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Cumulative dominance and probabilistic sophistication

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Abstract

Machina and Schmeidler [Econometrica, 60 (1992) 745–780] gave preference conditions for probabilistic sophistication, i.e. decision making where uncertainty can be expressed in terms of (subjective) probabilities without commitment to expected utility maximization. This note shows that simpler and more general results can be obtained by combining results from qualitative probability theory with a 'cumulative dominance' axiom. © 2000 Elsevier Science B.V. All rights reserved.

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Machina and Schmeidler (1992) pose an interesting question: what conditions on an agent's preferences justify probabilistic sophistication; that is, when can beliefs be represented by probabilities? A probabilistically sophisticated agent may deviate from expected utility in his preferences over lotteries but must express uncertainty in terms of additive probabilities. In this note, we show that a cumulative dominance condition can be used to simplify and extend the results of Machina and Schmeidler (1992). In a previous paper (Sarin and Wakker, 1992), we showed that cumulative dominance is a necessary condition for Choquet expected utility and that it can be used to extend expected utility under risk to Schmeidler's (1989) Choquet expected utility under uncertainty. It turns out that cumulative dominance is necessary for probabilistic sophistication as well, and here we show that this axiom can be used to extend qualitative probability for two-consequence acts to probabilistically sophisticated preferences for many-consequence acts.

Let S denote the *state space*; $\mathscr{A}$ is the algebra of subsets of S, the elements of which are called *events*; $\mathscr{C}$ is the set of *consequences*; $\mathscr{F}$ is the set of *acts*, i.e. maps from S to

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$\mathscr{C}$ that are finite-valued and measurable ($f^{-1}(x) \in \mathscr{A}$ for all consequences x); x denotes both a consequence and the related constant act. By $\succcurlyeq$ we denote the preference relation over acts, that also denotes the induced ordering of consequences. The notation $\succ$ (strict preference), $\sim$ (equivalence), $\preccurlyeq$, and $\prec$ is as usual. In this note, probability measures are finitely additive and need not be countably additive unless stated otherwise.

Following Machina and Schmeidler (1992), an agent is *probabilistically sophisticated* if there exists a probability measure P over S such that:

(i) the agent chooses between acts based on the probability distributions generated over the consequences;
(ii) first-order stochastic dominance is satisfied.

Stochastic dominance in (ii) is taken in the strict sense (Machina and Schmeidler, 1992, Section 3.1). It relates to general, possibly nonmonetary, consequences, and is defined as follows: $f \succ g$ whenever $P(f \succcurlyeq x) \geq P(g \succcurlyeq x)$ for all consequences x with strict inequality for at least one x. We have modified the definition of probabilistic sophistication of Machina and Schmeidler by omitting their 'mixture continuity' condition, in order to allow for general models with atoms (e.g. in Theorem 5 below). Machina and Schmeidler show that mixture continuity is implied by Savage's P6, hence it can be seen to be implied in our Theorem 2.

We next derive a more likely than relation on events from preferences between two-consequence acts. We write $A \succcurlyeq_\ell B$ if there exist consequences $x \succ y$ such that the act [x if A; y if not A] is preferred to the act [x if B; y if not B]. Obviously, probabilistic sophistication requires that $A \succcurlyeq_\ell B$ if and only if $P(A) \geq P(B)$, i.e. P is an *agreeing* probability measure for $\succcurlyeq_\ell$. For the case where $\mathscr{C}$ contains only two consequences, one immediately observes that probabilistic sophistication holds if and only if an agreeing probability measure exists. Machina and Schmeidler (1992, Section 3.4) demonstrate that the existence of an agreeing probability measure need not imply probabilistic sophistication when there are more than two consequences. To obtain probabilistic sophistication for general acts, Machina and Schmeidler assume the restrictions of Savage's axioms that guarantee the existence of a probability measure agreeing with $\succcurlyeq_\ell$, and then add an additional axiom (P4* in their paper) for acts with many consequences.

We will follow the first part of their analysis but use cumulative dominance instead of their P4* to ensure probabilistic sophistication. In line with the strict stochastic dominance condition of Machina and Schmeidler, we strengthen the definition of cumulative dominance of Sarin and Wakker (1992) somewhat and define a strict version thereof. *Cumulative dominance* holds if

$f \succcurlyeq g$ whenever $\{s \in S: f(s) \succcurlyeq x\} \succcurlyeq_\ell \{s \in S: g(s) \succcurlyeq x\}$ for all consequences x, where the preference between f and g is strict whenever one of the antecedent $\succcurlyeq_\ell$ orderings is strict ($\succ_\ell$).

This condition says that if one considers the cumulative event [x or more] more likely under f than under g for every consequence x, then f should be preferred to g, where the more likely than relation $\succcurlyeq_\ell$ is derived from choices. The following elementary lemma shows the way in which cumulative dominance ensures probabilistic sophistication for many-consequence acts in situations where an agreeing probability measure exists for $\succcurlyeq_\ell$.

Lemma 1. *The following two statements are equivalent.*

(i) *Probabilistic sophistication holds.*
(ii) *There exists an agreeing probability measure P for* $\succcurlyeq_\ell$ *and cumulative dominance holds.*

Proof. If a probability measure P agrees with $\succcurlyeq_\ell$ and if preferences depend only on probability distributions generated over consequences through P, then one immediately sees that cumulative dominance and stochastic dominance are equivalent. So we only have to show:

under (i), P agrees with $\succcurlyeq_\ell$: this is elementarily verified;
under (ii), preferences depend only on probability distributions generated over consequences: if f and g generate the same probability distribution over consequences, then $\{s \in S: f(s) \succcurlyeq x\} \sim_\ell \{s \in S: g(s) \succcurlyeq x\}$ for all consequences x, and hence $f \succcurlyeq g$ and $g \succcurlyeq f$, by twofold application of cumulative dominance. That is, $f \sim g$. □

Preference characterizations of probabilistic sophistication can be derived from the above lemma by substituting preference conditions, obtained from qualitative probability theory, to guarantee the existence of an agreeing probability measure P for $\succcurlyeq_\ell$. Several such conditions have been presented in qualitative probability theory. Hence, our approach shows a way to derive probabilistic sophistication from results of qualitative probability theory. As a first illustration, we invoke the qualitative probability conditions of Savage (1954). For definitions the reader is referred to Savage (1954) or Machina and Schmeidler (1992). We assume their P1 (weak ordering), P3 (tastes are independent of beliefs), P4 (beliefs are independent of tastes), P5 (nontriviality), and P6 (fineness of the state space). P7 is not needed because we only consider finite-valued acts. We note that P4 need not be stated because it is implied by cumulative dominance (see the proof below), and restrict P2 to two-consequence acts. P2 is Savage's sure-thing principle, requiring that preferences be independent from common consequences. To see that P2* is a restriction of P2, observe that in P2* a common outcome y under event H in the first preference is replaced by a common outcome x in the second preference.

Postulate P2*. (*sure-thing principle for two-consequence acts*). For all consequences $x \succ y$ and events A, B, H with $A \cap H = B \cap H = \varnothing$:

$$[x \text{ if } A;\ y \text{ if not } A] \succcurlyeq [x \text{ if } B;\ y \text{ if not } B]$$
if and only if
$$[x \text{ if } A \cup H;\ y \text{ if not } A \cup H] \succcurlyeq [x \text{ if } B \cup H;\ y \text{ if not } B \cup H]$$

Theorem 2. *Assume Savage's* (1954) *P*1, *P*3, *P*5, *and P*6. *Then the following two statements are equivalent*.

(i) *Probabilistic sophistication holds*.
(ii) *P*2* *and cumulative dominance hold*.

Proof. It is obvious that (i) implies (ii), therefore we assume (ii) and derive (i). P4 follows from the restriction of cumulative dominance to two-consequence acts. P1, P2*, P3, P4, P5, and P6 are all that Savage needs to derive an agreeing probability measure. His analysis, as well as that of Machina and Schmeidler (1992), assumes that $\mathscr{A}$ is the collection of all subsets of *S*. It is well-known that these proofs and results immediately extend to any sigma-algebra of events (Savage, 1954, Section III.4; Machina and Schmeidler, 1992, Section 6.2). Our theorem extends the result further to any algebra of events: Savage's axioms P1, P2*, P3, P4, P5, and P6 also imply the existence of an agreeing probability measure for that case (Wakker, 1981). Now the theorem follows from Lemma 1. □

Machina and Schmeidler (1992) eloquently pose the question what additional conditions are needed to characterize probabilistic sophistication over many-consequence acts once an agreeing probability measure exists for $\succcurlyeq_\ell$. Once one assumes qualitative probability theory, then our characterization seems to be the simplest way to obtain probabilistic sophistication in the Savage framework. Note here that qualitative probability theory is the special case of additive conjoint measurement with two-element component sets (Fishburn and Roberts, 1988).

For many results of probability theory, countable additivity of a probability measure is essential. Hence Machina and Schmeidler (1992, Section 6.2) address the characterization of countable additivity in probabilistic sophistication. They conjecture that conditions of Villegas (1964) and Arrow (1965) may be used to imply countable additivity. We prove that that is indeed the case. Following Villegas (1964), *monotone continuity* holds if for each sequence of events A_j that increases to an event A ($A_{j+1} \supset A_j$ and $\cup A_j = A$), $A_j \preccurlyeq_\ell B$ for all j implies $A \preccurlyeq_\ell B$. In the presence of this condition, Savage's structural condition P6 can be weakened to the requirement that no atoms exist. An event *A* is an *atom* if it is nonnull but it cannot be partitioned into two nonnull events.

Theorem 3. *Assume Savage's* (1954) *P*1, *P*3, *and P*5, *assume that* $\mathscr{A}$ *is a sigma algebra*, *and that there are no atoms*. *Then the following two statements are equivalent*.

(i) *Probabilistic sophistication holds with respect to a countably additive probability measure*.
(ii) *P*2*, *monotone continuity*, *and cumulative dominance hold*.

Proof. The implication (i)⇒(ii) is straightforward, hence we assume (ii) and derive (i). Cumulative dominance implies P4. P4 in turn implies weak ordering of $\succcurlyeq_\ell$ (i.e. Villegas (1964), Conditions Q1(a), Q1(b), and Q1(c)). Now, mainly because of P3 and P5, it follows that $\varnothing \prec_\ell S$ and, for any event A, $\varnothing \preccurlyeq_\ell A \preccurlyeq_\ell S$ (Villegas' Condition Q1(d)). P2* implies Villegas' condition Q2 for $\succcurlyeq_\ell$. Now, by Theorem 4.3 of Villegas, there exists a unique countably additive probability measure agreeing with $\succcurlyeq_\ell$. By our Lemma 1, statement (i) holds. □

By means of the above result, it can be demonstrated that the conjecture in Section 6.2 of Machina and Schmeidler (1992) is correct.

Corollary 4. *In Theorem* 2 *of Machina and Schmeidler* (1992), *the probability measure in* (*ii*) *is countably additive if in statement* (*i*) *their monotone continuity axiom is added, and the domain of events is assumed to be a sigma algebra*.

Proof. Monotone continuity as defined here and in Villegas (1964) follows from the first part of the monotone continuity axiom of Machina and Schmeidler (1992, Section 6.2) by defining, for consequences $x \prec y$, $f = [y$ if A, x if $A^c]$, $E_j = A \backslash A_j$, and $g = [y$ if B, x if $B^c]$. Their preference conditions imply probabilistic sophistication and hence, by Lemma 1, cumulative dominance. Also there exist no atoms in their model, mainly by P6. Now the result follows from Lemma 1 and Theorem 3. □

Finally, we give an example of an alternative derivation of probabilistic sophistication using a qualitative probability result different than that of Savage. We invoke the remarkable result provided by Chateauneuf (1985). Using techniques introduced by Jaffray (1974), Chateauneuf provided necessary and sufficient conditions for the existence of an agreeing probability measure in full generality, i.e. without assuming any restrictive condition such as Savage's P6 or nonatomicity. His result extends the earlier results by Kraft et al. (1959) and Scott (1964), who gave necessary and sufficient conditions for finite state spaces. For brevity, we do not describe the details of these conditions, but refer the reader to the respective papers.

Theorem 5. (*general probabilistic sophistication*). *The following two statements are equivalent*.

(i) *Probabilistic sophistication holds*.
(ii) *Cumulative dominance holds, and Chateauneuf's* (1985) *conditions hold for* $\succcurlyeq_\ell$.

Proof. See Chateauneuf (1985) and Lemma 1. □

By means of Chateauneuf's conditions and our Lemma 1, we thus obtain a characterization of probabilistic sophistication in full generality. By means of Chateauneuf and Jaffray (1984), the above result can be extended to the case where in (i) the probability measure is countably additive. Obviously, many alternative derivations of probabilistic sophistication can be obtained by applying Lemma 1 to other results of qualitative probability theory. Fishburn (1986) gives an extensive survey of such results. An alternative approach, using the model of Anscombe and Aumann (1963), is presented by Machina and Schmeidler (1995).

A possible objection to our result may be that cumulative dominance is too close to the functional form it seeks to characterize and therefore the proof is simple. We note, however, that the purpose of characterizations is to provide preference conditions that are simple to understand and that can be directly tested through observable choices. The focus should be on the insights that preference conditions generate and not on the complexity of proofs they entail. The condition of Machina and Schmeidler (1992) is a generalization of independence of beliefs from tastes and in that manner it provides new insights. Our cumulative dominance condition is an extension of stochastic dominance to the case of uncertainty and thus it reveals the nature of probabilistic sophistication in a transparent and empirically meaningful way.

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