{\pi}_\mathrm{I} U(\mathrm{m}) + \boldsymbol{\overline{\pi}_A} U(\mathbf{m}) + \pi_L U(0)$$

$$\geq \pi_H U(M) + \overline{\pi}_\mathrm{I} U(\mathrm{m}) + \boldsymbol{\overline{\pi}_A} U(\mathbf{0}) + \pi_L U(0),$$
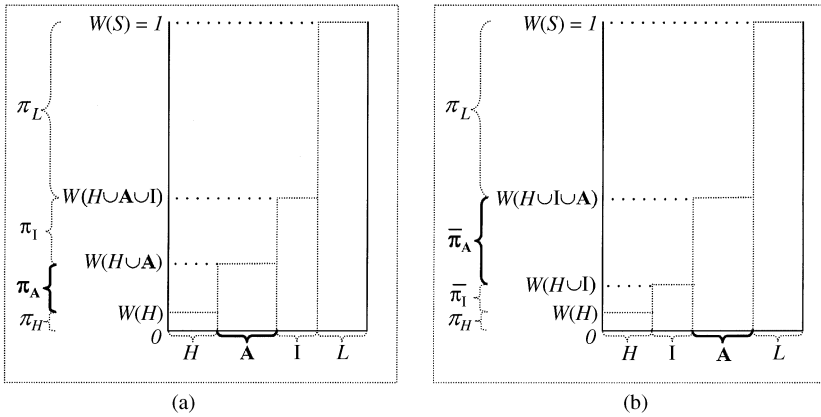
FIGURE 3: $\overline{\pi}_{\mathbf{A}} \geq \pi_{\mathbf{A}}$ for convex capacities.

where the π's, the decision weights, are depicted in Figure 3b. We rewrite equation (3.1) as

$$(3.3) \qquad \pi_{\mathbf{A}}(\mathbf{U(m)} - \mathbf{U(0)}) - \pi_H(U(M) - U(m)) = 0$$

and equation (3.2) as

$$(3.4) \qquad \overline{\pi}_{\mathbf{A}}(\mathbf{U(m)} - \mathbf{U(0)}) - \pi_H(U(M) - U(m)) \geq 0.$$

Equation (3.3) implies equation (3.4) whenever $\overline{\pi}_{\mathbf{A}} \geq \pi_{\mathbf{A}}$. Convexity of the capacity (Equation (2.2)) means exactly that $\overline{\pi}_{\mathbf{A}} \geq \pi_{\mathbf{A}}$; hence, convexity of the capacity implies the pessimistic preference behavior in Figure 2.

The reversed implication also holds under some technical conditions. That is, if the capacity is not convex, then a violation of the implication in Figure 2 can be constructed. The result is presented formally in Theorem 5.2.(i). Let me briefly comment here on the major technical difficulty in the derivation. Equations (3.3) and (3.4) imply $\overline{\pi}_{\mathbf{A}} \geq \pi_{\mathbf{A}}$ only if $U(m) - U(0) > 0$. For some events $H, \mathbf{A}, \mathrm{I}, L$, however, no outcomes $M \geq m > 0$ exist to give the indifference in Figure 2a, for instance if H is empty. In such cases, other solutions must be found, based on convergence arguments that invoke events for which appropriate $M \geq m > 0$ can be found.

4. THE GENERAL METHOD OF WEAKENING THE SURE-THING PRINCIPLE

This section presents the general method of analyzing capacities in rank-dependent theories. The characteristic feature of these theories is the rank dependence of decision weights, leading to special predictions of deviations from expected utility. Hence, our preference conditions are based on that rank dependence. They will all describe special kinds of violations of the sure-thing principle.

Figure 4 presents a general method of comparing the decision weight of an event $\mathbf{A}$ in one rank-ordering position to its weight in another rank-ordering
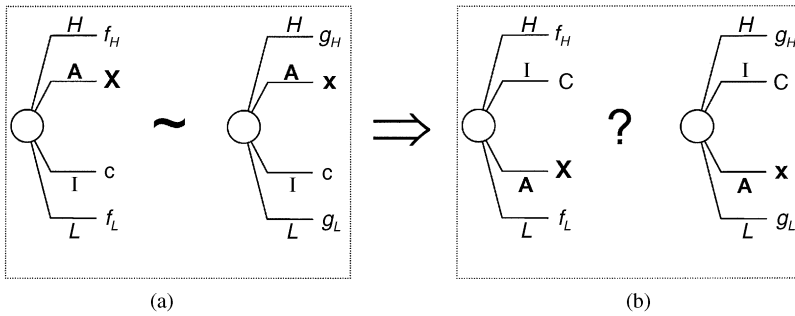
FIGURE 4: $(X \geq x)$. At the question mark a preference is to be substituted.

position, by letting an event I with a common outcome "pass by event **A**" in rank-ordering. The method was described by Wakker (1986, Condition VI.11.1). The left act in Figure 4a is described by the restriction of an act f for the events $H(f_H)$ and $L(f_L)$ (assigning $f(s)$ to each s in H and L); it is constant and equal to outcome **X** in event **A**, and it is constant and equal to outcome c in event I. The other acts are defined similarly. For the left act in Figure 4a, $f(s) \geq \mathbf{X} \geq \mathrm{c} \geq f(t)$ for all $s \in H$ and $t \in L$. Similar orderings of outcomes hold for other acts. These orderings follow from our convention of depicting events higher as they are rank-ordered higher. The acts in Figure 4a are comonotonic and so are the ones in Figure 4b.

Assume that $\mathbf{X} > \mathbf{x}$. The analysis of Figure 4 in terms of CEU is similar to the analysis of Figure 2 and is described briefly. The CEU difference between the left and right act in Figure 4b differs from that in Figure 4a (which is zero) only regarding the contribution of event **A**, for which the decision weight has changed from $\pi_\mathbf{A}$ to $\overline{\pi}_\mathbf{A}$ (see Figure 3). Hence, the following substitutions are possible for the question mark in Figure 4b, corresponding to a nonnegative, zero, or nonpositive CEU difference, respectively:

$$[? = \succcurlyeq] \Leftrightarrow \overline{\pi}_\mathbf{A} \geq \pi_\mathbf{A};$$

$$[? = \sim] \Leftrightarrow \overline{\pi}_\mathbf{A} = \pi_\mathbf{A};$$

$$[? = \preccurlyeq] \Leftrightarrow \overline{\pi}_\mathbf{A} \leq \pi_\mathbf{A}.$$

Figure 4 thus gives a precise measurement tool for the change of decision weights and thereby, indirectly, for the convexity and concavity of the capacity. Several empirical investigations derived such implications and then tested them, without studying reversed implications; see Camerer (1989, Section 1.7), Starmer (1992, Section 2.iv), Wakker, Erev, and Weber (1994, Section 4), Wu and Gonzalez (1996, Propositions 1 and 3; 1998), and Birnbaum and McIntosh (1996, p. 93−94) for risk, and Fennema and Wakker (1996, Section 1) and Wu and Gonzalez (1999, Section 2.3) for uncertainty. A few papers, only for risk, derived reversed implications (Segal (1987), Wu and Gonzalez (1996)). That is, little is available to show that the conditions *characterize* the properties of probability

transformations and capacities and in this sense provide critical tests. Providing such characterizations and explaining the underlying general method of analyzing capacities is the purpose of this paper.

Theorem 4.1 summarizes the discussion for convexity. It is convenient to also consider the case $\mathbf{X} = \mathbf{x}$ in the preference conditions of this paper. Indifference then always results in Figure 4b, and the preference conditions are vacuously satisfied under CEU. Adding this case therefore does not change the logical force of the preference conditions and, hence, the case is also mentioned in Theorem 4.1.

THEOREM 4.1 (Central Theorem): *Assume that CEU holds*.

(i) *Convexity of the capacity implies weak preference for the left act in Figure 4b* (*i.e.*, ? *is* $\succcurlyeq$, *also if $X = x$*). *Conversely, for every triple of disjoint events $H, \mathbf{A}, I$ for which $f, g, \mathbf{X} > \mathbf{x}$, and c can be found such that the indifference holds in Figure 4a, weak preference for the left act in Figure 4b implies $d(\mathbf{A}, H \cup I) \geq d(\mathbf{A}, H)$.*

(ii) *A similar claim holds for concave instead of convex capacities* (*substitute concave for convex*, $\leq$ *for* $\geq$, *and reverse preferences*).

The next section considers special cases of Figure 4. One variation of Figure 4 is particularly useful for experiments. It results when there is a reversed preference $\preccurlyeq$ instead of indifference $\sim$ in Figure 4a, and a preference $\succcurlyeq$ in Figure 4b. By reasonings similar to those given before, $d(\mathbf{A}, H \cup I) \geq d(\mathbf{A}, H)$ is implied, providing evidence for convexity. Such observations are easier to find experimentally because preferences are easier to observe than indifferences. Hence, most experimental papers have tested such versions of Figure 4. The resulting preference condition (a weak preference in Figure 4a, rather than just indifference, should imply the *same* preference in Figure 4b) is logically stronger than the conditions used in our theorems but is still necessary. Accordingly, it gives weaker characterization theorems and therefore it has not been used. Similar observations, with preferences reversed, apply to concavity.

5. CHARACTERIZING VARIOUS FORMS OF CAPACITIES FOR RICH STRUCTURES

In the absence of a rich structure of outcomes and events, there may not exist many configurations of the type depicted in Figure 4 (Wakker (2001)). In such a case, a requirement of specific preferences in Figure 4b may be too weak to imply the desired properties of the capacity. We therefore add a continuum assumption. As a preparation, the capacity W is *solvable* if, for each pair of events $B \subset D$ and $W(B) < \gamma < W(D)$, there exists an event C such that $B \subset C \subset D$ and $W(C) = \gamma$.

ASSUMPTION 5.1 (Continuum Assumption): *CEU holds with a solvable capacity and with a continuous and strictly increasing utility.*

Solvability implies that the state space is infinite. It is satisfied in Gilboa (1987, "convex-ranged") and also in Savage (1954), and holds for decision under

risk if the probability transformation is continuous. Preference conditions characterizing Assumption 5.1 are obtained by imposing the axioms of both Gilboa (1987) and Wakker (1993, Theorem 2.13). Given continuity in outcomes, solvability of the capacity is equivalent to the existence of an event B with $A \supset B \supset C$ and $(B, x; S - B, y) \sim f$ whenever $(A, x; S - A, y) \succ f \succ (C, x; S - C, y)$, $x \succ y$, and $A \supset C$.

Under Assumption 5.1, pessimism (violations of the sure-thing principle only as permitted in Figure 2) characterizes convexity of the capacity. Concavity is characterized by *optimism*, which requires that the implication in Figure 2 holds with reversed preference $\preccurlyeq$ instead of $\succcurlyeq$ in Figure 2b, whenever $M \geq m \geq 0$ and $\{H, \mathbf{A}, \mathbf{I}, L\}$ partitions the state space. The condition reflects the optimistic attitude of paying more attention to events as they are rank-ordered higher.

THEOREM 5.2 (Characterizing Convexity and Concavity): *Under the Continuum Assumption* 5.1:

(i) *The capacity is convex if and only if* $\succcurlyeq$ *exhibits pessimism* (*the implication in Figure* 2).

(ii) *The capacity is concave if and only if* $\succcurlyeq$ *exhibits optimism*.

Most papers assume convexity because of its importance in theoretical derivations. However, whereas convexity would predict an underweighting of unlikely events when associated with highest outcomes, experiments suggest the opposite: Such events are overweighted. The phenomenon suggests an "inverse-S shaped" capacity (Tversky and Fox (1995), Fox, Rogers, and Tversky (1996), Fox and Tversky (1998), Kilka and Weber (1998), Viscusi and Chesson (1999), and Wu and Gonzalez (1999) for uncertainty; see the end of Appendix A for risk). A remarkable exception among the theoretical studies is Karni and Safra (1990), who already discussed inverse-S shapes.

Inverse-S means that the capacity assigns relatively high values to unlikely events and relatively low values to likely events. For risk, the condition entails an inverse-S shaped graph of the probability transformation indeed. The condition implies extremity-oriented behavior, where high decision weights are assigned to the highest and lowest outcomes and low decision weights to the intermediate outcomes. The inverse-S pattern resolves the classical economic paradox of the simultaneous existence of gambling and insurance. Under cumulative prospect theory (Tversky and Kahneman (1992)), the two phenomena are explained by the same cause: the overestimation of unlikely events.

The inverse-S property is based on the observation that people deviate from expected utility not only because of a general dislike for risk and uncertainty (pessimism etc.), but also because of insensitivity. People are not sufficiently sensitive towards changes in intermediate levels of likelihood and, for instance, do not sufficiently distinguish between a .6 and a .8 level probability. This insensitivity is more pronounced for unknown probabilities than for known probabilities. It causes a sort of regression to the mean. This regression is not an artifact from our statistical analysis but it is a real psychological phenomenon

occurring in people's minds, and therefore worthwhile being incorporated into our models of human behavior.

A first way to model inverse-S was proposed by Bell (1985) and Cohen (1992), who deviated from expected utility by overweighing extreme outcomes. A second approach was adopted by Tversky and Wakker (1995) and Tversky and Fox (1995), who imposed "subadditivity" of the capacity for unlikely events and, similarly, subadditivity of the dual capacity. This paper follows a third approach, the one most commonly adopted (Wu and Gonzalez (1996, 1998, 1999)): Concavity is imposed on the unlikely events and convexity on the likely events. Henceforth, such capacities are called *cavex*. As a preparation, we discuss a technique for imposing conditions on "intervals" of events.

Event C is *revealed more likely* than $D, C \succcurlyeq D$, if there exist outcomes $Z > z$ such that $(C, Z; S - C, z) \succcurlyeq (D, Z; S - D, z)$. Under CEU, this relation on the events is a weak order, represented by the capacity. The notation $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ is as usual. For events $C \preccurlyeq D$, $[C, D]$ denotes the set of events E for which $C \preccurlyeq E \preccurlyeq D$. W is *convex on* $[C, D]$ if $d(\mathbf{A}, H \cup \mathbf{I}) \geq d(\mathbf{A}, H)$ whenever $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathbf{I} \preccurlyeq D$. The latter restriction guarantees that all arguments of W that play a role in the decision weights (events $\mathbf{A} \cup H \cup \mathbf{I}$, $H \cup \mathbf{I}$, $\mathbf{A} \cup H$, and H) are contained in $[C, D]$. W is *concave on* $[C, D]$ if $d(\mathbf{A}, H \cup \mathbf{I}) \leq d(\mathbf{A}, H)$ whenever $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathbf{I} \preccurlyeq D$. The preference conditions are adapted similarly; $\succcurlyeq$ satisfies:

- *pessimism on* $[C, D]$ if the implication in Figure 2 holds whenever $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathbf{I} \preccurlyeq D$;
- *optimism on* $[C, D]$ if the implication in Figure 2, with reversed preference in Figure 2b, holds whenever $C \preccurlyeq H \preccurlyeq H \cup \mathbf{A} \cup \mathbf{I} \preccurlyeq D$.

Restrictions of preference conditions to intervals were presented by Wu and Gonzalez (1996, 1998), Prelec (1998), and Abdellaoui (2001) for risk, and by Fennema and Wakker (1994) and Tversky and Wakker (1995) for uncertainty.

LEMMA 5.3: *Under the Continuum Assumption 5.1:*
(i) *W is convex on $[C, D]$ if and only if $\succcurlyeq$ exhibits pessimism on $[C, D]$;*
(ii) *W is concave on $[C, D]$ if and only if $\succcurlyeq$ exhibits optimism on $[C, D]$.*

Characterizations of cavex capacities immediately follow from the preceding lemma.

THEOREM 5.4 (Characterizing Cavexity): *Under the Continuum Assumption 5.1: W is concave on $[\varnothing, C]$ and convex on $[C, S]$ if and only if $\succcurlyeq$ exhibits optimism on $[\varnothing, C]$ and pessimism on $[C, S]$.*

Empirical relevance, and several related and dual results, are discussed by Tversky and Wakker (1995). I note here only that their preference conditions all have the structure of Figure 4.

We next apply the preceding results to decision under risk, where preferences are defined over probability distributions $(p_1, x_1; \ldots; p_n, x_n)$ yielding outcome x_j

with probability p_j, $j = 1, \ldots, n$. Formally, decision under risk can be considered the special case of decision under uncertainty where the state space is $[0, 1]$, endowed with the uniform distribution P, and acts are indifferent whenever they generate the same probability distribution over outcomes. Rank-dependent utility is the special case of Choquet expected utility where the capacity W on the state space is $w \circ P$ for a strictly increasing continuous weighting function w with $w(0) = 0$ and $w(1) = 1$ (Wakker (1990)). It is elementarily verified that convexity, concavity, and, similarly, the inverse-S condition, of w are then equivalent to the same conditions of W. Hence, the preference axiomatizations of this section immediately result in corresponding preference axiomatizations for risk, simply by replacing events by their probabilities. For example, w is convex if and only if the implication of Figure 5 holds for all $M \geq m \geq 0$. The other results for uncertainty can similarly be applied to risk.

The preceding discussion of risk may highlight two points. First, uncertainty is important not only in a practical sense, as probabilities are rarely known, but also in a formal sense, because the results for uncertainty immediately imply the corresponding results for risk. Once this point is understood, the extension of the Allais paradox to uncertainty becomes natural, leading to the second point: The Allais paradox reveals a pessimistic attitude that pertains as much to uncertainty as to risk.

6. ABSOLUTE AND RELATIVE PESSIMISM AND THEIR LOGICAL INDEPENDENCE

It has sometimes been thought that the Allais paradox exhibits a phenomenon characteristic of risk and the Ellsberg paradox a phenomenon characteristic of uncertainty. This paper has demonstrated that the Allais paradox reflects pessimism in an "absolute" sense, for risk as well as for uncertainty, leading to convex capacities. The Ellsberg paradox suggests that there is more pessimism for unknown probabilities than for risk, i.e. it exhibits pessimism in a relative sense. The next two examples illustrate the difference.

EXAMPLE 6.1 (Relative Pessimism Does Not Imply Absolute Pessimism): Assume that an urn contains 100 balls and that x is an unknown number between 0 and 50. The urn contains $50-x$ purple balls, x blue balls, x green
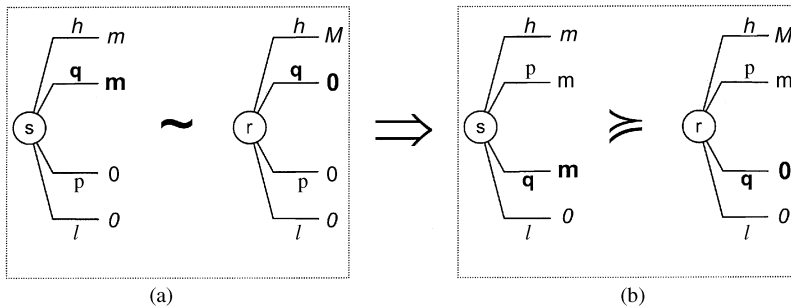

FIGURE 5: ($M \geq m \geq 0$). Pessimism for risk.

balls, and 50−x red balls. One ball will be drawn at random and its color (P, B, G, or R) observed. Assume that the decision maker maximizes expected utility with respect to known probabilities and is more averse to unknown probabilities than to known probabilities.

Let $M > 1$ be such that the indifference in Figure 6a holds. The left gamble in Figure 6a yields $1 with probability .5, and the right gamble yields M with an unknown probability between 0 and .5. Aversion to unknown probabilities implies that M must be relatively high to obtain the indifference. The preference in Figure 6b, violating our pessimism preference condition (Figure 2), seems natural. In Figure 6b, both gambles yield an outcome of at least 1 with a probability of .5 or more. Here, because of aversion to unknown probabilities, event B does not have the impact it had in Figure 6a. Under CEU, it seems natural that $W(P) < .25$, $W(P,B) = W(P,G) = .5$, and $.5 < W(P, G, B) < .75$. Then the decision weight of event B in Figure 6a, $d(\mathbf{B}, P)\,(= W(P, B) - W(P))$, exceeds .25, but in Figure 6b it is $d(\mathbf{B}, P \cup G)\,(= W(P, G, B) - W(P, G))$ which is below .25. An increase of the second argument of $d(\mathbf{B}, H)$ has led to a decrease of the weight, i.e. equation (2.3) is violated. W is not convex.

EXAMPLE 6.2 (Absolute Pessimism Does Not Imply Relative Pessimism): Let the urn and balls be as in Example 6.1. Consider the following capacity. $W(\varnothing) = 0$, $W(P) = W(B) = W(G) = W(R) = 1/100$, $W(P, B) = W(P, G) = W(B, R) = W(G, R) = 1/20$, $W(P, R) = W(B, G) = 1/10$, $W(P, B, G) = W(P, B, R) = W(P, G, R) = W(B, G, R) = 1/2$, and $W(P, B, G, R) = 1$. Because of the small capacities of small events, the decision weight $d(A, H)$ of each event A is increasing in H. W is therefore convex and pessimism holds. However, any bet on a two-color event with unknown probability is preferred to any bet on a two-color event with known probability. The decision maker is pessimistic but prefers unknown probabilities to known probabilities.

Pessimism as defined in this paper can also be formulated as a relative concept, i.e. being more pessimistic than subjective expected utility. Similarly, absolute risk aversion is sometimes formulated as being more risk averse than expected value maximization. There are, however, some drawbacks to a subjective model as a neutrality benchmark. The subjective expected utility maximiza-
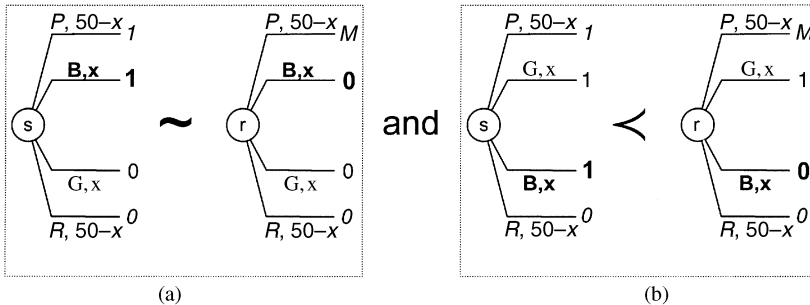

FIGURE 6: ($M > 1$). Relative pessimism but no absolute pessimism.

tion is not directly observable because neither is it objectively given, as is expected value maximization, nor can it be directly revealed from choices of the decision maker under consideration. The benchmark should probably be ascribed to another, hypothetical, agent. This other agent may have the same utility function as the decision maker but it is not clear what his *subjective* probabilities should be. Ghirardato and Marinacci (1999) therefore chose multiple levels of neutrality, i.e. all subjective expected utility maximizers. The present paper instead adopts the absolute interpretation of pessimism.

Contrary to the interpersonal comparisons of capacities on the same events to which we alluded above, the Ellsberg paradox concerns an intrapersonal comparison, comparing behavior revealed by one decision maker with respect to different events (hence, directly observable). It suggests that people are more pessimistic for events with unknown probabilities than for events with known probabilities. Tversky and Wakker (1995, Section 6) considered intrapersonal comparisons of different events regarding two aspects: "source sensitivity" and "source preference." The two Ellsberg paradoxes suggest source preference for known probabilities over unknown probabilities, a condition that corresponds but is not identical with being more convex (Epstein (1999)). In addition, Tversky and Wakker (1995, Sections 4 and 5) analyzed interpersonal comparisons of chance attitude.

The terms uncertainty aversion and ambiguity aversion have been used in the literature both for the absolute version of pessimism, designating convexity of the capacity (or, similarly, upper subadditivity or source preference in Tversky and Wakker (1995), or nonempty core in Ghirardato and Marinacci (1999)), and for the relative version, designating more pessimism for unknown probabilities than for known probabilities (Epstein (1999), who used subjective instead of known probabilities). Which term is most appropriate for which concept is a terminological issue. If uncertainty is taken to comprise both risk and ambiguity, then uncertainty aversion seems to be most suited for the absolute concept and ambiguity aversion for the relative concepts.

In Schmeidler's (1989) two-stage model, as in the multiple priors model, expected utility is assumed for risk. Then the absolute and relative versions of pessimism "happen to be" equivalent, and the two-urn Ellsberg paradox also leads to convex capacities. This equivalence may have given rise to misunderstandings and confusions about the absolute and relative concepts. In general, these concepts must be distinguished. Preference conditions for relative "subadditivity," comparing risk to uncertainty, were analyzed theoretically by Tversky and Wakker (1995) and were tested empirically by Tversky and Fox (1995). The present paper has concentrated on pessimism and other conditions in an absolute sense.

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APPENDIX A: EMPIRICAL DEMONSTRATIONS

The general method of analyzing capacities, explained in Section 4 and used in the theoretical analysis of Section 5, is well suited for empirical investigations. In fact, many conditions tested in the literature are special cases of this general method. Some examples are described next. The empirical evidence presented in this paper concerns only gain-outcomes unless explicitly stated otherwise.

EXAMPLE A.1 (The Allais Paradox for Uncertainty): Tversky and Kahneman (1992) demonstrated that the Allais paradox and upper subadditivity hold not only for risk, but also for uncertainty with events for which no probabilities are given. For example, they conducted the following experimental test of Figure 1. The design was within-subjects. The participants were 156 money managers during a workshop. Let d denote the difference between the closing value of the Dow-Jones on the day of the experiment and on the day after, I the event of $d < 30$, A the event of $30 \leq d \leq 35$, H the event of $d > 35$, $M = \$75,000$, and $m = \$25,000$. Of the participants, 77% preferred the risky option in Figure 1a, but 68% preferred the safe option in Figure 1b. This example and many similar examples, e.g. in Wu and Gonzalez (1999), confirm the Allais paradox for uncertain events. An early example is described by MacCrimmon and Larsson (1979, pp. 364–365).

EXAMPLE A.2 (Figure 2, Pessimism, for Uncertainty): Wu and Gonzalez (1999, Questions 2.1 and 2.2) considered participants' prior expectation of the 1996 U.S. national election. The design was between-subjects, with 70 participants answering each question. The events are defined in Table I.
Because event L is nonempty, this configuration tests the general pessimism condition of Figure 2 but not upper subadditivity of Figure 1. With $M = \$350$ and $m = \$300$, 35% of the participants chose the risky gamble in Figure 2a, but 67% chose the risky gamble in Figure 2b (significant at the .05 level). That is, the tendency to choose the risky gamble in Figure 2b was larger, rather than smaller as pessimism requires. This finding provides a violation of pessimism and suggests the dual, optimism. It illustrates once more that pessimism (and, hence, convexity of capacities) is not as universal as has often been thought.

EXAMPLE A.3 (Figure 5, Pessimism for Risk): Camerer (1989) tested Figure 5 with $M = \$25,000$, $m = \$10,000$, $h = .1$, $\mathbf{q} = .1$, p $= .4$, and $l = .4$; see choices 12 and 9 for large gains and Table 7 in his paper. Out of 30 participants, 20 chose the same in Figures 5a and 5b, 9 shifted from a risky to a safe choice, and 1 from a safe to a risky choice. This finding demonstrates a significant shift from risky to safe, supporting pessimism.

EXAMPLE A.4 (Figure 5 with $l = 0$, i.e. Figure 1 for given probabilities; Violation of the Certainty Effect): Starmer (1992, set 1, Questions 4 and 1) tested Figure 5 with $h = .1$, $q = .1$, $p = .8$ (and $l = 0$), $M = £.7$, and $m = £.3$. The design was within-subjects. The participants were 124 students from various disciplines. In Figure 5a, 65% chose safe whereas only 46% did so in Figure 5b. Remarkably, this finding constitutes a significant shift from safe to risky, in violation of upper subadditivity and therefore of the certainty effect.

There have been many empirical investigations for decision under risk. It is mostly found that w is concave on an interval $[0, b]$ and convex on $[b, 1]$ where b ranges between .2 and .35 (Wu and Gonzalez (1996, 1998)). Wakker, Erev, and Weber (1994) reported a negative finding where

TABLE I

	Democratic Control House of Representatives	Republican Control House of Representatives
Democratic president	H	A
Republican president	I	L

violations of the sure-thing principle were mostly caused by random choice and did not suggest rank dependence. Starmer (1992), who also tested preference conditions of the kind presented in this paper, did not find universal support for convex w. Birnbaum and Navarrete (1998) found confirmation of pessimism and violation of inverse-S shapes.

There is overwhelming evidence for the inverse-S shape, usually based on parametric fitting (Preston and Baratta (1948), Yaari (1965), Cohen and Jaffray (1988), Kachelmeier and Shehata (1992), Lattimore, Baker, and Witte (1992), Camerer and Ho (1994), Tversky and Fox (1995), Fox, Rogers, and Tversky (1996), Bleichrodt, van Rijn, and Johannesson (1999)). Parameter-free tests, confirming inverse-S, are given by Abdellaoui (2000), Bleichrodt and Pinto (2000), Gonzalez and Wu (1999), and Wu and Gonzalez (1996).

An interesting version of the method of this paper can be found in the often-cited Chew and Waller (1986). They used a "HILO" design to test special cases of Chew's weighted utility theory. The "O" questions of the design will not be discussed here. The "I," "L," and "H" questions can be obtained from Figure 5 by assuming $l = 0$ and replacing the outcome 0 by a general lowest outcome z. Then the L question is Figure 5a and the I question is Figure 5b. The H question can be obtained by taking the p-probability common outcome not equal to 0 (z) or m, but equal to M. The preceding substitutions show that the L and I questions constitute a special case of Figure 4 so that these questions test pessimism. The I and H questions also do so. They follow from Figure 4 with probabilities instead of events and with the following specifications: Event H is empty, g_L is constant and equal to c, $\mathbf{x} = c$, and f_L is constant and below c. Then the I question is Figure 4a and the L question is Figure 4b.

The substitutions show that the HILO design can be used to test the probability transformation function. Chew and Waller's data contained antecedent preferences instead of indifferences, as explained following Theorem 4.1. The overall risky choices in the L, I, and H questions were 67%, 56%, and 50%, respectively (see Chew and Waller's Table 4), suggesting pessimism from L to I (likely gain-events) and optimism from I to H (unlikely gain-events). These data therefore support an inverse-S probability transformation.

Only a few papers have dealt with loss outcomes. These papers also found the inverse-S shape, both for risk (Abdellaoui (2000), Currim and Sarin (1989), Tversky and Kahneman (1992)) and for uncertainty (Tversky and Kahneman (1992)). For losses, more risk seeking and optimism were found than for gains, once again casting doubt on the universality of pessimism and risk aversion (Camerer (1989), Dobbs (1991), Eraker and Sox (1981), Fennema and Van Assen (1998), Hershey and Schoemaker (1980), Laughhunn, Payne, and Crum (1980), Lopes and Oden (1999), Mangelsdorff and Weber (1994), Viscusi and Chesson (1999)). An exception is Keren and Gerritsen (1999), who found pessimism for losses as well.

APPENDIX B: Mathematical Proofs

We start with some notation. Because the rank-ordering of outcomes and events is important in partitions, ordered partitions, denoted $(E_1, \ldots, E_n)$, are invoked, assuming the rank-ordering with best outcomes for $E_1, \ldots,$ and worst outcomes for E_n. (E_1, E_2) is therefore to be distinguished from (E_2, E_1). When no misunderstanding can arise, the term ordered is suppressed. A useful property of decision weights is additivity. Consider a partition $(E_1, \ldots, E_n)$ and the decision weights $d(E_j, E_1 \cup \cdots \cup E_{j-1})$ of events in this partition. For the decision weight of $E_j \cup E_{j+1}$ in this partition, we have

$$\text{(B.1)} \qquad d(E_j \cup E_{j+1}, E_1 \cup \cdots \cup E_{j-1}) = d(E_j, E_1 \cup \cdots \cup E_{j-1}) + d(E_{j+1}, E_1 \cup \cdots \cup E_j).$$

To apply equation (B.1), the rank-ordering of E_j and E_{j+1} must have been specified, as in the partition $(E_1, \ldots, E_n)$. Figure 3 gives examples of equation (B.1). The decision weight of $\mathbf{A} \cup \mathbf{I}$, $W(H \cup \mathbf{A} \cup \mathbf{I}) - W(H)$, equals $\boldsymbol{\pi}_{\mathbf{A}} + \pi_{\mathbf{I}}$ in Figure 3a and $\overline{\pi}_{\mathbf{I}} + \overline{\pi}_{\mathbf{A}}$ in Figure 3b; these are therefore the same. Equation (B.1) will often be used without explicit mention.

PROOF OF NECESSITY OF THE PREFERENCE CONDITIONS IN ALL RESULTS: For all results, necessity follows from Theorem 4.1 and the reasoning preceding it.

Henceforth, only sufficiency of the preference conditions needs to be established.

PROOF OF SUFFICIENCY IN THEOREM 5.2.(i): We assume pessimism and derive convexity of the capacity. One more notation is introduced. For a partition $(E_1, \ldots, E_n)$,

(B.2) $\qquad (E_1, \ldots, \{E_i; E_{i+1}\}, \ldots, E_n)$ means that

$$d(E_i, E_1 \cup \cdots \cup E_{i-1}) \leq d(E_i, E_1 \cup \cdots \cup E_{i-1} \cup E_{i+1}).$$

In words, it means that E_{i+1} loses decision weight to E_i if E_{i+1} "passes by E_i" in the rank-ordering. By equation (2.3), convexity of W is equivalent to $(E_1, \ldots, \{E_i; E_{i+1}\}, \ldots, E_n)$ for all E_i in all partitions, and also equivalent to the condition $(H, \{A; I\}, L)$ for all four-fold partitions. Consider any partition (H, A, I, L). We prove $(H, \{A; I\}, L)$, i.e. $d(A, H) \leq d(A, H \cup I)$. Figure 3 illustrates $(H, \{A; I\}, L)$, with I losing decision weight to A as I passes by A in rank-ordering from Figure 3a to Figure 3b. The total decision weight of $A \cup I, W(H \cup A \cup I) - W(H)$, remains unaffected.

CASE 1: $d(A, H) = 0$: Then the result is trivial.

CASE 2: $d(A, H) > 0$.

CASE 2.a: $W(H) > 0$: Then we can find outcomes $M \geq m > 0$ such that the indifference in Figure 2a holds, i.e. equation (3.3) holds. By Theorem 4.1, $(H, \{A; I\}, L)$.

CASE 2.b: $W(H) = 0$: This case is derived from Case 2.a by a convergence argument. The argument does not invoke any kind of continuity of W (or anything beyond the set of events being an algebra) other than solvability. Assume for now that A_1 and A_2 partition A, and both have positive decision weight in the partition (H, A_1, A_2, I, L). By Case 2.a, $(H \cup A_1, \{A_2; I\}, L)$, i.e. I loses decision weight if it passes by A_2 in rank-ordering. So I has less decision weight in (H, A_1, I, A_2, L). When passing by A_1, I may gain decision weight, but never more than the total decision weight of A_1. Hence, in (H, I, A_1, A_2, L) and thus in (H, I, A, L), the decision weight of I may exceed that in (H, A, I, L), but not by more than the decision weight of A_1. Because of solvability, the latter can be taken arbitrarily small. We conclude that $(H, \{A; I\}, L)$.

PROOF OF SUFFICIENCY IN THEOREM 5.2.(ii): We assume optimism and derive concavity of the capacity. Because of the asymmetric role of event L in Figure 2, this proof is not a dual copy of the derivation of (i) and requires an independent elaboration. For a partition $(E_1, \ldots, E_n)$,

(B.3) $\qquad (E_1, \ldots, [E_i; E_{i+1}], \ldots, E_n)$ means that

$$d(E_i, E_1 \cup \cdots \cup E_{i-1}) \geq d(E_i, E_1 \cup \cdots \cup E_{i-1} \cup E_{i+1}).$$

Concavity of W is equivalent to $(E_1, \ldots, [E_i; E_{i+1}], \ldots, E_n)$ for all E_i in all partitions, which is equivalent to the condition $(H, [A; I], L)$ for all four-fold partitions. Take any such partition.

CASE 1: $W(H) > 0$: Then we can find outcomes $M \geq m > 0$ such that the indifference in Figure 2a holds. By Theorem 4.1, $(H, [A; I], L)$.

CASE 2: $W(H) = 0$.

CASE 2.a: $d(A, H) = 0$: Then the indifference in Figure 2a holds, where $m > 0$ can be taken, because both H and A have decision weight zero. Because of optimism, we get the reversed

preference of Figure 2b which, together with $U(m) > U(0)$, implies that $d(\mathbf{A}, H \cup \mathrm{I})$ cannot be positive. It follows that $d(\mathbf{A}, H \cup \mathrm{I}) = 0$, implying $(H, [\mathbf{A}; \mathrm{I}], L)$.

CASE 2.b: $d(\mathbf{A}, H) > 0$: If $d(\mathbf{A}, H \cup \mathrm{I}) = 0$ we are done, so assume $d(\mathbf{A}, H \cup \mathrm{I}) > 0$. Assume that A_1, A_2 partition $\mathbf{A}$, with $d(A_1, H \cup \mathrm{I}) > 0$. Then $d(A_1, H) > 0$, for if it were zero then, by a reasoning similar to Case 2.a, $d(A_1, H \cup \mathrm{I}) = 0$ would follow, contradicting $d(A_1, H \cup \mathrm{I}) > 0$.[3] $d(A_1, H) > 0$ implies $W(H \cup A_1) > 0$. Consequently, by Case 1, $(H \cup A_1, [A_2; \mathrm{I}], L)$.

Consider the following three rank-orderings of events: $(H, \mathrm{I}, A_1, A_2, L)$, $(H, A_1, \mathrm{I}, A_2, L)$, and $(H, A_1, A_2, \mathrm{I}, L)$. Going from the first to the second partition, I may gain decision weight when A_1 passes by I, but not more than the total decision weight of A_1. Because of $(H \cup A_1, [A_2; \mathrm{I}], L)$, I loses decision weight when A_2 passes by I. The decision weight of I in $(H, \mathrm{I}, A_1, A_2, L)$ (i.e. in $(H, \mathrm{I}, \mathbf{A}, L)$) therefore cannot exceed that in $(H, A_1, A_2, \mathrm{I}, L)$ (i.e. in $(H, \mathbf{A}, \mathrm{I}, L)$) by more than $d(A_1, H \cup \mathrm{I})$. Because of solvability, the positive decision weight $d(A_1, H \cup \mathrm{I})$ can be arbitrarily small. Hence, the decision weight of I in $(H, \mathrm{I}, \mathbf{A}, L)$ cannot exceed that in $(H, \mathbf{A}, \mathrm{I}, L)$. $(H, [\mathbf{A}; \mathrm{I}], L)$ follows.

PROOF OF SUFFICIENCY IN LEMMA 5.3: The proof of Theorems 5.2.(i) and 5.2.(ii) can be invoked. The restriction regarding D never causes any complication because no variations in L need to be invoked in these proofs; we deal only with cases in which $S - L \preccurlyeq D$. The restriction regarding C is vacuous if $W(C) = 0$. If $W(C) > 0$, then the proof is in fact simpler because the most difficult cases, Cases 2.b and 2, respectively, then can be skipped.

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[3] This is the only place where we use the separation of Cases 2a and 2b.

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