SOURCE THEORY: A TRACTABLE AND POSITIVE AMBIGUITY THEORY

Aurélien Baillon, baillon@em-lyon.com, Emlyon business school, Lyon, France
Han Bleichrodt, hanbleichrodt@ua.es, Department of Economics (FAE), University of Alicante, Spain
Chen Li[a], c.li@ese.eur.nl
Peter P. Wakker[a], wakker@ese.eur.nl
a: Erasmus School of Economics, Erasmus University Rotterdam, Rotterdam, the Netherlands

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This paper introduces source theory, a new theory for decision under ambiguity. It shows how probability weighting functions can be used to model ambiguity. It can do so in the Savage (& Gilboa) framework, and does not need complex two-stage gambles, multistage optimization principles, expected utility for risk (descriptively problematic), or any linear algebra. Still the mathematical analysis is simple, with intuitive preference axioms, tractable empirical implementations and calculations, and convenient graphical representations of ambiguity attitudes. It gives new ways to compare weighting functions, not between persons as is common, but within one person and between sources, giving Arrow-Pratt-like transformations "within" rather than "outside" the functions. Within-person between-sources comparisons are the main novelty of ambiguity over risk, first demonstrated by Ellsberg's paradox.
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1. INTRODUCTION

Subjective expected utility has been the classical model for measuring uncertainty attitudes, widely used for over a century (Arrow 1951; Farquhar 1984). The most popular underlying theoretical framework, with acts mapping states to outcomes, as in "Arrow-Debreu securities," was first formalized by Arrow (1953) and Debreu (1953).

Such a formalization is needed prior to developing axiomatizations and theoretical and mathematical tools.[1] Savage (1954) subsequently used this formalization for his famous preference axiomatization. Preference axiomatizations are needed to establish the theoretical soundness of a model and its normative status. Thus, original prospect theory (Kahneman & Tversky 1979) could never become a full-blown *theory* because it lacked a preference axiomatization. And Wald's (1950) maxmin expected utility only became popular after Gilboa & Schmeidler (1989) provided a preference axiomatization and thus, according to non-Bayesians, a normative status. Regarding empirical status, preference axioms show what the critical tests of a theory are.

This paper introduces source theory (ST), a new ambiguity theory, providing axiomatic foundations to many empirical studies lacking such. Ambiguity theories generalize Savage's (1954) expected utility (EU) so as to accommodate deviations first pointed out by Ellsberg (1961). Many theories have been proposed[2] that are, however, sometimes difficult to measure empirically. The source method, by contrast, is a widely used *empirical* method to measure ambiguity attitudes[3], but it has as yet lacked theoretical foundations. ST, first, formalizes the underlying theoretical framework of the source method. It then shows how probability weighting functions can be used to capture ambiguity attitudes. This use of probability weighting functions had been suggested since the origins of the ambiguity literature (Fellner 1961 p. 672; Kahneman & Tversky 1979 p. 289) but it had never yet been formalized. We then, further, provide a preference axiomatization for ST and its main attitudinal components, which is necessary to establish this theory and make it normatively acceptable for non-Bayesian prescriptive applications. Our axiomatizations give theoretical support to the empirical finding (Trautmann & van de Kuilen 2015) that,

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[1] It was absent from Ramsey (1931), making it impossible to specify exactly what he did. His contemporary de Finetti (1931) used part of the formalization.

[2] Surveys include Gilboa & Marinacci (2016) and Machina & Siniscalchi (2014).

[3] See, for instance, Abdellaoui et al. (2011), Anantanasuwong et al. (2022), Baillon et al. (2018), Bleichrodt, Grant, & Yang (2022), Einhorn & Hogarth (1985), Grevenbrock et al. (2021), Ivanov (2011), Kemel & Gutierrez (2022), and Spiliopoulos & Hertwig (2023).

besides aversion, also insensitivity ("perception") towards ambiguity is needed to fully capture ambiguity attitudes.

Many studies have demonstrated the importance of distinguishing between different sources of uncertainty; i.e, groups of events with similar uncertainty.[4] Three main insights were put forward. First, even though Ellsberg (1961) challenged the general existence of subjective probabilities, they can still be defined within sources (Chew & Sagi 2008). Second, people are not always ambiguity averse but can simultaneously be ambiguity averse for some sources and ambiguity seeking for others (Tversky & Fox 1995). Surely for empirical purposes, source dependence of ambiguity attitudes is needed. Third, ambiguity does not only lead to aversion or attraction, but also to insensitivity to likelihood information ("perception"), which is also source-dependent and is higher for more ambiguous sources (Trautmann & van de Kuilen 2015). Insensitivity is a component orthogonal to the aversion-attraction component.

ST accommodates the aforementioned insights. Ambiguity attitudes are modeled using source-dependent transformations of Chew & Sagi's subjective probabilities[5] within a source. This allows for convenient graphical representations of ambiguity attitudes. For instance, Figure 4 will visually and completely capture the risk, uncertainty, and ambiguity attitudes of a given subject for the CAC40 stock index. This feature makes ST attractive to a broad audience. For instance, policy makers can easily express their desired ambiguity attitude graphically: extra perceived uncertainty about probabilities by moving weight ("probability mass") from the expected middle to the unexpected extremes (insensitivity/perception: as-if increasing variance) and extra aversion by pushing the curve down.

A crucial property of ST is that the weighting functions (transformations) are source dependent, for instance, different for an unknown Ellsberg urn than for a known urn, and different for foreign stocks than for domestic stocks (homebias). This accommodates the aforementioned source dependence of ambiguity attitudes. Further,

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[4] They were put central by Tversky & Fox (1995) and were used by Georgalos (2019), l'Haridon et al. (2018), Maafi (2011), and many studies cited later.

[5] Later, these probabilities will formally be called a-neutral.

ST gives a central place to insensitivity, which captures ambiguity perception as defined in many ambiguity models, while adding dependence on cognitive ability, a subjective element. Our formalization of insensitivity thus unifies two seemingly separate concepts: the ambiguity perception of ambiguity models and the inverse-S probability weighting of risk models. For the latter property, so widely documented in empirical studies, it is extra remarkable that no formal mathematical definition had been provided before in the literature.

Section 8 compares our framework with some other ones, in particular the Anscombe-Aumann framework. It points out that we do not need to assume EU for risk or complex multi-stage stimuli with a commitment to multistage optimization procedures. We, instead, introduce ambiguity concepts in Savage's (1954) framework. Our starting point is, more precisely, Gilboa's (1987) generalization of Savage (1954) that accommodates ambiguity. Yet more precisely, we use a biseparable generalization (Ghirardato & Marinacci 2001), adapted to the Gilboa-Savage framework, which includes many ambiguity models as special cases. Our formal results are, consequently, valid not only under rank-dependent utility/Choquet expected utility for uncertainty (Gilboa 1987; Schmeidler 1989; Tversky & Kahneman's 1992 prospect theory for gains), but also under various multiple prior models (maxmin EU; α-maxmin). For risk, a special case of uncertainty and part of our model, our results are valid under the models of Kahneman & Tversky (1979), Quiggin (1982), Tversky & Kahneman (1992), and several others (Wakker 2010 Observation 7.11.1). ST generalizes Chateauneuf, Eichberger, & Grant's (2007) neo-additive utility.

The outline of this paper is as follows. Standard concepts are in §2. Section 3 formalizes the theoretical framework underlying ST, including sources. Section 4 defines the two main components of uncertainty attitudes, aversion and insensitivity, through comparative conditions (absolute conditions in §6). Insensitivity, widely studied empirically, had never yet been fully formalized.[6] Preference axiomatizations are in §5. Our preference axiomatization of ST captures the meaning of probabilistic

[6] Compare Tversky & Wakker (1995 Footnote 7). Our definition captures inverse-S shapes for risk as a special case.

sophistication directly in terms of ambiguity. Section 7 provides a new technique for comparing weighting functions so as to compare ambiguity/uncertainty attitudes. Unlike common Arrow-Pratt results, our comparisons are within-subject-between-sources rather than between-subjects and our transformations are "inside" rather than "outside" functions. Based on intuitive arguments, Kemel & Gutierrez (2022) used such "inside" transformations and demonstrated their empirical usefulness. We provide a theoretical basis, i.e., a formalization and preference axiomatization, for their intuitive claims. Section 7, further, presents an empirical demonstration of ST to illustrate its tractability and empirical relevance. Section 8 discusses related frameworks and shows that ST uses simple formulas and calculations that are convenient for applications. Because of their mathematical tractability the functionals that we use, nonlinear transformations of subjective additive probabilities, provide the most popular risk measures nowadays (Artzner et al. 1999), called law-invariant or distorted risk measures (Liu, Schied, & Wang 2021; Nguyen, Pham, & Tran 2012; Wang, Wei, & Willmot 2020). Hence, this paper also serves as a preference foundation of those risk measures, and provides new tools for analyzing them. Section 9 concludes. Appendix A reformulates some preference conditions to make them more generally testable. Proofs are in Appendix B.

2. BASIC DEFINITIONS

S denotes a *state space*. Its subsets are *events*.[7] Later assumptions will imply that S is infinite. A *weighting function* W is defined on all events and satisfies $W(\emptyset) = 0$, $W(S) = 1$, and $A \supset B \Rightarrow W(A) \geq W(B)$. *Probability measures* P are weighting functions that satisfy additivity. Following Savage (1954), we only impose finite additivity. Countable additivity can easily be characterized in all results below by an extra preference condition (Wakker 1993, Proposition 4.4).

Γ denotes a set of consequences, or *outcomes*, and can be finite or infinite. We assume that all outcomes are gains, leaving reference-dependence to future studies.

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[7] Our results do not change if we endow S with an algebra or a σ-algebra of events and consider only measurable acts.

An *act* maps S to Γ and is finite-valued. We denote acts by $x = (E_1: x_1, \ldots, E_n: x_n)$, mapping every $s \in E_j$ to x_j. It is implicit in this notation that the E_js partition S and that the x_js are outcomes. $\alpha_E \beta$ denotes $(E: \alpha, E^c: \beta)$. A *preference relation* $\succcurlyeq$ of an agent is given over acts, with $\preccurlyeq$, $\succ$, $\prec$, and $\sim$ as usual. We assume that $\succcurlyeq$ is a *weak order* (transitive and complete). As usual, we identify constant acts with outcomes, so that $\succcurlyeq$ also denotes a preference relation over outcomes.

We assume *biseparable utility*: there exist a *utility function* $U: \Gamma \to \mathbb{R}$ and a weighting function W such that preferences over binary acts $\gamma_E \beta$, with $\gamma \succcurlyeq \beta$, maximize

$$W(E)U(\gamma) + (1 - W(E))U(\beta). \tag{1}$$

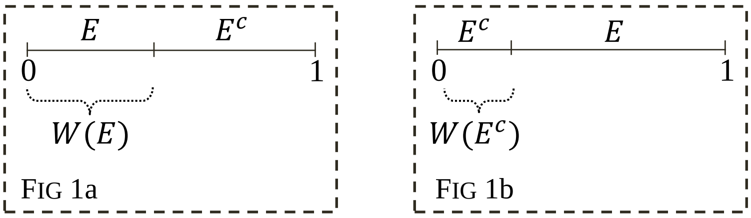
An event is *null* if its outcomes never affect preference. We assume that $\succcurlyeq$ satisfies *strong monotonicity:* strictly improving an outcome of an act on a nonnull event strictly improves the act. As explained in the introduction, biseparable utility includes many models as a special case. The primary model of interest for our analysis is rank-dependent utility, i.e., prospect theory for gains. Intuitive discussions that need specific models will be targeted to this model. However, our results hold for all biseparable models.

We interpret $W(E)$ as the share of the agent's attention given to event E if it is *ranked best*, i.e., it yields the best outcome(s), as for instance in Eq. 1. Formally, $W(E)$ is the *decision weight* then. The complementary share of attention, $1 - W(E)$, is given to event E^c if it yields the worst outcome(s). That is, if it is *ranked worst*. See Figure 1. For acts $(E_1: x_1, \ldots, E_n: x_n)$ with more than two outcomes $x_1 \succ \cdots \succ x_n$ there are "middle" events with neither best nor worst outcomes. The remaining share of attention is divided among them. Empirical studies usually find that the attention paid to a fixed event E_i when ranked middle is more or less constant (even though theoretically it could depend on the act). In informal interpretations we will, therefore, use the term "middle" attention/weight without further specification.

Rank-dependent utility (*RDU*) (or Choquet expected utility), a special case of our biseparable utility, holds if $(E_1: x_1, \ldots, E_n: x_n)$, with $x_1 \succcurlyeq \cdots \succcurlyeq x_n$, is evaluated by $\sum_{j=1}^{n} \pi_j U(x_j)$ with $\pi_j = W\big(E_j \cup \cdots \cup E_1\big) - W\big(E_{j-1} \cup \cdots \cup E_1\big)$; here, $\pi_1 = W(E_1)$.

Expected utility (*EU*) holds if W is a probability measure. EU holds if and only if the decision weight of a fixed event is the same for all acts, in particular, when ranked best or worst. This weight then always is the probability of that event. Deviations from EU are characterized by the way in which the decision weight of a fixed event varies over different acts (Figure 1), and this is what we investigate later.

FIGURE 1. Decision weights of events



Events are ranked from best (left part of line) to worst (right part). The abcissa indicates W values. In Fig. 1a, the distance from the bar to 1 indicates how much of W is lost if E^c is subtracted from certainty; i.e., the decision weight of this event when ranked worst. Fig. 1b considers changed ranks. The weight of E^c has changed (indicating violation of expected utility). It has in fact decreased, suggesting ("pessimistic") reduced attention for favorable events and a general dislike of the source.

We assume Gilboa's (1987) Savage-type richness: (i) there are at least three nonequivalent outcomes; (ii) *convex-rangedness* holds: for all events $A \subset C$ and $W(A) \leq \mu \leq W(C)$, there exists $A \subset B \subset C$ with $W(B) = \mu$. The theorems in this paper can be turned into complete preference foundations by adding the necessary and sufficient preference conditions that Gilboa gave for RDU. For brevity, we do not repeat them. Abdellaoui & Wakker (2005) argued for the intuitive appeal of Gilboa's axioms. His main axiom P2*, generalizing Savage's (1954) P2 and P4, can be seen to require consistent orderings of decision weights.

For events A, B, we define $A \succcurlyeq B$ (A is *preferred* to B) if $\gamma_A\beta \succcurlyeq \gamma_B\beta$ for some $\gamma \succ \beta$. Under biseparable utility, $\succcurlyeq$ is represented by W. The *event interval* $[E, G]$ contains all events F with $W(E) \leq W(F) \leq W(G)$, i.e, $E \preccurlyeq F \preccurlyeq G$.

Although duality is not needed in our formal analysis, it facilitates conceptual understanding. For W, $W^d(E) = 1 - W(E^c)$ is the *dual* of W, and we have a *dual preference relation* on events defined by $A \succcurlyeq^d B$ if $A^c \preccurlyeq B^c$, maximizing W^d. W^d

gives decision weights of events when ranked worst, and $\succcurlyeq^d$ derives likelihood orderings from bets against events: one disprefers betting against more likely events. Duality is symmetric: W is the dual of W^d and $\succcurlyeq$ is the dual of $\succcurlyeq^d$. Property P of weighting functions is *dual* to property Q if P holds for W if and only if Q holds for W^d, and for properties of $\succcurlyeq$ over events duality is defined similarly. It is useful to have in mind, before studying definitions given later, that source preference is dual to source dispreference and a source being liked is dual to it being disliked. The most complex property in this paper, insensitivity, always involves two conditions that are, simply, each others' dual. Thus, insensitivity is self-dual: W is insensitive if and only if W^d is, and source $\mathcal{A}$ is more insensitive than $\mathcal{B}$ under W ($\succcurlyeq$) if and only if it is under W^d ($\succcurlyeq^d$). Insensitivity, illustrated in Figure 3 below, treats bets on and bets against events the same, because it deals with middle events relative to extreme events (good or bad). This insight will simplify the study of insensitivity.

3. SOURCE THEORY

We first formalize the concept of a source. Formally, *sources* are algebras of events. Uncertainty attitudes may differ for different sources. For instance, Tversky & Fox (1995) showed that basketball fans are ambiguity averse for Ellsberg urns but ambiguity seeking for basketball games. This illustrates that ambiguity theories have to reckon with source dependence. Our formal results will always assume that weighting functions satisfy convex-rangedness when restricted to sources. That is, we only consider "rich", infinite, sources. We sometimes use finite sources in illustrations. An act x is *from a source* if it is measurable w.r.t. that source, i.e., $x^{-1}(\alpha)$ is in the source for each outcome α.

Our results can in principle be applied to any algebra of events, taking it as a source. In applications, people will usually specify sources that are of special interest to them, and then sources are exogenous. In Ellsberg's paradox they are also exogenous, determined by information about urns irrespective of preference. Therefore, sources will mostly be exogenous, similarly as commodities in consumer theory are, and this is our primary interpretation. Other authors have preferred endogenous interpretations of sources. Most of our results concern homogenous sources (defined later), and they can be taken endogenous. Observation 19 will list the related results, valid irrespective of whether the sources were endogenous or exogenous. Grant, Rich, & Stecher (2022) adopted a similar flexible interpretation of sources.

Risk, denoted $\mathcal{R}$, is a special source of uncertainty for which the probabilities of its events R are known, denoted $K(R)$. *Risky acts* are acts from that source. We throughout assume *stochastic dominance*, i.e., for $P = K$ and all risky acts x, y:

$$[\text{For all } \alpha \in \Gamma: P(x \succcurlyeq \alpha) \geq P(y \succcurlyeq \alpha)] \quad \Rightarrow \quad x \succcurlyeq y. \tag{2}$$

It implies that risky acts that induce the same probability distribution over outcomes are indifferent. (Apply Eq. 2 both ways.) In other words, preferences over risky acts depend only on the probability distribution they induce over outcomes. We usually identify risky acts with their induced probability distributions. w is a *probability weighting function* if $w: [0,1] \to [0,1]$, w is strictly increasing, $w(0) = 0$, and $w(1) = 1$. Our assumptions imply:

OBSERVATION 1. There exists a continuous probability weighting function w such that, for all risky events R,

$$W(R) = w(K(R)) \, . \tag{3}$$

□

Because $R_1 \succcurlyeq R_2 \Leftrightarrow K(R_1) \geq K(R_2)$ for risky events R_1 and R_2, we can rewrite Eq. 2 for risky acts x, y as:

$$[\text{For all } \alpha \in \Gamma: \{s \in S: x(s) \succcurlyeq \alpha\} \succcurlyeq \{s \in S: y(s) \succcurlyeq \alpha\}] \quad \Rightarrow \quad x \succcurlyeq y. \tag{4}$$

Cumulative dominance holds if Eq. 4 also holds for acts x, y that are not risky. We assume cumulative dominance throughout the paper.

DEFINITION 2. *Source theory* (*ST*) holds for a source $\mathcal{S}$ if, besides biseparable utility, *local probabilistic sophistication* holds for $\mathcal{S}$. That is, there exists a probability measure $P_{\mathcal{S}}$ on $\mathcal{S}$ such that Eq. 2 holds with $P = P_{\mathcal{S}}$ for all acts x, y from that source. □

We will later define a preference condition, uniformity, that characterizes local probabilistic sophistication. We, therefore, call such sources *uniform* henceforth. $P_{\mathcal{S}}$ may be subjective probabilities, or merely a mathematical device without any particular interpretation. Risk is a special case of local probabilistic sophistication. As with risk, preferences over acts from a uniform source $\mathcal{S}$ are entirely determined by the probability distributions over outcomes induced by $P_{\mathcal{S}}$ under ST (but depending on the source $\mathcal{S}$). Consequently, as for risk, we can define a continuous source-dependent probability weighting function $w_{\mathcal{S}}$ under ST such that:

$$\text{On } \mathcal{S}\text{:}\ \ W = w_{\mathcal{S}} \circ P_{\mathcal{S}}. \tag{5}$$

The proof is identical to Observation 1 and is omitted. We call $w_{\mathcal{S}}$ the *source function* of $\mathcal{S}$. $w = w_{\mathcal{R}}$ is the source function for risk. The source $\mathcal{S}$ is *ambiguity neutral* if it satisfies local probabilistic sophistication, the risky source is available, and $w_{\mathcal{S}} = w$. That is, the probabilities $P_{\mathcal{S}}$ are treated as if objective. An agent is *ambiguity neutral* if all sources are ambiguity neutral. In general, for any uniform source $\mathcal{S}$, we call $P_{\mathcal{S}}$ the *a-neutral probability measure* because it would serve as a regular probability measure had the agent been ambiguity neutral. It can be interpreted as the beliefs of a "Bayesian twin" of the agent.

Local probabilistic sophistication only concerns comparisons of acts from $\mathcal{S}$. By restricting general, "global" probabilistic sophistication to local probabilistic sophistication we can use probabilities but still accommodate Ellsberg's paradoxes and still model ambiguity attitudes (Chew & Sagi 2008). Two different acts that induce the same probability distribution over outcomes need not be equivalent if they are from different sources. For instance, in Ellsberg's two-color paradox, a gamble on the known urn may be preferred to a gamble on the unknown urn even though both induce the same fifty-fifty probability distribution over outcomes. *Expected utility* holds *in a source* $\mathcal{S}$ if EU represents preferences over all acts from $\mathcal{S}$. Then the source is uniform and $w_{\mathcal{S}}$ is the identity function. EU for risk thus means that w is the identity.

We summarize the assumptions made so far. They are assumed explicitly in theorems, and implicitly elsewhere.

ASSUMPTION 3 [Structural Assumption]. S is a state space, and Γ an outcome set. Acts $x = (E_1: x_1, \ldots, E_n: x_n)$ are finite-valued maps from S to Γ, endowed with a weak order $\succcurlyeq$, the preference relation. Γ contains at least three nonindifferent outcomes. Sources are sub-algebras of events. Biseparable utility holds, with a utility function $U: \Gamma \to \mathbb{R}$, a weighting function W on S, and a representation $W(E)U(\gamma) + (1 - W(E))U(\beta)$ over binary acts $\gamma_E\beta$ $(\gamma \succcurlyeq \beta)$. W is convex-ranged for every source. The relation $\succcurlyeq$ is extended to outcomes via constant acts, maximizing U, and to events via bets on them, maximizing W. Strong monotonicity and cumulative dominance hold. □

4. COMPARATIVE UNCERTAINTY AND AMBIGUITY ATTITUDES

This section introduces the main novelty of uncertainty relative to risk: within-person-between-sources comparisons. This was a big but not always sufficiently appreciated novelty in Ellsberg (1961).

4.1. Introduction and Matching Partitions

Matching partitions, generalizing matching probabilities, will provide a useful new tool for comparing uncertainty attitudes for general sources. We take risk as a single source of uncertainty, partly for tractability reasons. The domain of ambiguity, however, is too rich to be taken as one single source, in the same way as the domain of nonmonetary commodities is too rich to be taken as one. We will, therefore, distinguish between different sources of ambiguity. This leads to within-person between-source comparisons.

We consider within-person comparisons between two sources $\mathcal{A}$ and $\mathcal{C}$. Although our analysis is symmetric between the sources, an asymmetric presentation is more convenient. In elucidations, we take $\mathcal{C}$ as an established source used for calibration, and $\mathcal{A}$ as a new source to be compared with $\mathcal{C}$. Ambiguity concerns the special case of $\mathcal{C} = \mathcal{R}$. Generic elements of $\mathcal{A}$ are A, A_i, A_j, and those of $\mathcal{C}$ are C, C_i, C_j. Two partitions $(A_1, \ldots, A_n)$ and $(C_1, \ldots, C_n)$ of S are *matching* if $W(A_1 \cup \cdots \cup A_i) = W(C_1 \cup \cdots \cup C_i)$ for all i; i.e., $A_1 \cup \cdots \cup A_i \sim C_1 \cup \cdots \cup C_i$ for all i. This concerns cases where the "left" events in the partition are ranked best, i.e., have best outcomes, and the right events are ranked worst with worst outcomes.[8] If $n = 2$ and $\mathcal{C} = \mathcal{R}$, then $K(C_1)$ is the *matching probability* of A_1. Dimmock, Kouwenberg, & Wakker (2016) demonstrated how this classical Bayesian concept (Arrow 1951 Footnote 4; Raiffa 1968) serves well to analyze modern non-Bayesian ambiguity attitudes. Matching partitions generalize them so as to compare general sources of uncertainty.

As explained before, typical of nonexpected utility is that the decision weight of a fixed event may change if the act changes (Figure 1). The following subsections will introduce and intuitively explain preference conditions from this perspective.

4.2. Source Preference

Figure 2 compares changes of decision weights for two matching partitions (A_1, A_2) and (C_1, C_2). If we change the ranks of A_2 and C_2 from worst (Fig. 2a) to best (Fig. 2b), and we get $W(A_2) < W(C_2)$, then A_2 has lost more weight than C_2; or it has gained less.[9] In other words, there is less attention for favorable events (so, more for unfavorable eventes) for source $\mathcal{A}$ than for source $\mathcal{C}$. It leads to more dislike of $\mathcal{A}$. Formally, source preference (or preference for short) for $\mathcal{C}$ over $\mathcal{A}$ allows for such inequalities but precludes any reversed inequalities. Eq. 7 below will show that this definition agrees with common concepts in the literature.

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[8] Following mathematical conventions, we denote partitions using brackets rather than braces here, to indicate that the ordering of the events may sometimes be relevant.

[9] This violates EU because, under EU, the weights of the events should be invariant and, hence, so should be their equality.

FIGURE 2. Source preference for $\mathcal{C}$ over $\mathcal{A}$

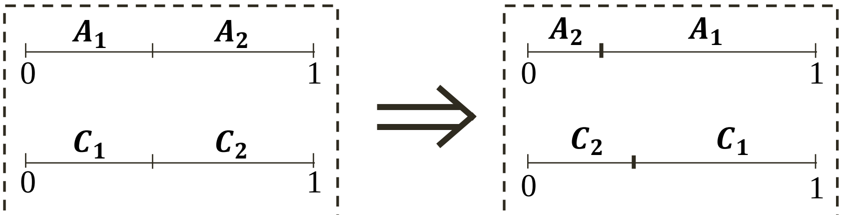

FIG 2a. Two matching partitions

FIG 2b. A_2 lost more weight than C_2

Decision weights are depicted as in Figure 1;
$W(A_1) = W(C_1) \Rightarrow W(A_2) \leq W(C_2)$;
upper fat bar to left of lower fat bar in Fig. 2b;
less attention for favorable events for $\mathcal{A}$.

DEFINITION 4. (*Source) preference* for $\mathcal{C}$ over $\mathcal{A}$ holds, or $\mathcal{C}$ is *preferred* to $\mathcal{A}$, if for all partitions (C_1, C_2) from $\mathcal{C}$ and (A_1, A_2) from $\mathcal{A}$ we have

$$W(A_1) = W(C_1) \;\Rightarrow\; W(A_2) \leq W(C_2). \tag{6}$$

Source indifference means source preference both ways; we then also say that the two sources are *equally preferred*. □

Absolute and relative ambiguity attitude results readily follow. Thus, *ambiguity aversion* for source $\mathcal{A}$ means source preference for $\mathcal{R}$ over $\mathcal{A}$. Comparative results for ambiguity and uncertainty simply coincide: *more ambiguity aversion* for source $\mathcal{A}$ than for source $\mathcal{C}$ is the same as source preference for $\mathcal{C}$ over $\mathcal{A}$. Corresponding definitions apply to ambiguity seeking instead of aversion. *Ambiguity indifference* for source $\mathcal{A}$ means that there is both ambiguity aversion and ambiguity seeking for $\mathcal{A}$. Then $\mathcal{A}$ is liked equally preferred as .

As we will elaborate on later, we should distinguish between ambiguity indifference and ambiguity neutrality. In Corollary 17 and Example 18, we will see that, for ambiguity neutrality, we should not only have ambiguity indifference, i.e., the same source preference as for risk, but also the same insensitivity, defined later.

Given the richness and monotonicity that we assume, source preference for $\mathcal{C}$ over $\mathcal{A}$ is equivalent to the following condition (Appendix A):

$$W(A_1) \geq W(C_1) \Longrightarrow W(A_2) \leq W(C_2). \tag{7}$$

4.3. Insensitivity

Source preference (including ambiguity aversion) is a motivational component of ambiguity attitudes. It makes the agent focus more on the best or on the worst outcomes. However, ambiguity attitudes also include a cognitive component, insensitivity, reflecting lack of discriminatory power and perception. It does not concern focusing on good versus bad, but focusing on middle versus extremes. Focusing on extremes decreases discriminatory power and blurs perception. We will show how matching partitions can be used to study this component, in a similar way as for ambiguity aversion and source preference. The insensitivity component plays an important role in empirical studies of ambiguity (Trautmann & van de Kuilen 2015). Theorem 9 and Corollary 17 will give further, theoretical, support, confirming that this component is needed to fully capture ambiguity attitudes. Hence, this paper introduces a formalization of the insensitivity component.

For risk, insensitivity is modeled through inverse-S shaped probability weighting (Fehr-Duda & Epper 2012) and is the prevailing pattern. Insensitivity is even stronger for ambiguity.[10] The uncertainty about probabilities under ambiguity leads to yet more attention for deviations and outliers and, thus, to extra weight for tail events and less for intermediate events[11], like increased variance. Its behavioral implication is extra extremity orientedness. Cognitively, there is less attention and discriminatory power for the intermediate events, which are more treated as one blur, moved towards a default of "just don't know."

Many ambiguity models in the literature contain a component of ambiguity perception, determined by the vagueness of information. For instance, in the α-maxmin model, the size of the set of priors can be interpreted this way. This model

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[10] In general, phenomena for risk also occur for ambiguity, but to a more pronounced degree. See Fellner (1961 p. 684) and many other references in Wakker (2010 p. 292). More recent references include Gayer (2010), Kemel & Paraschiv (2013), Maafi (2011), and Vieider et al. (2015).

[11] Following Savage (1954), events in themselves carry no value. Their favorability and extremity is determined by the outcomes they yield, which depends on the act considered. This is implicitly understood throughout this paper.

has the empirically desirable feature that it can accommodate insensitivity. Increasing the set of priors indeed increases insensitivity, with more weight for extreme outcomes. Insensitivity generalizes perception because it can also depend on the cognitive ability of the agent, a subjective element. Many studies[12] have shown that, empirically, insensitivity indeed does depend on subjective cognitive abilities. Hence, for empirical work, this subjective generalization of ambiguity perception is useful. Baillon et al. (2021) theoretically analyzed indexes of insensitivity and showed that in many ambiguity models (that, if taken normative, involve no cognitive limitations) they agree with indexes of perception, e.g., sizes of sets of priors in several multiple priors models. An important feature of insensitivity is that it brings together perception in ambiguity and inverse-S shaped probability weighting in risk (Fehr-Duda & Epper 2012).

To compare $\mathcal{A}$ to $\mathcal{C}$ regarding insensitivity, we compare events when ranked extreme (best or worst) versus when ranked middle; see Figure 3. We now take a threefold partition (A_1, A_2, A_3) and find a matching partition (C_1, C_2, C_3). We see what happens to A_2 and C_2 when their rank turns from middle (as in Fig. 3a = Fig. 3a´) to extreme, by switching best rank with the bold A_1 and C_1 to the left in Fig. 3b, and switching worst rank with the bold A_3 and C_3 to the right in Fig. 3c. Consider $W(A_2) > W(C_2)$ (Fig. 3b) and $W(A_2^c) < W(C_2^c)$ (Fig. 3c).[13] That is, A_2 adds more to impossibility or subtracts more from certainty than C_2 does. In either case, A_2 gains more weight than C_2 when ranked extreme. There is more attention for extreme events (and less for middle events) for source $\mathcal{A}$ than for source $\mathcal{C}$. This reflects higher insensitivity for $\mathcal{A}$ than for $\mathcal{C}$. This higher insensitivity allows for the aforementioned violations but precludes any reversed ones. That is, we require both

$$W(A_2) \geq W(C_2) \tag{8}$$

and

[12] These include Dimmock et al. (2021), Grevenbrock et al. (2021), and Zhang & Maloney (2012).

[13] These are two ways to violate EU. Under EU, the weights of the events should be invariant and, hence, so should be their equality.

$$W(A_2^c) \leq W(C_2^c)\,. \tag{9}$$

FIGURE 3. $\mathcal{A}$ is more insensitive than $\mathcal{C}$

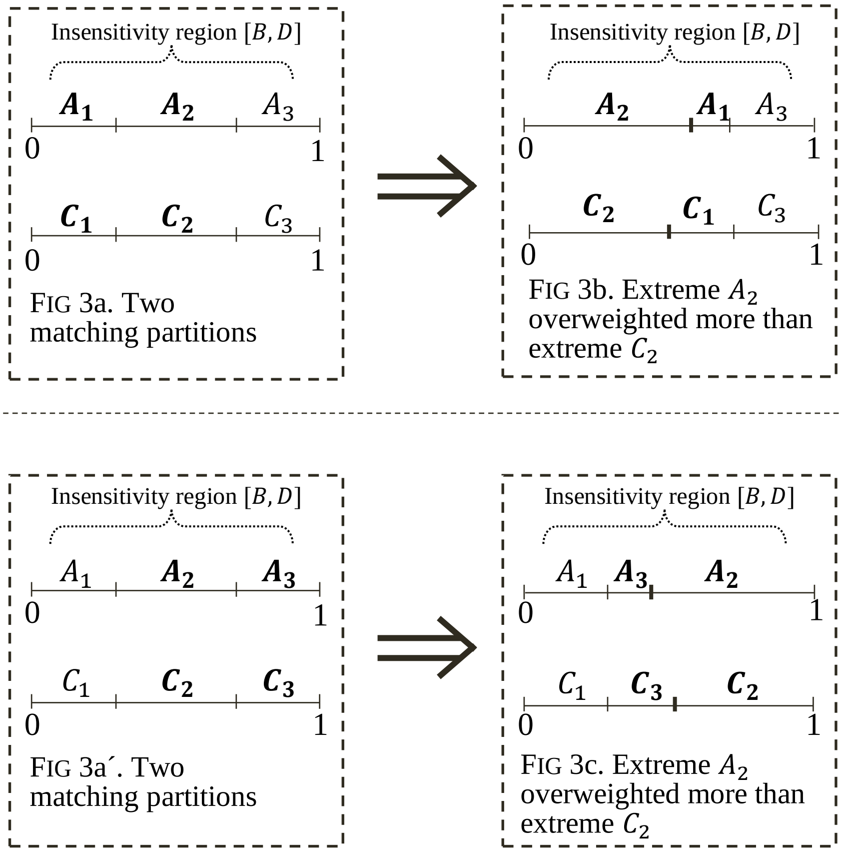

FIG 3a. Two matching partitions

FIG 3b. Extreme A_2 overweighted more than extreme C_2

FIG 3a´. Two matching partitions

FIG 3c. Extreme A_2 overweighted more than extreme C_2

Decision weights as in Figure 1.
In Fig. 3a =3a´, $W(A_1) = W(C_1)$ and $W(A_1 \cup A_2) = W(C_1 \cup C_2)$.
More attention for extreme events for $\mathcal{A}$ than for $\mathcal{C}$:
In Figs 3b and 3c, C_2's and A_2's ranks changed from middle to extreme. A_2 *gained more weight than* C_2: ordering of fat bars as in Fig. 3a and as in Fig. 3b.

When comparing events at one extreme, they should be safely separated from the other extreme by not crossing the border of the insensitivity region there. Thus, $A_1 \cup A_2$ should not cross the right bound D of the insensitivity region in the upper figures, and $A_2 \cup A_3$ should not cross the left bound B of the insensitivity region in the lower figures. Same then holds for C events.

The lower figures are dual to the upper ones by interchanging left and right.

We have to add boundary restrictions to the two requirements. We want to compare events when ranked middle and when ranked extreme (best or worst; left or right in Figure 3), but we want to avoid commitment to any direct comparison between the two extremes, i.e., between (almost) best and (almost) worst ranks. Imagine in Fig. 3a that A_3 were empty (so that C_3 would be null). Then A_2, ranked best in Fig. 3b, would be ranked worst in Fig. 3a and we would run into a comparison to be avoided.[14] We, therefore, impose the inequalities only if the middle event, to be moved to one extreme, is safely bounded away from the other extreme. That is. $W(A_3^c)$ and, equivalently, $W(C_3^c)$ should not be too large in the upper half of Figure 3, and $W(A_1)$ and $W(C_1)$ should not be too small in the lower half. We should accept that there will be an upper bound, denoted $max\delta$, such that Eq. 8 only holds if $W(C_3^c) \leq max\delta$, and there is a lower bound $min\beta$ such that Eq. 9 only holds if $W(C_1) \geq min\beta$. In applications, it is often sufficient to impose Eq. 8 for a "good enough" upper bound $\delta \leq \; max\delta$ for $W(C_3^c)$ and Eq. 9 for a "good enough" lower bound $\beta \geq min\beta$ for $W(C_1)$, and we do not need to know the exact $max\delta$ and $min\beta$. For later purposes, it is convenient to specify events B, D with $W(B) = \beta$ and $W(D) = \delta$. The conditions ensure that W is more shallow and "insensitive" for $\mathcal{A}$-event than for $\mathcal{C}$-event between B and D, which is why we call $[B, D]$ the insensitivity region.

DEFINITION 5. There is *more insensitivity* for source $\mathcal{A}$ than for source $\mathcal{C}$ (or $\mathcal{A}$ is *more insensitive than* $\mathcal{C}$) with *insensitivity region* $[B, D]$, if for all partitions (C_1, C_2, C_3) from $\mathcal{C}$ and (A_1, A_2, A_3) from $\mathcal{A}$:

$$W(A_1) = W(C_1) \;\&\; W(A_3^c) = W(C_3^c) \leq W(D) \;\Rightarrow\; W(A_2) \geq W(C_2) \qquad (10)$$

and

$$W(B) \leq W(A_1) = W(C_1) \;\&\; W(A_3^c) = W(C_3^c) \;\Rightarrow\; W(A_2^c) \leq W(C_2^c)\,. \qquad (11)$$

□

---

[14] Then Eq. 8 in fact reflects source preference.

B is the *Best-rank bound*, D is the *worst-rank ("Disliked") bound*, $minB$ is the *minimal best-rank bound*, and $maxD$ is the *maximal worst-rank bound*. Further, $[minB, maxD]$ is the *maximal insensitivity region*.[15] Taking events B and D with matching probabilities 0.05 and 0.95 usually works well in applications, and we can often even take B empty.

Again, absolute and relative ambiguity attitude results readily follow. Thus, *ambiguity-generated insensitivity*, or *a-insensitivity* for short, holds for a source if it is more insensitive than $\mathcal{R}$. Comparative results for ambiguity and uncertainty again coincide: more a-insensitivity for $\mathcal{A}$ than for $\mathcal{C}$ is the same as more insensitivity.

5. COMPARATIVE PREFERENCE CONDITIONS TO AXIOMATIZE SOURCE THEORY

Preference conditions to capture the aforementioned comparative properties readily follow because all conditions were in terms of inequalities and equalities for W that immediately translate into preferences and indifferences between events. We, therefore, use the same terms. *Source preference holds for $\mathcal{C}$ over $\mathcal{A}$* if, for all partitions (A_1, A_2) from $\mathcal{A}$ and (C_1, C_2) from $\mathcal{C}$:

$$A_1 \sim C_1 \Rightarrow A_2 \preccurlyeq C_2 \ . \tag{12}$$

There is *more insensitivity* for source $\mathcal{A}$ than for source $\mathcal{C}$ (or $\mathcal{A}$ is *more insensitive than $\mathcal{C}$*) with *insensitivity region* $[B, D]$, if for all partitions $\{C_1, C_2, C_3\}$ from $\mathcal{C}$ and $\{A_1, A_2, A_3\}$ from $\mathcal{A}$:

$$A_1 \sim C_1 \ \& \ A_3^c \sim C_3^c \preccurlyeq D \ \Longrightarrow \ A_2 \succcurlyeq C_2 \tag{13}$$

and

$$B \preccurlyeq A_1 \sim C_1 \ \& \ A_3^c \sim C_3^c \ \Longrightarrow \ A_2^c \preccurlyeq C_2^c \ . \tag{14}$$

---

[15] By solvability of W, $minB$ and $maxD$ always exist.

The following result is immediate.

OBSERVATION 6. W shows more source preference (or insensitivity) for one source over another if and only if preferences do. □

Thus, by taking the conditions below as preference conditions, we will obtain a preference axiomatization of ST in Theorem 9.

OBSERVATION 7. The source preference and source insensitivity relations are transitive. For insensitivity, the new insensitivity region is the intersection of the other two. □

Two sources are equally preferred (Definition 4), or *equally insensitive*, if the comparative relations hold in both directions. In what follows, insensitivity regions should be large enough to avoid triviality. An insensitivity region $[B, D]$ is *regular* for source $\mathcal{S}$ if for every threefold partition (C_1, C_2, C_3) of S from $\mathcal{S}$ we have $C_j \succcurlyeq B$ and $C_j^c \preccurlyeq D$ for at least one j. Intuitively, the region should capture at least the middle 1/3 of the event domain. If we take $\mathcal{S}$ uniform and B and D from $\mathcal{S}$, then $P_{\mathcal{S}}(B) \leq \frac{1}{3} \leq \frac{2}{3} \leq P_{\mathcal{S}}(D)$ follows (take $P_{\mathcal{S}}(C_j) = \frac{1}{3}$ for all j). Empirically, insensitivity regions are commonly found to be larger and regularity is not restrictive.

DEFINITION 8. A source $\mathcal{S}$ is *uniform* if it is equally preferred and insensitive to itself w.r.t. a regular insensitivity region. □

In the two-color Ellsberg urns, both the known and the unknown urn can be taken to be uniform. For Ellsberg's three-color urn, if all events defined by this urn are taken as one source, then it is not uniform.[16]

[16] It may better be modeled as two sources. We leave the study of interactions and intersections of different sources to future research.

We need one technical condition. The *Archimedean axiom* holds for source $\mathcal{S}$ if there is no infinite sequence of disjoint nonnull event $E_1, E_2, \ldots$ in S with $E_i \sim E_j$ for all i, j. The axiom is, in the presence of the other axioms, necessary and sufficient to ensure that probabilities are real-valued.

THEOREM 9. Under Structural Assumption 3, [source theory (Def. 2) holds for a source $\mathcal{S}$] if and only if [$\mathcal{S}$ is uniform and satisfies the Archimedean axiom]. □

There have been several axiomatizations of probabilistic sophistication (Chew & Sagi 2008; Grant, Rich, & Stecher 2022 Theorem 5; Machina & Schmeidler 1992). The main feature of our axiomatization (within one source) is that it captures the meaning of probabilistic sophistication directly in terms of ambiguity attitudes: the source must be equal to itself regarding ambiguity aversion and a-insensitivity. This observation suggests that our two components capture the essence of ambiguity attitudes. Theorem 16 and Corollary 17 will further support this point.

If ST holds for a source, then, given the preference relation and its biseparable utility representation, the representing functional over nonbinary acts is uniquely determined via equivalent binary acts. It is called the *source-theory (ST)* functional. The following corollary readily follows because, under ST, $P_{\mathcal{S}}$ represents both the preference relation $\succcurlyeq$ over events from $\mathcal{S}$ and its dual $\succcurlyeq^d$ (seen through complementary events).

COROLLARY 10. Under source theory for $\mathcal{S}$, when restricted to events from $\mathcal{S}$, $\succcurlyeq = \succcurlyeq^d$. $\mathcal{S}$ is equally insensitive to itself where we can take the maximal insensitivity region $[\emptyset, S]$. □

6. ABSOLUTE UNCERTAINTY ATTITUDES

The following proposition follows from substitution.

PROPOSITION 11. If EU holds for two sources, then they are equally preferred, and equally insensitive with the maximal insensitivity region $[\emptyset, S]$. □

Absolute conditions follow from comparative conditions by choosing a neutrality point, and EU is a natural choice here. This is justified by Proposition 11 and comparative results presented later. It underlies the central role of EU in Alon & Gayer's (2016) aggregation result. We obtain the following definitions. W is *liked* if always $W(E) \geq 1 - W(E^c)$ (Figure 2). If there is a source available on which EU is maximized, then W is liked if and only there is source preference for the EU source over W's entire domain, as follows from substitution. This illustrates how EU serves as neutrality point. *Disliked* results from the reversed inequality. W is *insensitive with insensitivity region* $[B, D]$ if for all partitions (E_1, E_2, E_3):

$$W(E_2) \geq W(E_1 \cup E_2) - W(E_1) \text{ whenever } W(E_1 \cup E_2) \leq W(D) \quad (15)$$

and

$$1 - W(E_2^c) \geq W(E_1 \cup E_2) - W(E_1) \text{ whenever } W(E_1) \geq W(B)\,. \quad (16)$$

The inequalities compare the decision weight of E_2 when ranked middle and when ranked extreme, safely bounded away from the other extreme. The conditions ensure that W is shallow and "insensitive" for events between B and D, i.e., on the insensitivity region $[B, D]$. Again, if there is a source available on which EU is maximized, then W is insensitive if and only if its entire domain is more insensitive than the EU source. Eq. 15 (and similarly 17 below) without the boundary restriction is sometimes called subadditivity. Insensitivity amounts to imposing subadditivity and its dual, and imposing boundary conditions to avoid the two conditions "biting" each other.

Similarly, for risk, w is *liked* if $w(p) \geq 1 - w(1 - p)$ for all p and *disliked* if the reversed inequality holds. Further, w is *insensitive with insensitivity region* $[b, d]$ $(0 \leq b < d \leq 1)$ if, for all probabilities p_1, p_2, p_3 summing to 1:

$$w(p_2) \geq w(p_1 + p_2) - w(p_1) \text{ whenever } p_1 + p_2 \leq d \quad (17)$$

and

$$1 - w(p_1 + p_3) \geq w(p_1 + p_2) - w(p_1) \text{ whenever } p_1 \geq b \ . \quad (18)$$

For w, the insensitivity region $[b, d]$ is a subinterval of the reals. If risk is available as a source with objective probability measure K, then $E[b, d]$ denotes the corresponding event interval $[B, D]$, i.e., $K(B) = b$ and $K(D) = d$. In other words, it contains the events with matching probabilities between b and d.

Under the assumption of EU for risk (or another source), commonly made in the literature on ambiguity, the absolute conditions can readily be axiomatized by applying Observation 6 to comparisons with those EU preferences.

7. COMPARING UNIFORM SOURCES

This section provides comparative uncertainty/ambiguity results for ST. We here obtain tractable and appealing characterizations in terms of transformation functions.

Matching partitions generalized matching probabilities. We now provide another generalization that will facilitate comparisons between different uniform sources. The *pmatcher* from a uniform source $\mathcal{A}$ to a uniform source $\mathcal{C}$ specifies, for each probability $P_{\mathcal{A}}(A)$ from $\mathcal{A}$, the gambling equivalent probability $P_{\mathcal{C}}(C)$ from $\mathcal{C}$. That is, C is taken such that $C \sim A$. The pmatcher is a function from [0,1] to [0,1] denoted φ. By solvability and strong monotonicity, it is well-defined, continuous, and strictly increasing. We have

$$\varphi = w_{\mathcal{C}}^{-1} \circ w_{\mathcal{A}} \quad \text{and} \quad w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi \ . \quad (19)$$

If $\mathcal{C} = R$, then the pmatcher is the matching probability function. Thus, pmatchers generalize matching probabilities. Uniform partitions (C_1, C_2, C_3) and (A_1, A_2, A_3) are matching if and only if $P_{\mathcal{C}}(C_1) = \varphi(P_{\mathcal{A}}(A_1))$ and $P_{\mathcal{C}}(C_3^c) = \varphi(P_{\mathcal{A}}(A_3^c))$.

ASSUMPTION 12 [for this section]. ST holds for sources $\mathcal{C}$ and $\mathcal{A}$ with generic events $C, C_1, C_2, \ldots$ and $A, A_1, A_2, \ldots$, a-neutral probability measures $P_{\mathcal{C}}$ and $P_{\mathcal{A}}$, and source functions $w_{\mathcal{C}}$ and $w_{\mathcal{A}}$. Further, φ denotes the pmatcher from $\mathcal{A}$ to $\mathcal{C}$. □

OBSERVATION 13. Under Structural Assumption 3 and Assumption 12, $\varphi(P_{\mathcal{A}}(A)) = P_{\mathcal{C}}(C)$ if and only if $A \sim C$. Consequently, $\varphi(P_{\mathcal{A}}(A)) > P_{\mathcal{C}}(C)$ if and only if $A \succ C$ and $\varphi(P_{\mathcal{A}}(A)) < P_{\mathcal{C}}(C)$ if and only if $A \prec C$. □

The following two theorems show that pmatchers directly capture comparative uncertainty attitudes and, hence, comparative ambiguity attitudes. The results deviate from traditional Arrow-Pratt results because the transformation φ applies to arguments rather than to images of $w_{\mathcal{C}}$.

THEOREM 14. Under Structural Assumption 3 and Assumption 12, [$\mathcal{C}$ is preferred to $\mathcal{A}$] if and only if [the pmatcher from $\mathcal{A}$ to $\mathcal{C}$ is disliked, i.e., $w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi$ with φ disliked]. Ambiguity aversion holds if and only if matching probabilities are disliked. □

THEOREM 15. Under Structural Assumption 3 and Assumption 12, [$\mathcal{A}$ is more insensitive than $\mathcal{C}$] if and only if [the pmatcher from $\mathcal{A}$ to $\mathcal{C}$ is insensitive, i.e., if $w_{\mathcal{A}} = w_{\mathcal{C}} \circ \varphi$ for an insensitive transformation φ]. Furthermore, if the insensitivity region for the preference condition is $[B, D]$ where the boundary events are from $\mathcal{A}$, then the insensitivity region for φ is $[P_{\mathcal{A}}(B), P_{\mathcal{A}}(D)]$. Ambiguity-generated insensitivity holds if and only if matching probabilities are insensitive. □

In the above theorem, for a general insensitivity region $[B, D]$, we can take B' and D' from $\mathcal{A}$ such that $B' \sim B$ and $D' \sim D$, and then the insensitivity region for φ is $[P_{\mathcal{A}}(B'), P_{\mathcal{A}}(D')]$.

The following theorem shows that we fully capture ambiguity and uncertainty attitudes through the two components of preference and insensitivity. We do not impose Assumption 12 in the following theorem because it is implied by the other conditions.

THEOREM 16. Under Structural Assumption 3, sources $\mathcal{A}$ and $\mathcal{C}$ are equally preferred and insensitive (with a regular insensitivity region) and satisfy the Archimedean axiom if and only both are uniform and $w_{\mathcal{A}} = w_{\mathcal{C}}$. We can take the maximal insensitivity region $[\emptyset, S]$. □

COROLLARY 17. Ambiguity neutrality holds for source $\mathcal{A}$ if and only if it is equally preferred[17] and insensitive as risk (with a regular insensitivity region) and satisfies the Archimedean axiom. We can take the maximal insensitivity region $[\emptyset, S]$. □

The following example shows that aversion/source preference alone is not enough to capture ambiguity and uncertainty attitudes, underscoring the importance of the insensitivity component.

EXAMPLE 18. Suppose an agent behaves according to EU for a known urn (risk), but has an insensitive, symmetric[18] but nonlinear, source function $w_{\mathcal{A}}$ for an unknown urn. Then $\gamma_A\beta \sim \gamma_p\beta \Rightarrow \gamma_{A^c}\beta \sim \gamma_{1-p}\beta$ so that ambiguity indifference holds. This has often been defined as ambiguity neutrality in the literature. However, the agent is less sensitive to the unknown urn and ambiguity greatly impacts the agent. □

As announced before, sources can be taken endogenous in our analysis. An *endogenous uniform source* is any algebra of events for which the uniformity preference condition holds, together with convex-rangedness of the restriction of W and Archimedeanity. Theorem 9 applies to such algebras and we get local probabilistic sophistication. This way, all results of this section can be applied.

OBSERVATION 19 [Endogenous results]. Observation 13, Theorems 14, 15, 16, and Corollary 17 remain valid if $\mathcal{A}$ and $\mathcal{C}$ are endogenous uniform sources. □

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[17] Being equally preferred as risk is also called ambiguity indifference.

[18] That is, $w_{\mathcal{A}}(p) + w_{\mathcal{A}}(1-p) = 1$ for all p, for instance if $w_{\mathcal{A}}(p) = \frac{p^2}{p^2+(1-p)^2}$.

FIGURE 4. Attitudes of subject 2 of Abdellaoui et al. (2011)

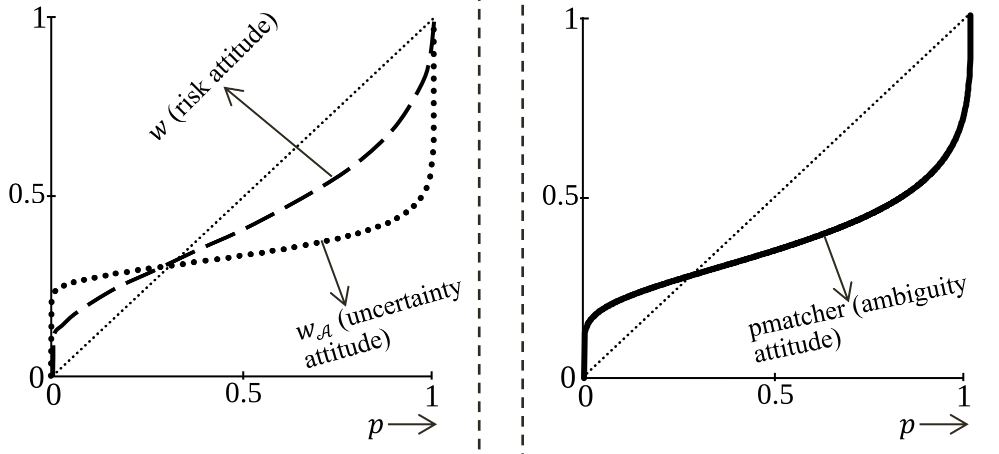

Fig. 4a. Source functions for CAC40 and for risk

Fig. 4b. Matching probabilities for CAC40

We, finally, illustrate the tractability of our tools. Figure 4 shows the ease with which uncertainty attitudes can be completely captured and compared visually under ST. It shows the data of subject 2 of Abdellaoui et al. (2011, Figure 10).[19] Source $\mathcal{A}$ is the CAC40 stock index, which passed a test for uniformity. Source $\mathcal{C}$ is risk. Fig. 4a shows the source functions $w_{\mathcal{A}}$ and w. Risk is disliked ($w(p) + w(1-p) \leq 1$) and also exhibits insensitivity. Dislike and insensitivity are reinforced by the extra uncertainty due to the ambiguity of CAC40. Fig. 4b shows that the pmatcher ($w^{-1} \circ w_{\mathcal{A}}$), i.e., the matching probability function, indeed has the corresponding properties, so that the comparative conditions of Theorems 14 and 15 hold. The extra insensitivity due to ambiguity reflects (possibly as-if) doubts about the a-neutral probabilities, shifting extra weight to the extremes, i.e., to deviations from standard expectations.[20] The area between the weighting function and the diagonal in Fig. 4a can be taken as an index of insensitivity/perception, similar to the variance of

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[19] Abdellaoui et al. have $w_{\mathcal{A}} = exp(-1.14(-ln(\text{p})^{0.15})$ and $w = exp(-1.06(-ln(\text{p})^{0.47})$. By Eq. 19, the pmatcher is $exp(-1.17(-ln(\text{p})^{0.32})$. The maximal insensitivity regions are [0, 0.9993], [0, 0.965], and [0, 0.975], respectively.

[20] Thus, it reinforces both the ambiguity seeking and the ambiguity aversion that are present. Many theoretical studies assume universal ambiguity aversion, and then insensitivity/perception only reinforces ambiguity aversion.

Izhakian (2020). It is also similar to the size of the sets of priors under the α-maxmin model. We note that we did not need multistage stimuli to obtain our measurements.

8. DISCUSSION

Related literature. Many studies on ambiguity used the Anscombe-Aumann (1963) (AA) framework. Here, acts do not assign outcomes to states but probability distributions over prizes. They are, thus, two-stage. EU is assumed for risk[21], and a backward induction evaluation is applied to the two-stage acts[22]. This framework makes it possible to use linear algebra to analyze ambiguity, which greatly facilitates the mathematical analysis and, thus, has propelled the ambiguity field. Multistage optimization should be studied for applications anyhow, and plays a big role in ambiguity. Yet, there is also interest in studying ambiguity in a single-stage framework such as Savage's (1954). Especially for empirical work, it is desirable to allow for violations of EU (Starmer 2000). Further, multistage stimuli are complex for tests and applications. Regarding backward induction and, in general, multistage optimization, as unproblematic and self-evident as they are under classical EU, so problematic and controversial they are under ambiguity and nonEU.[23] Many studies have, thus, criticized backward induction in the AA framework.[24] ST introduces its ambiguity concepts while avoiding dynamic complications and while allowing violations of EU for risk. As a price to pay, our results had to be derived without resorting to linear algebra as a tool to simplify the mathematics. As we have shown, our tools still remain tractable, and their validity is immune to violations of backward induction or EU for risk.

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[21] Hill (2019) and Wang (2022) allowed violations of EU for risk, sacrificing some tractability.

[22] Denti & Pomatto (2022) showed that this is equivalent to separable partitions of the state space on which to condition ambiguity of information. Li (2020) considered a similar approach.

[23] Normative criticisms include Jaffray (1998), McClennen (1990), and Machina (1989). Empirical criticisms are referenced later.

[24] See Eichberger, Grant, & Kelsey (2007), Ellis (2018), Georgalos (2021), König-Kersting, Kops, & Trautmann (2023 pp. 2-3), and Schneider & Schonger (2018).

ST is a specification of Choquet expected utility (Gilboa 1987; Schmeidler 1989) and prospect theory (Tversky & Kahneman 1992). Those theories use nonadditive measures. However, it has often been argued that nonadditive measures are too general to be tractable.[25] Basu & Echenique (2020) showed that this holds even more for multiple prior models. Second-order distributions, as used in the smooth model (Klibanoff, Marinacci, & Mukerji 2005), are of yet higher cardinality and have this problem even more. It holds yet more for various generalizations of these theories provided in the literature. Spiliopoulos & Hertwig (2023) pointed this problem out for their extensive empirical study and, hence, used the source method because of its tractability.[26] This problem was also pointed out by Chew, Bin, & Zhong (2017).

Besides Gilboa (1987), axiomatizations of ambiguity models that do not use the Anscombe-Aumann framework include Alon (2022), Borie (2023), and Casadesus-Masanell, Klibanoff, & Ozdenoren (2000) for maxmin EU, and Tversky & Kahneman (1992) and Wakker (2010 Ch. 10) for nonadditive measures. As explained, these models are too general to be tractable. Further, maxmin EU assumes EU for risk and cannot accommodate insensitivity.

The aforementioned ambiguity models, using high-dimensional parameters, have been used in empirical studies, but then strong parametric assumptions had to be added, especially if the underlying models were very general. Then the extra assumptions, rather than the underlying model, may have driven the results to a large extent (Polisson, Quah, & Renou 2020 p. 1783). ST uses nonadditive weighting functions but adds uniformity restrictions and, thus, achieves better parsimony. Abdellaoui et al. (2011) and Dimmock, Kouwenberg, & Wakker (2016) showed that the source method is tractable enough to even allow for nonparametric measurements, i.e., without any parametric assumption added. The source method outperformed other

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[25] See Basu & Echenique (2020), Grabisch & Labreuche (2008 §2.7 and §7), Ivanov (2011 p. 367), and Tversky & Kahneman (1992 p. 311).

[26] They used the term two-stage model (Fox & Tversky 1998), but this uses a decomposition $w(P)$ where w is the risk-probability weighting function and ambiguity is captured solely through a, nonadditive, P, usually based on introspective measurements. Spiliopoulos & Hertwig (2023) instead used w to capture ambiguity attitudes. That is, they used the source method.

ambiguity theories in prediction tests (Georgalos 2019; Kothiyal, Spinu, & Wakker 2014), underscoring its good parsimony. It provided a better fit than the smooth model in Abdellaoui et al. (2021).

Chandrasekher et al. (2022) generalized the multiple priors models and, like us, emphasized the importance of insensitivity and source preference. Their model is very general and needs specifications before being empirically applicable. Gul & Pesendorfer (2015) developed a deep mathematical ambiguity theory that does not use the Anscombe-Aumann framework. However, its requirement that all ideal events (from maximally preferred sources, all interpreted as unambiguous[27]) should be elicited from preferences and their probabilities measured, needed to obtain inner and outer measures for ambiguous events, is problematic. So is the requirement that diffuse events exist that involve extreme decision attitudes that are neither empirically nor normatively plausible and violate monotonicity (Grant, Rich, & Stecher 2022 p. 10).

In many ambiguity theories, ambiguity attitudes depend mainly on the set of outcomes and not on the events. Examples include Chew et al. (2008), Grant, Rich, & Stecher (2022), and Skiadas (2013). The most well-known theory of this kind is Klibanoff, Marinacci, & Mukerji's (2005) smooth model. Such theories cannot accommodate the fourfold pattern of ambiguity, or insensitivity (König-Kersting, Kops, & Trautmann 2023). Like Dillenberger, Postlewaite, & Rozen (2017) and Machina (2009 p. 390), we think that ambiguity attitudes are mainly event-driven rather than outcome-driven. Chew, Bin, & Zhong (2017) found that event-driven models fit data better than the smooth model. Giraud (2014) and Izhakian (2020) presented an event-driven variation of the smooth model that can accommodate insensitivity. They assumed EU for risk and backward induction, and used high-dimensional subjective parameters. Chateauneuf, Eichberger, & Grant's (2007) neo-additive model is a popular and efficient special case of ST that focuses only on overweighting infimum and supremum values, as does α-maxmin EU. This ignores attitudes towards intermediate values, relevant for instance in values at risk and their generalizations in finance.

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[27] This is violated by Tversky & Fox's (1995) empirical findings and by our Example 18.

This paper extends concepts, source preference and insensitivity, of Tversky & Wakker (1995), the theoretical counterpart to Tversky & Fox (1995). Tversky & Wakker used traditional between-subject comparisons (except their §9, discussed further below). Nascimento & Ng (2021) extended their results to conditions on weighting functions and derived advanced comparative results. Nascimento, Ng, & Gonzalez (2022) provided detailed numerical analyses of various parameters of source preference and insensitivity. We, to the contrary, focussed on the main novelty of uncertainty: within-subject between-source comparisons. Observation 6, our, trivial, starting result, was given by Tversky & Wakker (1995 §7). Other than that, our results are new. In particular, we compare the same function in different subdomains. All preference-axiomatizations of Pratt-Arrow-type transformations in the literature, including Tversky & Wakker (1995) and Nascimento & Ng (2021), compared different functions on the same domain and applied transformations to images of functions ("outside"). We instead apply transformations to arguments of functions ("inside"), formalizing and justifying Kemel & Gutierrez's (2022) empirical implementation. Wakker (2004) axiomatized a simple version of an "inside" transformation.

Mathematical tractability. Besides being empirically tractable, ST also provides tractable calculations. The formula

$$\int_0^\infty w_{\mathcal{S}}\left(G_{f,U}(\alpha)\right)d\alpha \tag{20}$$

captures the rank-dependent utility value of an act f from a uniform source $\mathcal{S}$, assuming nonnegative outcomes and utilities. Here, with $P_{\mathcal{S}}$ the a-neutral probability measure on $\mathcal{S}$, $G_{f,U}$ denotes the dual of the distribution function that f induces over outcome utilities by $P_{\mathcal{S}}$, and $w_{\mathcal{S}}$ denotes the source function. Except for $w_{\mathcal{S}}$, everything in this formula is standard under EU and, thus, under any ambiguity theory—because they all generalize EU. The mere addition of the transformation $w_{\mathcal{S}}$ is straightforward and, we think, easier than, for instance, adding higher-order distributions and calculating double integrals as in the smooth model, or solving maximization and minimization problems as in multiple prior models. Our representation theorems in §7 involve simply replacing $w_{\mathcal{S}}$ by $w_{\mathcal{S}} \circ \varphi$. Alternative formulas for rank-dependent utility require a ranking of outcomes and authors have complained about that (Spiliopoulos & Hertwig 2023 Footnote 12). However, again, ranking outcomes is easier than solving optimization problems or calculating extra integrals.

Further topics. Several authors proposed cavexity (first concave overweighting and then convex underweighting) as an alternative definition of insensitivity. This definition critically depends on the specification of one reflection point, rather than of two (noncritical!) boundaries of the insensitivity region. The essence of insensitivity, however, is the overweighting of extreme outcomes, and not the nature of curvature in the middle region. This curvature is usually close to linear anyhow, making the reflection point highly unstable. Thus, cavexity does not capture the essence of insensitivity. Wakker (2010 Appendix 7.12) provided further criticisms of cavexity. Araujo et al. (2018) established the existence of equilibria while reckoning with insensitivity.

Following Tversky & Fox (1995 p. 271), we assumed one fixed w for all objective probabilities. We let parsimony prevail over fit here for tractability reasons. The assumption holds approximately for emotion-neutral risky events and outcomes and we focus on those.[28] Objective probabilities served as the neutrality benchmark for ambiguity attitudes.

9. CONCLUSION

Source theory shows, in one sentence, how to use probability weighting functions to capture ambiguity and uncertainty attitudes. It does so by specifying sources of events and letting the weighting functions depend on those sources. Those weighting functions can easily be measured and displayed in graphs. This makes source theory suitable for applied work. It does not assume expected utility for risk (desirable for empirical purposes) and does not commit to assumptions about dynamic decisions. Source theory captures the main findings on decision under ambiguity and risk,

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[28] We assume a fixed outcome set. Violations have been found for events and outcomes inducing particular emotions, e.g., if referring to complex arithmetics (Armantier & Treich 2016) or particular familiarities (Chew, Ebstein, & Zhong 2012).

including the empirically prevailing insensitivity. It is specific enough to allow for measurements and predictions, even without parametric assumptions. We have thus shown that ambiguity can be tractably formalized and applied in the Savage-Gilboa framework.

APPENDIX A. REFORMULATIONS USING WEAK PREFERENCES

We give reformulations of some conditions using weak preferences instead of indifferences in the premises. The reformulations are more restrictive and would lead to less powerful axiomatizations, but may be better suited for empirical tests in coarse data sets where indifferences are not easy to obtain. The first reformulation was already given in the main text, in Eq. 7.

LEMMA 20. Eq. 6 is equivalent to Eq. 7.

PROOF. Eq. 7 immediately implies Eq. 6. We next assume Eq. 6 and derive Eq. 7. Assume $W(A_1) \geq W(C_1)$. By convex-rangedness, we can take a part of A_1 and move it to A_2 so that the premise of Eq. 6 follows. The resulting conclusion in that equation and set-monotonicity of W imply Eq. 7. □

We next give the corresponding reformulations of insensitivity, and of the prefefence conditions.

LEMMA 21. Eq. 10 is equivalent to:

$$W(C_1) \geq W(A_1) \;\&\; W(C_3^c) \leq W(A_3^c) \leq W(D) \implies W(A_2) \geq W(C_2)\,. \tag{21}$$

Eq. 11 is equivalent to:

$$W(C_3^c) \leq W(A_3^c) \;\&\; W(C_1) \geq W(A_1) \geq W(B) \implies W(A_2^c) \leq W(C_2^c)\,. \tag{22}$$

Eq. 12 is equivalent to:
For all partitions (A_1, A_2) from $\mathcal{A}$ and (C_1, C_2) from $\mathcal{C}$:

$$A_1 \succcurlyeq C_1 \Rightarrow A_2 \preccurlyeq C_2\,. \tag{23}$$

Eq. 13 is equivalent to:

For all partitions (C_1, C_2, C_3) from $\mathcal{C}$ and (A_1, A_2, A_3) from $\mathcal{A}$:

$$C_1 \succcurlyeq A_1 \ \&\ C_3^c \preccurlyeq A_3^c \preccurlyeq D \implies A_2 \succcurlyeq C_2 \,. \tag{24}$$

Eq. 14 is equivalent to:

For all partitions (C_1, C_2, C_3) from $\mathcal{C}$ and (A_1, A_2, A_3) from $\mathcal{A}$:

$$B \preccurlyeq A_1 \preccurlyeq C_1 \ \&\ A_3^c \succcurlyeq C_3^c \implies A_2^c \preccurlyeq C_2^c \,. \tag{25}$$

PROOF. It is immediate that Eq. 21 implies Eq. 10, so, we assume Eq. 10 and derive Eq. 21. Assume $W(C_1) \geq W(A_1)$ and $W(C_3^c) \leq W(A_3^c)$. By convex-rangedness, we can move part of C_1 to C_2 such that $W(C_1) = W(A_1)$ results. Similarly, we move part of C_3 to C_2 such that $W(C_3^c) = W(A_3^c)$. (Note here that the previous move of part of C_1 to C_2 did not affect C_3^c.) Now Eq. 10 and set-monotonicity of W imply Eq. 21. The equivalence of Eqs. 11 and 26 follows by similar moves. It in fact is dual to Eqs. 10 and 21. The remaining results in terms of preferences follow from the corresponding results on W. □

APPENDIX B. PROOFS

PROOF OF OBSERVATION 1. We define $w(r) = W(R)$ for event R with $K(R) = r$. It is well-defined because $K(R) = K(R') = r$ implies, by stochastic dominance, $W(R) = W(R') = w(r)$. Convex-rangedness readily implies that w's domain is the entire [0,1], and further w is strictly increasing (Online Appendix). We now have Eq. 3 for all risky events R. We have $w(0) = 0,\ w(1) = 1$. Surjectivity, implied by convex-rangedness, implies that w is continuous. □

PROOF OF OBSERVATION 7. See Figures 5-7. In these figures, given matching $\mathcal{A}$ and $\mathcal{C}$ partitions, the required matching $\mathcal{B}$ partitions can be obtained using convex-rangedness. Then transitivity of equality, inequality, and logical implication implies all that is needed. For Figure 6 (7), the required boundary conditions on A_3 (A_1) for the two implications follow from the boundary condition in the observation. □

FIGURE 5. Transitivity of source preference

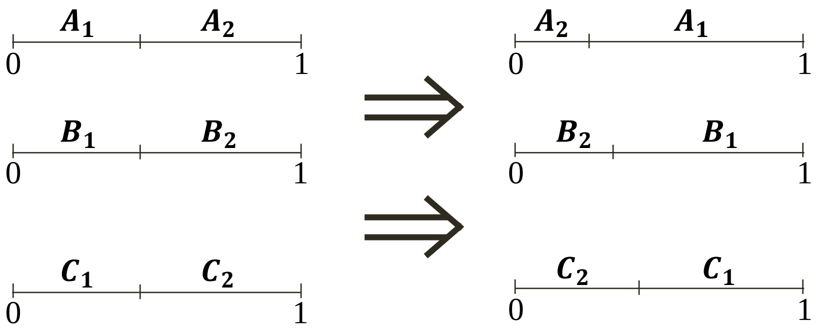


FIGURE 6. Transitivity of more insensitive

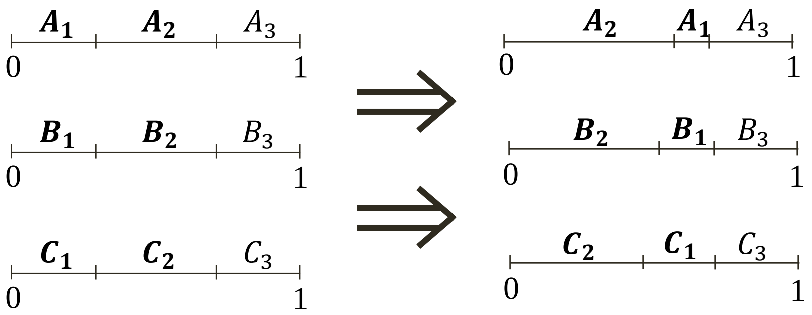
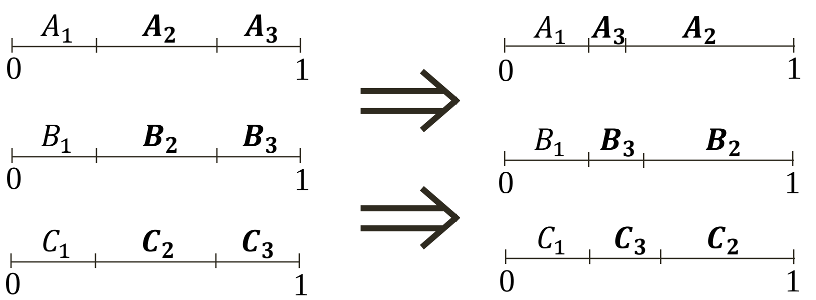


Figure 7. Transitivity of more insensitive



PROOF OF THEOREM 9. The proof will be stated in terms of preference conditions. Throughout the proof, all events A, B, C and so on are assumed to be from the source

$\mathcal{S}$. By convex-rangedness, boundary events can be assumed to be from $\mathcal{S}$. We write P for $P_{\mathcal{S}}$ throughout this proof. We first show that the axioms are necessary. For source preference, assume $A_1 \sim B_1$. This implies $P(A_1) = P(B_1)$, so, with subscript 2 indicating complementary events: $P(A_2) = P(B_2)$; $W(A_2) = W(B_2)$; $A_2 \sim B_2$ so $A_2 \succcurlyeq B_2$, as required by source preference. For insensitivity, consider threefold partitions as before. $A_1 \sim B_1$ and $A_3^c \sim B_3^c$ imply $P(A_1) = P(B_1)$ and $P(A_3) = P(B_3)$, so, $P(A_2) = P(B_2)$ and hence, both $A_2 \sim B_2$ and $A_2^c \sim B_2^c$. We have $A_2 \succcurlyeq B_2$ and $A_2^c \preccurlyeq B_2^c$, as required by more insensitivity. The boundary conditions were not needed here. This shows that the insensitivity region can be taken maximal: $\emptyset, S]$. The Archimedean axiom follows directly. It can also be directly seen that cumulative dominance is implied by biseparable utility and probabilistic sophistication.

The rest of this proof concerns the reversed implication, and assumes the Archimedean axiom, cumulative dominance, and uniformity. We throughout use the reformulations in Appendix A. By source preference, we have: $A \succcurlyeq B \Rightarrow A^c \preccurlyeq B^c$. The reversed implication, $A^c \preccurlyeq B^c \Rightarrow A \succcurlyeq B$ follows by taking complementary events. Hence, we have

$$A \succcurlyeq B \Leftrightarrow A^c \preccurlyeq B^c \tag{27}$$

within the uniform source $\mathcal{S}$. Assume a regular insensitivity region $[B, D]$.

LEMMA 22. Assume $E \cap G = F \cap G = \emptyset$. Then $E \succcurlyeq F \Leftrightarrow E \cup G \succcurlyeq F \cup G$.

PROOF OF LEMMA 22.

CASE 1. First assume $E \succcurlyeq F$. We derive $E \cup G \succcurlyeq F \cup G$. Define $A_1 = F,\ A_2 = (F \cup G)^c, A_3 = G$, and $C_1 = E,\ C_2 = (E \cup G)^c, C_3 = G$.

CASE 1.1. Assume $A_3^c \preccurlyeq D$. By Eq. 24, $A_2 \succcurlyeq C_2$, by Eq. 27 implying $A_2^c \preccurlyeq C_2^c$, i.e., $E \cup G \succcurlyeq F \cup G$.

CASE 1.2. Assume $A_1 \succcurlyeq B$. By Eq. 25, $A_2^c \preccurlyeq C_2^c$, i.e., $E \cup G \succcurlyeq F \cup G$.

CASE 1.3. Assume $A_1 \prec B$ and $A_3 \prec D^c$ (so that $B \succ \emptyset$ and $D \prec S$). Then, by regularity, G exceeds all these events. By convex-rangedness, there exists $G_1 \subset G$ such that $A_1 \cup G_1 \sim B$. Define $G_2 = G - G_1$ and $A_3' = A_3 \cup G_2$. Now ${A_3'}^c = A_1 \cup G_1 \sim B \preccurlyeq D$. Therefore, by Case 1.1 applied to $\{A_1, G_1, A_3'\}$ and $\{C_1, G_1, C_3'\}$ with $C_3' = C_3 \cup G_2$, we have that $A_1 \succcurlyeq C_1$ implies $A_1 \cup G_1 \succcurlyeq C_1 \cup G_1$.

Now define $A_1' = A_1 \cup G_1$, $C_1' = C_1 \cup G_1$. By Case 1.2 applied to $\{A_1', G_2, A_3\}$ and $\{C_1', G_2, C_3\}$ we have that $A_1 \cup G_1 \succcurlyeq C_1 \cup G_1$ implies $A_1 \cup G_1 \cup G_2 \succcurlyeq C_1 \cup G_1 \cup G_2$, i.e., $E \cup G \succcurlyeq F \cup G$. Case 1 is done.

CASE 2. We next assume $E \succ F$, and derive $E \cup G \succ F \cup C$. By convex-rangedness, there exists $A_1 \subset E$ with $A_1 \sim F$. By Case 1, $A_1 \cup G \succcurlyeq F \cup G$. $A_2 \coloneqq E - A_1$ is nonnull. By monotonicity, $E \cup G \succ A_1 \cup G$. Transitivity gives $E \cup G \succ F \cup G$. QED

By Lemma 22, weak ordering of $\succcurlyeq$, convex-rangedness, and Krantz et al. (1971 Theorem 5.2.2), there exists a probability measure P on source $\mathcal{S}$ that represents the preference relation $\succcurlyeq$ on events. So does W and, hence, $W = w_{\mathcal{S}} \circ P$ for a strictly increasing $w_{\mathcal{S}}$. Cumulative dominance implies Eq. 2 w.r.t. $P = P_{\mathcal{S}}$ for all x, y from $\mathcal{S}$. That is, local probabilistic sophistication holds for $\mathcal{S}$. □

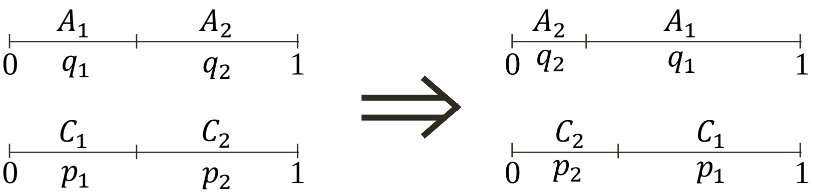
PROOF OF COROLLARY 23. That the insensitivity region can be taken maximal was demonstrated in the proof of Theorem 9. The rest is immediate. □

PROOF OF OBSERVATION 13. The first iff follows from the definition of φ. For the second iff, $A \succ C$ implies the existence of $A' \subset A$ with $A' \sim C$ and $\varphi(P_{\mathcal{A}}(A')) = P_{\mathcal{C}}(C)$. The second iff now mainly follows from monotonicity and transitivity. The third is similar. □

PROOF OF THEOREM 14. Figure 8 repeats Figure 2 with uniformity and probabilities added. The left side displays two matching partitions, which is equivalent to $\varphi(q_1) = p_1$. The implication $A_2 \preccurlyeq C_2$ is equivalent to $\varphi(q_2) \leq p_2 = 1 - p_1 = 1 - \varphi(q_1) =$

$1-\varphi(1-q_2)$. Thus, preference for $\mathcal{C}$ over $\mathcal{A}$ is equivalent to a disliked pmatcher from $\mathcal{A}$ to $\mathcal{C}$. The result on ambiguity follows from taking $\mathcal{C}=\mathcal{R}$. □

FIGURE 8. Preference for uniform $\mathcal{C}$ over uniform $\mathcal{A}$



In the left part: probability q_1 for source $\mathcal{A}$ gives the same decision weight (distance to 0) as probability p_1 for source $\mathcal{C}$. In the right part, q_2 and p_2 changed from worst to best rank: q_2 lost more weight than p_2.

PROOF OF THEOREM 15. See Figure 3. We first consider the upper half, dealing with Eqs. 10 and 11. Write q_j and p_j for the probabilities of A_j and C_j, respectively. (C_1,C_2,C_3) is matching with (A_1,A_2,A_3) iff $\phi(q_1)=p_1$ and $\varphi(q_1+q_2)=p_1+p_2$. Then $A_2 \succcurlyeq C_2$ if and only if $\varphi(q_2)\geq p_2=(p_1+p_2)-p_1=\varphi(q_1+q_2)-\varphi(q_1)$. The boundary condition $A_3^c \preccurlyeq D$ means $q_1+q_2\leq P_{\mathcal{A}}(D)$. The q probabilities are the arguments of φ. Hence, the worst-rank bound for φ is $P_{\mathcal{A}}(D)$. In general, if B, D are from a source $\mathcal{S}$, then the worst-rank bound for φ is $m(D)$ where m is the p-matcher from $\mathcal{S}$ to $\mathcal{A}$.

The lower half of Figure 3 is analyzed similarly. This case is the above case applied to the dual of W. The result on ambiguity follows from taking $\mathcal{C}=\mathcal{R}$. □

PROOF OF THEOREM THEOREM 16. We derive uniformity and $w_{\mathcal{A}}=w_{\mathcal{C}}$ from the other conditions. (The reversed implication is direct.) With R abbreviating source preference or source insensitivity, $\mathcal{C}R\mathcal{A}$ and $\mathcal{A}R\mathcal{C}$ and transitivity (Observation 7) imply $\mathcal{C}R\mathcal{C}$, similarly with $\mathcal{A}$. Hence, by Theorem 9, both sources are uniform. φ denotes the pmatcher from $\mathcal{A}$ to $\mathcal{C}$. By twofold source preference, $C{\sim}A \Rightarrow C^c{\sim}A^c$, i.e., $\varphi(P_{\mathcal{A}}(A)) = P_{\mathcal{C}}(C) \Rightarrow \varphi(1 - P_{\mathcal{A}}(A)) = 1 - P_{\mathcal{C}}(C)$; that is, $\varphi(p) + \varphi(1-p) = 1$. Hence, $\varphi\left(\frac{1}{2}\right) = \frac{1}{2}$ and we need to prove our result only on $[0, \frac{1}{2}]$.

Because $\mathcal{A}$ is more insensitive than $\mathcal{C}$, $\varphi(\varepsilon) \geq \varphi(p+\varepsilon) - \varphi(p)$ for every positive p and ("small") positive ε as long as $p + \varepsilon$ is below the upper bound of the insensitivity region which, by regularity, exceeds $\frac{1}{2}$. Hence, it holds on the entire $[0, \frac{1}{2}]$. Because $\mathcal{C}$ is more insensitive than $\mathcal{A}$, $\varphi^{-1}(\varepsilon) \geq \varphi^{-1}(p+\varepsilon) - \varphi^{-1}(p)$ for every positive p and ("small") positive ε as long as $p + \varepsilon$ is below the upper bound of the insensitivity region which, by regularity, exceeds $\frac{1}{2}$. Hence, it holds on the entire $[0, \frac{1}{2}]$. The two inequalities can only hold if φ is linear on $[0, \frac{1}{2}]$. Because $\varphi\left(\frac{1}{2}\right) = \frac{1}{2}$, φ must be the identity on $[0, \frac{1}{2}]$ and, then, on $[0,1]$. That the insensitivity region can be taken maximal, follows from substitution. □

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ONLINE APPENDIX TO
"SOURCE THEORY: A TRACTRABLE AND POSITIVE AMBIGUITY THEORY

Aurélien Baillon, Han Bleichrodt, Chen Li, & Peter P. Wakker

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ADDITION TO PROOF OF OBSERVATION 1

We assume w well-defined as in Observation 1. We first show that the domain of w is entire [0,1]. This proof would be easy if K were countably additive and defined on a sigma algebra. However, K is only finitely additive and defined only on an algebra. Assume, for contradiction, that the w-range RK of K is a strict subset of [0,1]. We can standardly define w well on RK as indicated above, and it is nondecreasing. By convex-rangedness of W, the image $w(RK)$ is the entire [0,1]. So, RK is uncountable. For each "small" $\epsilon > 0$ there exist probabilities $p < q$ in RK with $q - p \leq \varepsilon$. Take event A with $K(A) = q$. By convex-rangedness, there is a subset $B \subset A$ with $W(B) = w(p)$, i.e., $K(B) = p$. So, $K(A - B) < \varepsilon$. Using convex-rangedness of W, we can keep on extending a disjoint array A_1 , …,A_i with $K(A_i) = K(A - B)$ as long as $K(A_1 \cup \cdots \cup A_i)^c \geq\ K(A - B)$ so that also $W(A_1 \cup \cdots \cup A_i)^c \geq\ W(A - B)$. Such standard sequences for smaller and smaller ε readily show that KR is dense in [0,1]. Now, if a p is missing from KR, then we must have 0<p<1, and w([0,p)) and w((p,1]) provide a partition of [0,1] of two open nonempty sets, violating connectedness of [0,1]. This shows that w's domain is the entire [0,1].

We next show that w is strictly increasing. If w is constant on $[q, q + \varepsilon]$ for $\varepsilon > 0$, then an event with K value $p = \varepsilon$ is null, but then so is every event with K value $\frac{1}{n} < \varepsilon$, but then so are all their finite unions including the entire $[0,1]$. Then all outcomes are equivalent, and we have a contradiction. □