A powerful Tool for Analyzing Concave/Convex Utility and Weighting Functions

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Abstract

This paper shows that convexity of preference has stronger implications for weighted utility models than had been known hitherto, both for utility and for weighting functions. Our main theorem derives concave utility from convexity of preference on the two-dimensional comonotonic cone, without presupposing continuity. We then show that this, seemingly marginal, result provides the strongest tool presently available for obtaining concave/convex utility or weighting functions. We revisit many classical results in the literature and show that we can generalize and improve them all. The classical results include: risk aversion in expected utility, optimism/pessimism in rank-dependent utility and prospect theory, uncertainty aversion in Choquet expected utility, ambiguity aversion in the smooth ambiguity model, and inequality aversion in utilitarianism. We provide some surprising relations between well-known conditions, e.g.: in Yaari's dual theory, convexity/concavity in ("horizontal") outcome mixing is not only dual, but also logically equivalent, to concavity/convexity in ("vertical") probability mixing.

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1 Introduction

Convexity of preference is a standard condition in many fields (De Giorgi & Mahmoud 2016; Mas-Colell, Whinston, & Green 1995 p. 44). We examine it for weighted utility models, where its potential has not yet been fully recognized. Our first theorem shows its equivalence to concave utility on the two-dimensional comonotonic cone. This generalizes existing results by not presupposing continuity and by providing flexibility of domain. With this result, as thin and marginal as it may seem at first, we can in one blow generalize virtually all existing theorems on convex or concave utility or weighting functions, making them more appealing.

The aforementioned theorems concern: (1a) risk aversion for expected utility not only for risk (von Neumann-Morgenstern 1944) but also for (1b) uncertainty (Savage 1954); (1c) Yaari's (1969) comparative risk aversion generalized by allowing for different beliefs; (2a) concave utility and optimistic/pessimistic probability/event weighting for Quiggin's (1982) rank-dependent utility for risk, (2b) Gilboa's (1987) and Schmeidler's (1989) rank-dependent utility for ambiguity, and (2c) Tversky & Kahneman's (1992) prospect theory for both risk and ambiguity; (3a) corresponding results for Miyamoto's (1988) biseparable utility for risk[1]; (3b) corresponding results for Ghirardato & Marinacci's (2001) biseparable utility for ambiguity[2]; (4) smooth ambiguity aversion (Klibanoff, Marinacci, & Mukerji (2005); (5) loss aversion in Köszegi & Rabin's (2006) reference dependent model; (6) inequality aversion for welfare theory (Ebert 2004).

The main contribution of this paper is not to generalize some theorems, which would constitute a marginal contribution. The main contribution is to provide a general technique to obtain convex/concave utility or weighting functions in a more general and

[1] This includes many risk theories, such as disappointment theory (Bell 1985; Loomes & Sugden 1986) for a disappointment function kinked at 0, RAM and TAX models (Birnbaum 2008), disappointment aversion (Gul 1992), original prospect theory (Kahneman & Tversky 1979) for gains and for losses, Luce's (2000) binary RDU, and prospective reference theory (Viscusi 1989). See Wakker (2010 Observation 7.11.1).

[2] This includes many ambiguity theories, such as maxmin expected utility (Alon & Schmeidler 2014; Gilboa & Schmeidler 1989), the α-Hurwicz criterion (Arrow & Hurwicz 1972), the NEO-additive model (Chateauneuf, Eichberger, & Grant 2007), multiple priors-multiple weighting (Dean & Ortoleva 2017), contraction expected utility (Gajdos et al. 2008), alpha-maxmin (Ghirardato et al. 2004; Jaffray 1994; Luce & Raiffa 1957 Ch. 13), Hurwicz Expected Utility (Gul & Pesendorfer 2015), binary RDU (Luce 2000) including his rank-sign dependent utility, and binary expected utility (Pfanzagl 1959 pp. 287-288).

appealing manner than done before. As corolaries, we can generalize and improve virtually all existing results on this topic in the literature. To limit the size of this paper, we focus on uncertainty henceforth. Our theorems can readily be applied, though, to discounted utility for intertemporal choice with aversion to variation in outcomes, utilitarian welfare models with aversion to inequality, and other weighted utility models.

outcome-mixing provides a characterization of smooth ambiguity aversion (see §8) that, unlike the original characterization, is directly observable from preference.

The outline of the paper is as follows. Section 2 presents elementary definitions and our main result. The following sections revisit classical results in the literature, generalizing and improving them all, to demonstrate that we have provided a general tool for analyzing concave/convex utility and weighting. Section 3 presents implications for uncertainty focusing on classical expected utility. Section 4 turns to ambiguity models. The results for uncertainty presented up to this point provide useful preparations for decision under risk, presented in following sections. For risk, besides outcome mixing (§5), we can also consider probabilistic mixing (§6), which, as we show, is dual to outcome mixing. Some further implications for risk are in §7. Implications for uncertainty that are not directly relevant for risk are in §8, followed by a concluding section and an appendix with proofs. In each proof, we first find a substructure isomorphic to our main theorem, and then extend the desired result to the whole domain considered.

2 Definitions and our main theorem

S is the *state space* that can be finite or infinite. A denotes an algebra of subsets called *events*. The *outcome set* is a nonpoint interval I, $I \subset \mathbb{R}$. I can be bounded or not, and open, closed, or neither. $\mathcal{F}$ denotes a set of functions from S to I called *acts*, which are assumed measurable (inverses of intervals are events). Outcomes are identified with constant acts. We assume that $\mathcal{F}$ contains all *simple*, i.e., finite-valued, measurable functions. Other than that, $\mathcal{F}$ can be general, with however the restriction added that all RDU values (defined later) are well-defined. In particular, $\mathcal{F}$ may consist exclusively of simple acts, or contain all bounded acts. By $x = (E_1 : x_1, \ldots, E_n : x_n)$ we denote the simple act assigning outcome x_j to all states in E_j. It is implicitly understood that the E_j are events partitioning S.

A *preference relation*, i.e. binary relation, $\succcurlyeq$ is given on $\mathcal{F}$; $\succ, \preccurlyeq, \prec, \sim$ are as usual. V *represents* $\succcurlyeq$ on $F' \subset F$ if V is real valued with $\mathcal{F}'$ in its domain and $x \succcurlyeq y \Leftrightarrow V(x) \geq V(y)$ for all acts $x, y \in F'$. This implies weak ordering on $\mathcal{F}'$, i.e., $\succcurlyeq$ is transitive and complete there. If we omit "on $\mathcal{F}'$", then $\mathcal{F}' = \mathcal{F}$. Central in this paper are convex combinations $\lambda x + (1-\lambda)y$. Here x and y are acts, $0 \leq \lambda \leq 1$, and the combination concerns the statewise mixing of outcomes. We do not assume that $\mathcal{F}$ is closed under convex combinations. The set of simple acts is, and this provides enough richness for all our theorems.

DEFINITION 1 We call $\succcurlyeq$ *convex* if $x \succcurlyeq y \Rightarrow \lambda x + (1-\lambda)y \succcurlyeq y$ for all $0 \leq \lambda \leq 1$ and acts $x, y, \lambda x + (1-\lambda)y$. □

The condition is only imposed if the mix indeed is an act; i.e., is contained in the domain. Unfortunately, terminology in the literature is not uniform, and sometimes terms concave, quasi-convex, or quasi-concave have been used. We use the most common term, convex. Convexity of preference is equivalent to quasi-concavity of representing functions.

An *(event) weighting function* W maps events to $[0,1]$ such that: $W(\emptyset) = 0, W(S) = 1$, and $A \supset B \Rightarrow W(A) \geq W(B)$. *Finitely additive probability measures* P are additive weighting functions. They need not be countably additive. Preference conditions necessary and sufficient for countable additivity are well known (Wakker 1993 Proposition 4.4), and can optionally be added in all our theorems.

For a weighting function W and a function $U : I \rightarrow \mathbb{R}$, the *rank-dependent utility* (*RDU*) of an act x is

$$\int_{\mathbb{R}^+} W\{s \in S : U(x(s)) > \alpha\} d\alpha - \int_{\mathbb{R}^-} (1 - W\{s \in S : U(x(s)) > \alpha\}) d\alpha. \tag{1}$$

An alternative term used in the literature is Choquet expected utility. We impose one more restriction on $\mathcal{F}$: RDU is well defined and finite for all its elements. A necessary and sufficient statement directly in terms of preferences—requiring preference continuity with respect to truncations of acts— is in Wakker (1993). A sufficient condition is that all acts are bounded (with an upper and lower bound contained in I). For a simple act $(E_1 : x_1, \ldots, E_n : x_n)$ with $x_1 \geq \cdots \geq x_n$, the RDU is

$$\sum_{j=1}^{n} (W(E_1 \cup \cdots \cup E_j) - W(E_1 \cup \cdots \cup E_{j-1})) U(x_j). \tag{2}$$

Rank-dependent utility (*RDU*) holds on $\mathcal{F}' \subset \mathcal{F}$ if there exist W and strictly increasing U such that RDU represents $\succcurlyeq$ on $\mathcal{F}'$. Then U is called the *utility function*. Again, if we omit "on $\mathcal{F}'$," then $\mathcal{F}' = \mathcal{F}$. If $\mathcal{F}'$ contains all constant acts, then strict increasingness of U is equivalent to *monotonicity*: $\gamma > \beta \Rightarrow \gamma \succ \beta$ for all outcomes. We do not require continuity of U. The special case of RDU with W a finitely additive probability measure P is called *expected utility* (*EU*). We sometimes write *subjective EU* if P is subjective. We assume existence of a *nondegenerate event* E, meaning $(E : \gamma, E^c : \gamma) \succ (E : \gamma, E^c : \beta) \succ (E : \beta, E^c : \beta)$ for some outcomes $\gamma > \beta$ with the acts contained in the relevant domain $\mathcal{F}'$. Under sufficient richness, satisfied in all cases considered in this paper, nondegenerateness means $0 < W(E) < 1$.[3] We summarize the assumptions made.

ASSUMPTION 2 [Structural assumption for uncertainty] S is a state space, $\mathcal{A}$ an algebra of subsets (events), and I a nonpoint interval. $\mathcal{F}$, the set of acts, is a set of measurable functions from S to I containing all simple functions, endowed with a binary (preference) relation $\succcurlyeq$. RDU represents $\succcurlyeq$ on a subset $\mathcal{F}'$ of $\mathcal{F}$ (default: $\mathcal{F}' = \mathcal{F}$). There exists a nondegenerate event E. □

To obtain complete preference axiomatizations in the theorems in this paper, we should state preference conditions for the decision models assumed. Such conditions were surveyed by Köbberling & Wakker (2003) and will not be repeated here.

W is *convex* if $W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$. Elementary manipulations show that this holds if and only if

$$W(A \cup B) - W(B) \leq W(A \cup B') - W(B') \text{ whenever } A \cap B' = \emptyset \text{ and } B \subset B'. \quad (3)$$

The latter formulation shows the analogy with increasing derivatives of real-valued convex functions. An interesting implication of convexity of W is that RDU then belongs to the popular maxmin EU model (Wald 1950; Gilboa & Schmeidler 1989) with the set of priors equal to the *Core*, i.e., the set of probability measures that dominate W (Schmeidler 1986 Proposition 3; Shapley 1971).

[3]If no nondegenerate event exists, then an RDU representation exists with a linear, so surely concave, utility, and in this sense all results below hold true—also regarding convexity of weighting functions as can be demonstrated—without this extra requirement. However, then utility is ordinal (Wakker 1989 Observation VI.5.1') so that utility can also be chosen nonconcave, and we should formulate all our results as existence results. We avoid complicating our formulations this way.

The following theorem is our main result. Virtually all preceding results in the literature—Debreu & Koopmans (1982) excepted— assumed continuity and even often differentiability, but we do not.

THEOREM 3 *[Main theorem] Assume: (a) Structural Assumption 2; (b)* $S = \{s_1, s_2\}$*; (c)* s_1 *is nondegenerate; (d) EU (= RDU) holds on* $\mathcal{F}' = \{x : x_1 \geq x_2\}$*. Then utility is concave if and only if* $\succcurlyeq$ *is convex on* $\mathcal{F}'$*.* □

In the theorem, nondegeneracy of s_1 is equivalent to nondegeneracy of s_2. The proof of the theorem is more complex than of its analogs on full product spaces. On the latter, we can use hedging, as in the half-half mixture of $(1,0)$ and $(0,1)$ resulting in the sure $(0.5, 0.5)$. Hedging provides a powerful tool for analyzing convex preferences, extensively used in the literature. However, we cannot use hedging because all acts in our domain are maximally correlated. This complicates our proof relative to, for instance, Debreu & Koopmans (1982), its simplification Crouzeix & Lindberg (1986), its generalization Monteiro (1999), and most other predecessors. Therefore, unlike Debreu & Koopmans, we need strictly increasing utility. Example 19 shows that our theorem does not hold for nondecreasing utility. Because strictly increasing utility is natural in most applications, it does not entail a serious restriction. In return, the flexibility of domain provided by our theorem allows us to apply it to utility functions when expected utility is violated, and to apply it dually so that it speaks to weighting functions. Chateauneuf & Tallon (2002), Ghirardato & Marinacci (2001), and Wakker (1994) did consider comonotonic sets of acts. Their results are generalized by showing that continuity/differentiability is redundant. §8 gives further details.

3 Implications for decision under uncertainty: expected utility

This section considers applications of the main Theorem 3 to classical EU for decision under uncertainty.

COROLLARY 4 *If Structural Assumption 2 and EU hold, then* U *is concave if and only if* $\succcurlyeq$ *is convex.* □

The end of the introduction explained that Corollary 4 is useful for capturing risk aversion because convexity is directly observable, not involving subjective probabilities. Remarkably, the early Yaari (1965) already pointed out that the traditional definitions of risk aversion, relating to expected value or mean-preserving spreads, cannot be used for subjective EU. He hence tested convexity instead. However, he did not observe that convexity is actually equivalent to the traditional definitions.

Although an early version of Corollary 4 appeared in Debreu & Koopmans (1982 p. 4) and has been used in some works (Section 8), the result did not yet receive the attention it deserves and has not been generally known. Alternative, more complex, preference conditions for concave utility under subjective EU are in Baillon, Driesen, & Wakker (2012), Harvey (1986 Theorem 3), Wakker (1989), Wakker (2010 Eq. 4.8.2), and Wakker & Tversky (1993 §9).

We next turn to comparative results. In what follows, superscripts refer to decision makers. Yaari (1969) provided a well-known characterization of comparative risk aversion under subjective EU, where decision maker $\succcurlyeq^2$ with utility U^2 is more risk averse than decision maker $\succcurlyeq^1$ with utility U^1 if her certainty equivalents are always lower. Then U^2 is a concave transformation of U^1. Unfortunately, Yaari's condition is not necessary and sufficient, but only holds if the two decision makers have the same subjective probabilities. Decision makers with different beliefs cannot be compared because Yaari's condition then is never satisfied. The basic problem is that certainty equivalents depend on probabilities and, thus, involve not only risk attitudes but also beliefs. Yaari's method of comparing certainty equivalents has become a common tool in ambiguity theories, for instance to compare ambiguity aversion across decision makers.[4] Then invariably all other attitude components of the decision makers except the one compared have to be identical. This is implied by the fundamental problem of certainty equivalents of involving all components of decision attitudes. This limits the scope of application. We now show how outcome mixing avoids the aforementioned limitations and works for general beliefs, for Yaari's original EU framework. Generalizations to nonexpected utility theories are left to future work.

[4] See, for instance, Epstein (1999 Definition 2.3), Ghirardato & Marinacci (2001 §4.1), Ghirardato & Marinacci (2002 Definition 4), Izhakian (2017 Definition 4), and Klibanoff, Marinacci, & Mukerji (2005 Definition 5).

A preparatory notation: $\alpha_E\beta$ denotes the binary act $(E : \alpha, E^c : \beta)$. Mathematically, we will describe the case where $\succcurlyeq^2$ is risk averse if outcomes are expressed in U^1 units, i.e., units that make $\succcurlyeq^1$ risk neutral. In these outcome units, $\succcurlyeq^2$ should be convex. To capture this idea in a preference condition, we have to avoid the explicit use of theoretical constructs such as U^1. We have to reveal mixtures of acts in U^1 units directly from preferences. Gul (1992) showed a way to do this. Assume, for any event event A:

$$(x_{1_A} z_1) \sim^1 y_1 \textit{ and } (x_{2_A} z_2) \sim^1 y_2. \tag{4}$$

This shows that, in U^1 units, y_1 is a mixture of x_1 and z_1, and y_2 is so of x_2 and z_2, with as mixing weights the subjective probabilities $P^1(A)$ and $1 - P^1(A)$. These weights are not directly observable but this is no problem. All we need for what follows is that these weights are the same in both mixtures. This is enough to infer that, for any event B, the act $(y_{1_B} y_2)$ is a convex mixture of $(x_{1_B} x_2)$ and $(z_{1_B} z_2)$ in U^1 units. Our convexity condition requires that the mixture $(y_{1_B} y_2)$ is preferred to the other two acts if they are indifferent. That is, $\succcurlyeq^2$ is *more outcome-risk averse*[5] than $\succcurlyeq^1$ if, for all events B:

$$(x_{1_B} x_2) \sim^2 (z_{1_B} z_2) \Rightarrow (y_{1_B} y_2) \succcurlyeq^2 (z_{1_B} z_2). \tag{5}$$

is implied by Eq. 4. Eq. 5 is the convexity condition in terms of U^1 units, weakened to the case where the antecedent preference is actually an indifference, and where the mixtures weights are $P^1(A)$ and $1 - P^1(A)))$ for some event A. Under continuity, this weakened version is strong enough to imply full-force convexity, as Corollary 5 will show. To see intuitively that our condition captures comparative risk aversion, first note that Eq. 4 implies, for all events B'

$$(x_{1_{B'}} x_2) \sim^1 (z_{1_{B'}} z_2) \Rightarrow (y_{1_{B'}} y_2) \sim^1 (z_{1_{B'}} z_2) \tag{6}$$

because of linearity in probability mixing of the EU^1 functional.[6] The event B' to bring indifference for decision maker 2 may be different than B^1 due to different beliefs and/or state spaces. Comparing Eqs. 5 and 6 shows that, whenever we mix outcomes in ways that make decision maker 1 indifferent, then decision maker 2 has extra preference for that mix. This reveals stronger risk aversion, as formalized next.

[5] To avoid confusion with Yaari's widely accepted terminology, we add "outcome" to our term.

[6] Gul (1992) used a strengthened version of the implication Eq. 4 $\Rightarrow$ Eq. 6 to axiomatize subjective expected utility.

COROLLARY 5 *Assume that $\succcurlyeq^1$ and $\succcurlyeq^2$ both satisfy Structural Assumption 2 with the same outcome interval I, and both maximize subjective EU with continuous utility functions U^1 and U^2. Then $\succcurlyeq^2$ is more outcome-risk averse than $\succcurlyeq^1$ if and only if $U^2(.) = \varphi(U^1(.))$ for a concave transformation φ. The two decision makers may have different probabilities and may even face different state spaces.* □

Both the ambiguity aversion of Klibanoff, Marinacci, & Mukerji's (2005) smooth model, and the preference for early resolution of uncertainty of Kreps & Porteus (1978) amount to having one EU utility function more concave than another. Corollary 5 shows a way to obtain these results without involving probabilities as inputs in the preference condition. Note that we have to be able to deal with different state spaces in this application.

Our outcome-risk aversion condition is more complex, and less appealing, than Yaari's certainty equivalent condition. Its pro is that it delivers a clean comparison of utility and risk attitude, not confounded by beliefs. Both conditions deserve study. Baillon, Driessen, & Wakker (2012) provided other characterizations of comparative risk aversion that, like our result, do not require same beliefs or given probabilities. They use an endogenous midpoint operation for utilities. Heufer (2014) showed how Yaari's certainty equivalent condition can be elicited from revealed preferences. Our paper propagates the use of preference convexity. Heufer (2012) showed how this can be elicited from revealed preferences.

4 Implications for decision under uncertainty: ambiguity

We first discuss applications to RDU, and start with basic definitions. Acts x, y are *comonotonic* if there are no states s, t with $x(s) > x(t)$ and $y(s) < y(t)$. A set of acts is *comonotonic* if each pair of its elements is comonotonic. A *comoncone* is a maximal comonotonic set. It corresponds with an ordering ρ, called *ranking*, of S and contains all acts x with $s\rho t \Rightarrow x(s) \geq x(t)$. Every event E satisfying $s \in E, t\rho s \Rightarrow t \in E$ is called a *goodnews* event or, more formally, a *rank*. Intuitively, it reflects the good news of receiving all outcomes better than some outcome. A set of acts is comonotonic if and only if it is a subset of a comoncone. A comoncone is *nondegenerate* if it has a nondegenerate goodnews event. On every comoncone with related ordering ρ, RDU agrees with an EU functional with the finitely additive probability measure P_ρ agreeing with W on the goodnews events. The following corollary is a straightforward generalization of the main Theorem 3.

COROLLARY 6 *If Structural Assumption 2 holds and RDU holds on a nondegenerate comoncone $\mathcal{F}'$, then U is concave if and only if $\succcurlyeq$ is convex on $\mathcal{F}'$.* □

A first version of the following result was in Chateauneuf & Tallon (2002), which did not receive the attention it deserves.

COROLLARY 7 *[Main corollary] If Structural Assumption 2 and RDU hold, then U is concave and W is convex if and only if $\succcurlyeq$ is convex.* □

The result is especially appealing because the two most important properties of RDU jointly follow from one very standard preference condition. The only proof available as yet, by Chateauneuf and Tallon, assumes differentiability of utility, which is problematic for preference foundations. Differentiability is not problematic, and useful, in most economic applications. For preference foundations the situation is different, though. Preference foundations seek for conditions directly observable from preferences. In this way, preference foundations make theories operational. For general differentiability, there is no clear and elementary preference condition.[7] Hence, differentiability assumptions are to be avoided in preference foundations. Strictly speaking, Corollary 7 is then the first joint preference foundation of the two most popular specifications of RDU. Chateauneuf &Tallon (2002) did not present their result very saliently[8]. This, and the use of differentiability, may explain why this result has not yet been as widely known as it deserves to be.

We, finally, give results for a large class of nonexpected utility models. By $\mathcal{F}_E$ we denote the set of binary acts $\gamma_E\beta = (E : \gamma, E^c : \beta)$, and by $\mathcal{F}_E^\uparrow$ we denote the subset with $\gamma \geq \beta$. *Biseparable utility* holds if there exist a utility function U, and a weighting

[7]Once concavity has been derived, we are close to differentiability (Observation 17). Then necessary and sufficient conditions for complete differentiability can be stated in terms of vanishing limits of risk premia (Nielsen 1999), a condition which has the same, commonly accepted, observability status as continuity. The task of our paper is, however, to derive concavity.

[8]One has to combine their Proposition 1 with the equivalence of (i) and (iv) in their Theorem 1.

function W, such that $RDU(x)$ represents $\succcurlyeq$ on the set of all binary acts x. That is, for all binary acts $\gamma_E\beta$ ($\gamma \geq \beta$) we have an RDU representation $W(E)U(\gamma) + (1 - W(E))U(\beta)$, but for acts with more than two outcomes the representation has not been restricted. Biseparable utility includes many theories (see the Introduction) and the following theorem therefore pertains to all these theories. Statement (ii) characterizes concave utility for all these models. Statement (i) additionally characterizes *subadditivity* of W: $W(A) + W(A^c) \leq 1$. It is equivalent to convexity on binary partitions $\{E, E^c\}$. It is the prevailing empirical finding and has sometimes been taken as definition of ambiguity aversion.

COROLLARY 8 *If Structural Assumption 2 and biseparable utility (i.e., $\mathcal{F}'$ is the set of all binary acts) hold, then:*

(i) U is concave and W is subadditive if and only if $\succcurlyeq$ is convex on every $\mathcal{F}_E$.

(ii) U is concave if and only if $\succcurlyeq$ is convex on every $\mathcal{F}_E^{\uparrow}$. This holds if and only if $\succcurlyeq$ is convex on one set $\mathcal{F}_E^{\uparrow}$ with E nondegenerate. □

5 Implications for decision under risk: outcome-mixing

In decision under risk we consider a set $\mathcal{P}$ of probability distributions, called *lotteries* (generic notation P, Q, R), over a set X of *outcomes* (generic notation α, β, γ, or x_j). We assume that $\mathcal{P}$ contains all *simple* probability distributions, assigning probability 1 to a finite subset of X, with generic notation $(p_1 : \alpha_1, \ldots, p_n : \alpha_n)$, and possibly more distributions. Outcomes α are identified with degenerate lotteries $(1 : \alpha)$. A *(probability) weighting function* w maps $[0,1]$ to $[0,1]$, is strictly increasing, and satisfies $w(0) = 0$ and $w(1) = 1$. We do not assume continuity of w. Discontinuities at $p = 0$ and $p = 1$ are of special empirical interest. For a probability weighting function w, and a function $U : X \to \mathbb{R}$, the *rank-dependent utility* (RDU) of a lottery P is

$$\int_{\mathbb{R}^+} w(P(U(\alpha) > \mu))d\mu - \int_{\mathbb{R}^-} (1 - w(P(U(\alpha) > \mu)))d\mu.$$

RDU for risk can be seen to be a special case of RDU for uncertainty (Wakker 2010 §§2.1-2.3 and Ch. 10), because of which it is convenient to use the same term for both.

No confusion will arise. For a simple lottery $(p_1 : x_1, \ldots, p_n : x_n)$ with $x_1 \geq \cdots \geq x_n$, the RDU is

$$\sum_{j=1}^{n}(w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1})U(x_j). \tag{7}$$

Rank-dependent utility (*RDU*) holds if there exist w and U such that RDU represents $\succcurlyeq$ on $\mathcal{P}$. Then U is the *utility function*. The special case of RDU with w the identity is called *expected utility* (*EU*). We assume that RDU is well-defined and finite for all lotteries. To analyze convexity for outcome mixing as before, we assume that X is a nonpoint interval I and U is strictly increasing. Again, we do not assume continuity. We summarize the assumptions made.

ASSUMPTION 9 [Structural assumption for monetary risk] I is a nonpoint interval. $\mathcal{P}$ is a set of lotteries over I containing all simple probability distributions. RDU holds on $\mathcal{P}$ with U strictly increasing. □

For simplicity, we restrict the definition of convex preferences to simple lotteries, which will be strong enough to give all the desired implications. Preceding papers defined the condition for risk by specifying an underlying state space and then extended it to nonsimple lotteries.[9] We prefer to avoid this complication.

Because it is common in decision under risk to let concavity and convexity refer to probabilistic mixing, considered in the next section, we use a different term for outcome mixing. We call $\succcurlyeq$ *outcome-convex* if for each probability vector $p_1, ., , , .p_n$ (assumed to add to 1) and $0 < \lambda < 1$ we have $(p_1 : x_1, \ldots, p_n : x_n) \succcurlyeq (p_1 : y_1, \ldots, p_n : y_n) \Rightarrow (p_1 : \lambda x_1 + (1-\lambda)y_1, \ldots, p_n : \lambda x_n + (1-\lambda)y_n) \succcurlyeq (p_1 : y_1, \ldots, p_n : y_n)$. We call $\succcurlyeq$ *comonotonic outcome-convex* if the preceding implication only holds if $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$. The following corollary is an analog of Corollary 7 for risk.

COROLLARY 10 *Under Assumption 9:*

(i) U is concave and w is convex if and only if $\succcurlyeq$ is outcome-convex.

[9] In the following theorems, the proofs of sufficiency of the preference conditions then remain unaltered. For necessity, we can show that the properties of U and w imply that the representing functional is concave and, hence, surely quasi-concave, similarly as in the proof of Observation 20, implying convexity of preference. This implies that convexity for all simple lotteries is equivalent to convexity for all lotteries under RDU.

(ii) U is concave if and only if $\succcurlyeq$ is comonotonic outcome-convex. □

Result (i) provides an interesting alternative to Chew, Karni, & Safra (1987). They showed, assuming differentiability, that concavity of U plus convexity of w is equivalent to aversion to mean-preserving spreads. Quiggin (1993 §6.2) provided an alternative proof, also assuming differentiability—we are not aware of a proof in the literature without differentiability. Strictly speaking, we are terefore the first to provide a genuine preference foundation. We obtain the following corollary, where for definitions of differentiability and aversion to mean-preserving spreads we refer to Chew, Karni, & Safra (1987).

COROLLARY 11 *Under Assumption 9 and differentiability, outcome-convex of $\succcurlyeq$ is equivalent to aversion to mean-preserving spreads.*

The result is remarkable because, at first sight, one condition concerns only outcome mixing whereas the other condition also involves probabilistic mixing. This surprising point was discussed by Quiggin (1993 §9.2) in a somewhat different context. Many papers have used aversion to mean-preserving spreads conditions in various forms. In Appendix C, after the proof of Corollary 10, we give details of several results relevant to the preceding analyses. Our paper shows that convexity conditions can serve as appealing alternatives. This holds especially if the probabilities involved in mean-preserving spreads are subjective, implying that they are not directly observable, contrary to our preference condition. Thus, our condition can, for instance, serve to make the conditions in §5 of Gul & Pesendorfer (2015), which used subjective probabilities as inputs, directly observable.

The following result, adapting Corollary 4 to risk, shows that outcome-convexity is an alternative way to characterize risk aversion under EU, besides the traditional weak risk aversion—preference for expected value— or strong risk aversion—aversion to mean-preserving spreads. Thus, all these conditions are equivalent.

COROLLARY 12 *Under Assumption 9 and EU, U is concave if and only if $\succcurlyeq$ is outcome-convex.* □

Ghirardato & Marinacci (2001) propagated biseparable utility for ambiguity. For risk, this was done by Miyamoto (1988), who used the term generic utility. He also emphasized that any result for his theory holds for the many models comprised, referenced in our introduction. We now apply our technique to his model.

By $\mathcal{P}_p$ we denote the set of binary lotteries $\gamma_p\beta = (p : \gamma, 1-p : \beta)$, and by $\mathcal{F}_p^{\uparrow}$ we denote the subset with $\gamma \geq \beta$. *Biseparable utility* holds if there exist a utility function U and a weighting function w such that $RDU(\gamma_p\beta) = w(p)U(\gamma)+(1-w(p))U(\beta)$ represents $\succcurlyeq$ on the set of all binary lotteries. In Statement (i) below, *subadditivity* of w means $w(p) + w(1-p) \leq 1$ for all p. As with ambiguity, it is the prevailing empirical finding for risk, implying that the certainty effect is stronger than the possibility effect.

COROLLARY 13 *If Structural Assumption 9 holds except that biseparable utility holds instead of RDU, then:*

(i) U is concave and w is subadditive if and only if $\succcurlyeq$ is outcome-convex on every $\mathcal{P}_p$.

(ii) U is concave if and only if $\succcurlyeq$ is convex on every $\mathcal{P}_p^{\uparrow}$. This holds if and only if $\succcurlyeq$ is convex on one set $\mathcal{P}_E^{\uparrow}$ with $0 < p < 1$. □

6 Implications for decision under risk: probabilistic mixing

We kept the most surprising applications for last. Whereas up to now we considered outcome mixing, we now consider probabilistic mixing. For lotteries P, Q, $\lambda P \oplus (1-\lambda)Q$ denotes the probability measure assigning probability $\lambda P(\alpha) + (1-\lambda)Q(\alpha)$ to each outcome α, with a similar probability mix for each subset of outcomes instead of $\{\alpha\}$. Probabilistic mixing can be defined for general outcome sets X. We, thus, assume:

ASSUMPTION 14 [Structural assumption for risk with general outcomes] $\mathcal{P}$ is a set of lotteries over outcome set X containing all simple probability distributions. RDU holds on $\mathcal{P}$. X contains at least three nonindifferent outcomes $\gamma \succ \beta \succ \alpha$. □

We call $\succcurlyeq$ *convex* if $P \succcurlyeq Q \Rightarrow \lambda P \oplus (1-\lambda)Q \succcurlyeq Q$. This property, suggesting a deliberate preference for randomization, has been widely studied in the literature (Agranov & Ortoleva 2017; Cerreia-Vioglio et al. 2017; Fudenberg, Lijima, & Strzalecki 2015; Machina 1985; Saito 2015; Sopher & Narramore 2000). The opposite condition is more common empirically: $\succcurlyeq$ is *concave* if $P \succcurlyeq Q \Rightarrow \ P \succcurlyeq \lambda P \oplus (1-\lambda)Q$.

FIGURE 1: Duality of probabilities versus outcomes, and w versus U.

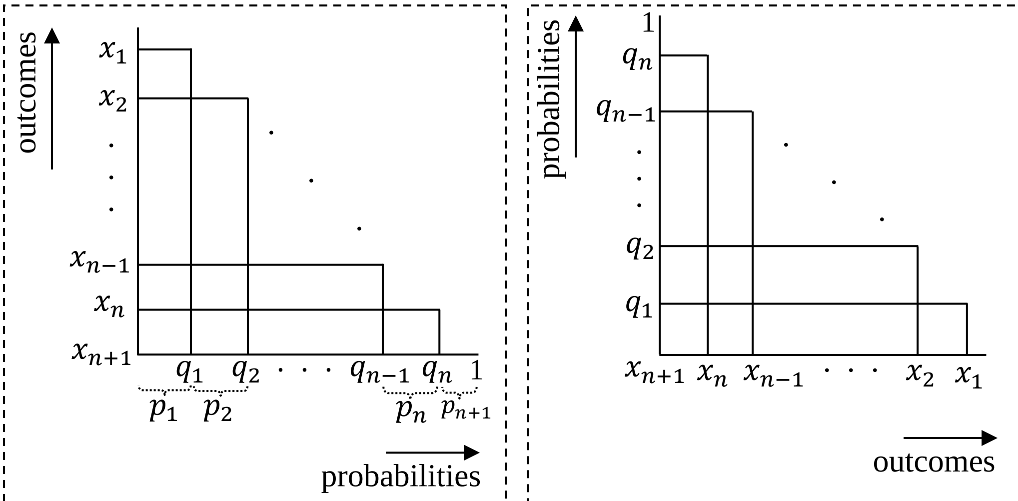


FIG. 1a: U is concave ⇔ convexity w.r.t. ordinate-mixing (Corollary 6)

FIG. 1b: w is concave ⇔ convexity w.r.t. ordinate-mixing (Corollary 15)

RDU results from Fig. a by transforming the abcissa by w and the ordinate by U, and then calculating the area of the figure.
RDU results from Fig b by transforming the abcissa by U and the ordinate by w, and then calculating the area of the figure.

COROLLARY 15 *Under Assumption 14, convexity of $\succcurlyeq$ is equivalent to concavity of w, and concavity of $\succcurlyeq$ is equivalent to convexity of w.* □

Figure 1 illustrates a duality between outcomes and probabilities—more precisely, goodnews probabilities or ranks q_j as defined in §4. It shows that Corollary 15 is the exact dual to Corollary 6. That is, if everywhere in Corollary 6 we replace outcome by goodnews probability, goodnews probability by outcome, weighting function by utility function, and utility function by weighting function, then Corollary 15 results. These theorems are essentially *identical* mathematical results. Formal details are in the proof. Similar dualities were exploited by Yaari (1987), Abdellaoui (2002), and Abdellaoui & Wakker (2005). Figure 1 also illustrates that Quiggin's (1982) insight, that one should use differences rather than absolute levels of weighting functions, is the dual of the insight of the marginal utility revolution (Jevons 1871; Menger 1871; Walras 1874) that differences of utility, rather than absolute levels are basic.

The axiomatization in Corollary 15 of convexity of w through a widely used preference condition is appealing. If there are only two nonindifferent outcomes then w is only ordinal and can be any strictly increasing function, thus can always be convex but never needs to be. Hence, Corollary 15 has characterized convexity of w as general as can be.

We present yet another surprising equivalence, combining Corollaries 10(i) and 15 for the special case of Yaari's (1987) dual model.

COROLLARY 16 *Under Yaari's (1987) dual model (Assumption 14, with X a nonpoint interval I and U the identity), $\succcurlyeq$ is outcome-convex (concave) if and only if it is concave (convex).* □

Thus the conditions, concerning mixing in two different dimensions—"horizontal" and "vertical"— are not only each others' duals, but they are also logically equivalent here. Röell (1987 §1) discussed these conditions in Yaari's model, but was not aware of their equivalence, nor was anyone else since then.

7 Further implications for existing results on risk in the literature

Yaari (1987) considered the special case of RDU for risk with linear utility. He characterized convexity of w through aversion to mean-preserving spreads, which is a special case of Chew, Karni, & Safra's (1987) theorem. Quiggin (1993 §9.1) and Röell (1987) similarly derived this result for linear utility. As our Corollary 15 showed, convexity with respect to probabilistic mixing provides an appealing alternative condition. It would have fitted better with the affinity condition for outcome addition that Yaari (1987) used, and the affinity condition for outcome mixing that Röell (1987) used, to axiomatize RDU with linear utility.

A surprising application concerns Köszegi & Rabin's (2006) reference dependent model. Masatlioglu & Raymond (2016) showed that Köszegi & Rabin's choice-acclimating personal equilibrium (CPE) is a special case of RDU. Loss aversion in Köszegi & Rabin's model then holds if and only of the probability weighting function in the equivalent RDU model is convex. Masatlioglu & Raymond's Propositions 3 and 10 used Wakker's (1994) version of our Corollary 15 to characterize loss aversion. They wrote: "we were able to demonstrate a previously unknown relationship between loss aversion/loving behavior and attitudes toward mixing lotteries within the CPE framework" (p. 2792) and "our results allow us to bring 20 years of existing experimental evidence to bear on CPE" (p. 2773). They required monetary outcomes and continuous utility. Our Corollary 15 shows that their result can be extended to noncontinuous utility and general outcomes. Their Proposition 6 uses aversion to mean-preserving spreads to characterize concave utilility and loss aversion. Our Corollary 10 shows that their mixture aversion would have provided an appealing alternative characterization.

8 Further implications for existing results on uncertainty in the literature

Schmeidler (1989), the most famous result in ambiguity theory, assumed an Anscombe-Aumann framework: a set of prizes is given, and the outcome set is the set of simple probability distributions over the prize space. That is, the outcome set is a convex subset of a linear space. Acts map states to outcomes. Utility over outcomes is assumed to be expected utility, i.e., it is linear with respect to probabilistic mixtures of outcomes. This is Structural Assumption 2 but with a multi-dimensional outcome space instead of our one-dimensional I. Schmeidler (1989, the Proposition) showed that W is convex if and only if $\succcurlyeq$ is convex—called uncertainty aversion. This follows from the special case of the main Corollary 7 with utility linear. That the outcome space is multi-dimensional changes nothing in our proofs.[10]

Most studies of multiple priors models, including $\alpha-$maxmin models, used the Anscombe-Aumann framework with linear utility. Exceptions without that restriction are Casadesus-Masanell, Klibanoff, & Ozdenoren (2000) and Alon & Schmeidler (2014). Our results characterize concave utility for the latter studies.

[10] For necessity, the proof of Observation 20 works with the first inequality an equality. For sufficiency, all o terms in the proof of Lemma 21 are exactly 0. Sufficiency can also be obtained by taking two outcomes $\gamma \succ \beta$ and equating lotteries over them with $I = [0, 1]$.

Several studies assumed linear utility as did Schmeidler (1989), and then gave various necessary and sufficient conditions for convex weighting functions alternative to our convexity: Chateauneuf (1991) and Kast & Lapied (2003) for monetary outcomes, and Wakker (1990) for the Anscombe-Aumann framework.

Three results in the literature come close to our main Theorem 3, in deriving concavity not on a full product set but on a comoncone. The first is Wakker (1994 Theorem 24). He considered the dual version as in Corollary 15. Whereas our Corollary allows for any outcome set with three or more nonindifferent outcomes, Wakker assumed an interval of outcomes plus continuous utility, and used this essentially in his proof. He used the term quasi-concave preference instead of our term convex preference. His proof was complex.[11] Our theorem is more general in not using the extra assumptions about X and U just stated.

The second result close to our main Theorem 3 or, more precisely, to our Corollary 6, is Theorem 3 in Chateauneuf & Tallon (2002). They assumed differentiable utility, whence they could skip Steps 2 and 3 of our proof in Appendix A.

The third result close to Theorem 3 is Ghirardato & Marinacci (2001; Theorem 17). They assumed continuity, and showed how Debreu & Koopmans (1982) can be used as in Step 1 of our proof in Appendix A. Our proof shows how to add Steps 2 and 3 to their proof.[12]

Strzalecki (2013, p. 62) criticized the use of subjective inputs in the ambiguity aversion condition in Klibanoff, Marinacci, & Mukerji (2005), and pointed out that convexity of preference (his Axiom A5') does not suffer from this problem. His Theorem 2 amounts to Corollary 4 for continuous utility, applied to second-stage acts in the second stage of the smooth ambiguity model. The observability problems of second-stage acts in the smooth model (Klibanoff, Marinacci, & Mukerji 2005 p. 1856) can be avoided by using our Corollary 5 to compare first- and second-stage overall utilities.

For the special case of linear utility (in an Anscombe-Aumann framework), Cerreia-Vioglio et al. (2011) characterized general preference functionals with convex preferences. Rigotti, Shannon, & Strzalecki (2008) examined general convex preferences and speci-

[11] Thus, Wakker (2010 p. 192 footnote 8) claimed that the result "is too hard to be proved ... by any of the readers."

[12] We thus show that the claim $X^0 \subset X^* \cup X_*$ on p. 887 line -13 in their proof holds true by ruling out the existence of β as in Figures 3 and 4.

fied results for several ambiguity models. Their Remark 1 discussed RDU with convex weighting functions, but did not specify how these are related to convex preferences. Our main Corollary 7 shows that concavity of utility is necessary and sufficient for that relation to be an equivalence.

9 Conclusion

We have provided a general technique to obtain convex/concave utility and weighting functions. Fields of application include intertemporal choice, utilitarian welfare aggregations, and, the context chosen in this paper, decision under risk and uncertainty. There we generalized and improved virtually all existing theorems. Knowledge of Corollaries 4, 7, and 15 will be useful for everyone working in decision theory. Convexity with respect to outcome-mixing is more powerful than had been known before.

Appendix A. Proof of our main Theorem 3

We first list some well-known properties of concave functions (Van Rooij & Schikhof (1982 §1.2).

OBSERVATION 17 *If U is concave and strictly increasing on I, then: (a) U is continuous on I except possibly at min(I) (if it exists). On int(I): (b) U has right derivative U'_r and left derivative U'_ℓ everywhere; (c) $U'_\ell(\alpha) \geq U'_r(\alpha) \geq U'_\ell(\alpha') > 0$ for all $\alpha' > \alpha$; (d) U is differentiable almost everywhere.* □

PROOF. As regards positivity in (c), if a left or right derivative were 0 somewhere in $int(U)$ then it would be 0 always after, contradicting strict increasingness of U. □

PROOF OF MAIN THEOREM 3. If U is concave then so is the EU functional, so that it is quasi-concave, implying convexity of $\succcurlyeq$. (This also follows from Observation 20.)

In the rest of this appendix we assume convexity of $\succcurlyeq$, and derive concavity of U. We write $\pi_1 = W(s_1), \pi_2 = 1 - \pi_1$. By nondegeneracy, $0 < \pi_1 < 1$. We suppress states from acts and write, for instance, (x_1, x_2) for $(s_1 : x_1, s_2 : x_2)$.

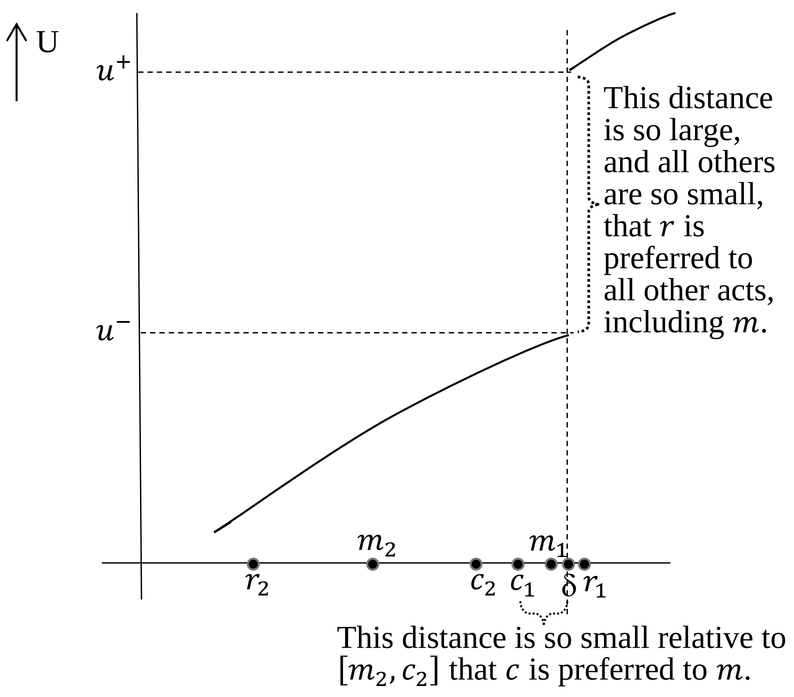
As explained in the main text, we cannot use hedging techniques in our proofs. Instead, we will often derive contradictions of convexity of p by constructing a "risky" act $r = (r_1, r_2)$ and a "close-to-certain" act $c = (c_1, c_2)$ with $r_1 > c_1 \geq c_2 > r_2$, such that

$$m := \lambda r + (1 - \lambda)c;\ \ r \succ m; c \succ m \tag{8}$$

for some $0 < \lambda < 1$ (mostly $\lambda = 0.5$), with m called the "middle" act.

LEMMA 18 *U is continuous except possibly at $inf(I)$.* □

FIGURE 2. Proof of Lemma 18



A discontinuity of U violates convexity of $\succcurlyeq$.

PROOF. See Figure 2. Assume, for contradiction, that U is not continuous at an outcome $\delta > inf(I)$. We construct r, m, c as in Eq. 8 with further $r_1 \geq \delta > m_1$.

Define $u^- = sup\{U(\beta) : \beta < \delta\}$. Define $u^+ = U(\delta)$ if $\delta = max(I)$ and $u^+ = inf\{U(\alpha) : \alpha > \delta\}$ otherwise. By discontinuity, $u^+ > u^-$. By taking $r_2 < \delta$ sufficiently close to δ we can get $U(r_2)$ as close to u^- as we want. We take it so close that $\pi_1(u^+) + \pi_2 U(r_2) > u^-$. This will ensure that r, the only act with its first outcome exceeding δ, is strictly preferred to all other acts, in particular, to m. We next choose c_2 strictly between r_2 and δ and define $m_2 = (r_2 + c_2)/2$. We then take c_1 strictly between c_2 and δ so close to δ that $\pi_1(u^-) + \pi_2 U(m_2) < \pi_1 U(c_1)) + \pi_2(U(c_2)$. This will ensure that $c \succ m$ if we ensure that $m_1 < \delta$. For the latter purpose we define $r_1 = \delta$ if $\delta = max(I)$, and otherwise $r_1 > \delta$ so close to δ that $m_1 := (r_1 + c_1)/2 < \delta$. In both cases, $U(r_1) \geq u^+$. The acts r, m, c are as in Eq. 8 with $\lambda = 1/2$. *QED*

Because U is strictly increasing, it suffices to prove concavity outside $inf(I)$. That is, we assume that I has no minimum. Assume for contradiction that U is not concave. Then there exist $0 < \lambda' < 1$ and outcomes $\alpha' < \gamma'$ such that $U(\lambda'\gamma' + (1 - \lambda')\alpha') < \lambda' U(\gamma') + (1 - \lambda')U(\alpha')$. Define ℓ as the line through $(\alpha', U(\alpha'))$ and $(\gamma', U(\gamma'))$. By continuity of U, we can define α as the maximum outcome between α' and $\lambda'\gamma' + (1 - \lambda')\alpha'$ with $(\alpha, U(\alpha))$ on (or above) ℓ, and γ as the minimum outcome between γ' and $\lambda'\gamma' + (1 - \lambda')\alpha'$ with $(\gamma, U(\gamma))$ on (or above) ℓ. We have $\gamma > \alpha$ and

$$U(\lambda\gamma + (1 - \lambda)\alpha) < \lambda U(\gamma) + (1 - \lambda)U(\alpha) \tag{9}$$

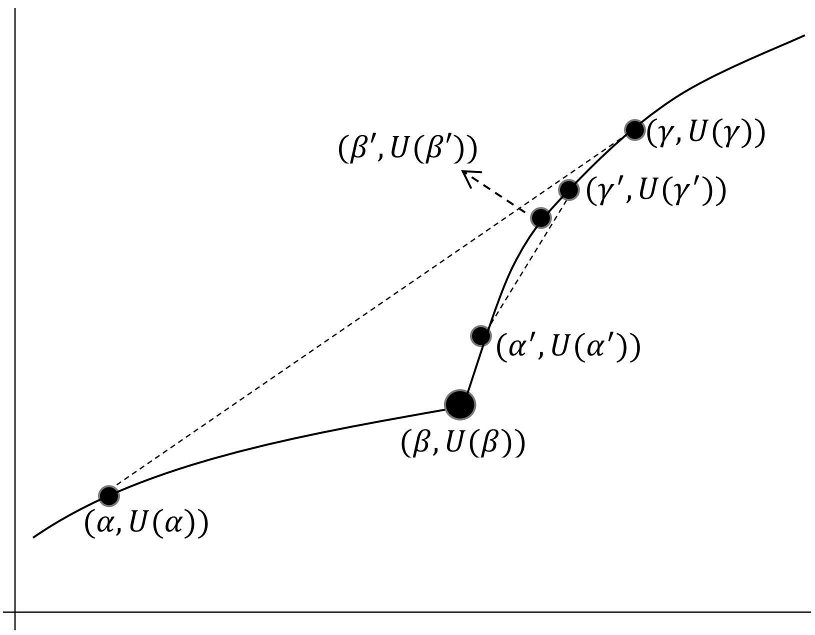
for all $0 < \lambda < 1$ (Figure 3).

Step 1 [At most one nonconcavity kink]. Assume that U is not concave on some "middle" interval $M \subset I$, with L and R the intervals in I to the left and right of M, possibly empty. Applying Debreu & Koopmans (1982 Theorem 2) to the additive representation $(1 - \pi_1)U(x_1) + \pi_1 U(x_2)$ on $L \times M$ and to the additive representation $(1 - \pi_1)U(x_1) + \pi_1 U(x_2)$ on $M \times R$ then implies that U is strictly concave on L and R. Applying this result to smaller and smaller subintervals M of $[\alpha, \gamma]$ there is one β in $[\alpha, \gamma]$ such that U is concave above and below β.

For the remainder of the proof, we could use Debreu & Koopmans (1982 Theorem 6). They provide a concavity index according to which U would be infinitely *convex* at the nonconcavity kink, then would have to be more, so infinitely, concave at every other point, but being concave there it can be infinitely concave at at most countably many points, and a contradiction has resulted. This reasoning is advanced and cannot be written formally very easily, because of which we provide a different proof.

Step 2 [Nonconcavity kink must be exactly at $\pi_1\gamma + \pi_2\alpha$]. We next show that β can

FIGURE 3. Contradiction from nonconcavity of U: Steps 1 & 2



STEP 1: There exists only one kink β; strict concavity outside β.
STEP 2: $\beta \neq \beta' := \pi_1\gamma + \pi_2\alpha$ cannot be.

only be $\beta' := \pi_1\gamma + \pi_2\alpha$. Assume, for contradiction, that $\pi_1\gamma + \pi_2\alpha$ were located at a point β' different than β (Figure 3). Then there would be a small interval around β', not containing α, β, or γ, where U would be strictly concave. We could then find $\gamma' > \beta' > \alpha'$ in the small interval with exactly $\pi_1\gamma' + \pi_2\alpha' = \beta'$ and satisfying the strict concavity inequality

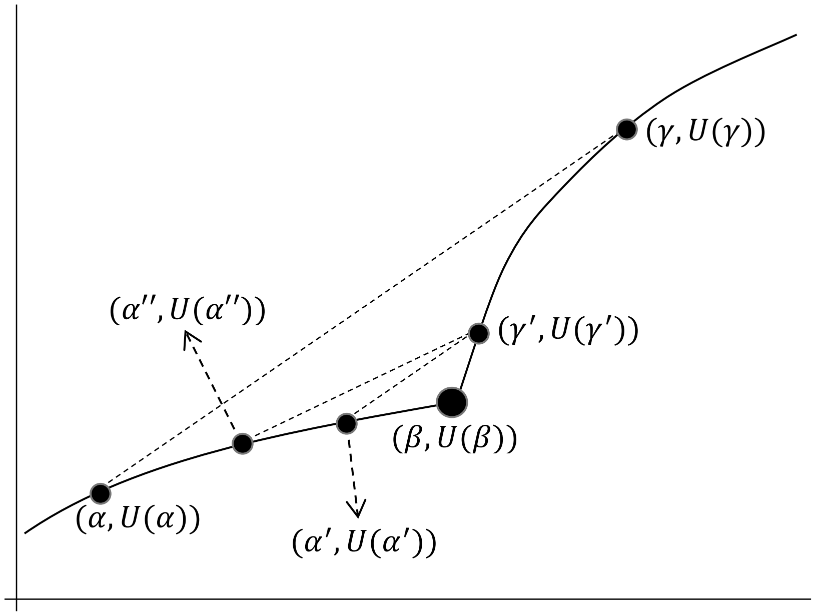
$$\pi_1 U(\gamma') + \pi_2 U(\alpha') < U(\beta'). \tag{10}$$

We define some acts, where, on our domain $\mathcal{F}'$ considered, always the best outcome is the first: $c = (\beta', \beta'), r = (\gamma, \alpha), m = (\gamma', \alpha')$, and we show that Eq. 8 holds. Because β' is the same π_1, π_2 mix of γ and α as of γ' and α', there exists $0 < \mu < 1$ such that $m = \mu r + (1-\mu)c$. We have $r \succ c \succ m$, the first preference by the convexity "risk-seeking" inequality Eq. 9 (with $\lambda = \pi_1$) and the second by the concavity "risk-averse" inequality Eq. 10. Eq. 8 is satisfied.

STEP 3 [Nonconcavity kink cannot be at $\pi_1\gamma + \pi_2\alpha$ either]. We construct, for contradiction, two outcomes γ', α' or γ', α'' that are as γ, α in Eq. 9 but for whom an analogous β as just constructed does not exist though. See Figure 4.

Consider the line through $(\alpha, U(\alpha))$ and $(\gamma, U(\gamma))$. Because of concavity of U on $[\alpha, \beta)$ and $[\beta, \gamma)$, the right derivative of U is decreasing on each of these two intervals. Because

FIGURE 4. Contradiction from nonconcavity of U: Step 3



STEP 3: We replace α by α'' and then Step 2 is violated.

the line mentioned lies above the graph of U, its slope exceeds the right derivative of U everywhere on $[\alpha, \beta)$ and is below it everywhere on $[\beta, \gamma)$. Now take any α' strictly between α and β, and take the line parallel to the line through $(\alpha, U(\alpha))$ and $(\gamma, U(\gamma))$. Its first intersection with the graph of U exceeding α' is γ'. Inspection of right derivates shows that $\gamma' > \beta$. If $\pi_1\gamma' + \pi_2\alpha' \neq \beta$ then we have a contradiction with Step 2 and we are done. If we have equality after all, then we take a line through $(\gamma', U(\gamma'))$ that is strictly between the two parallel lines. Its first intersection with the graph of U is $(\alpha'', U(\alpha''))$, and α'' and γ' are as in Eq. 9 but the outcome β has $\pi_1\gamma' + \pi_2\alpha'' \neq \beta$. Contradiction has resulted and we are done. The proof of Theorem 3 now is complete.
□

The following example shows that the main Theorem 3 does not hold if U is only assumed nondecreasing instead of strictly increasing.

EXAMPLE 19 In this example, all assumptions of Theorem 3 are satisfied except that U is nondecreasing instead of strictly increasing. $\succcurlyeq$ is convex but U is not concave.

$I = \mathbb{R}$, $W(s_1) = \pi_1 = \pi_2 = 0.5$, and U is the sign function. More precisely, $U(\alpha) = 1$ if $\alpha > 0$, $U(0) = 0$, and $U(\alpha) = -1$ if $\alpha < 0$. We show that $\succcurlyeq$ is convex. We consider an exhaustive list of cases of three acts m, r, c, with the cases ordered by the preference value of m. In each case we state some implications from $m \prec r$ and $m \prec c$ that readily preclude m from being a mixture of r and c. We suppress states and denote acts by the utilities of their outcomes. Thus, $(1, -1)$ denotes an act with a positive first outcome and a negative second outcome. We write $d'_j = U(d_j)$ for $d = c, m, r$ and $j = 1, 2$.

(1) If $m'_1 = -1$ then $r'_1 \geq 0$ and $c'_1 \geq 0$. (2) If $(m'_1, m'_2) = (0, -1)$ then $r'_1 \geq 0$ and $c'_1 \geq 0$ and either one of the latter is positive or $(r'_1, r'_2) = (c'_1, c'_2) = (0, 0)$. (3) If $(m'_1, m'_2) = (0, 0)$ or $(m'_1, m'_2) = (1, -1)$ then both (r'_1, r'_2) and (c'_1, c'_2) are $(1, 0)$ or $(1, 1)$. (4) If $(r'_1, r'_2) = (1, 0)$ then $(r'_1, r'_2) = (c'_1, c'_2) = (1, 1)$. (5) $(r'_1, r'_2) = (1, 1)$ cannot be. □

Appendix B. Proofs for uncertainty (§3 and §4)

For a weighting function W, we use the following notation: $\pi(E^R) = W(E \cup R) - W(R)$ is the *decision weight* of an outcome in RDU under event E if R is the *rank*, i.e., the event giving outcomes ranked better than the one under E. $\pi^b(E) = W(E)$ and $\pi^w(E) = 1 - W(E^c)$. W is additive (EU) if and only if we have rank independence, i.e., decision weights $\pi(E^R)$ do not dependend on the rank R. W is convex if and only if decision weights $\pi(E^R)$ are nondecreasing in the rank R, which reflects a pessimistic attitude. If W is convex, then its Core consists of all probability measures P_ρ (defined in §4) and

$$\text{RDU is the infimum EU with respect to all } P_\rho. \tag{11}$$

The following observation provides sufficiency of the preference conditions in all our results.

OBSERVATION 20 *If U is concave and W is convex, then RDU is concave and, hence, $\succcurlyeq$ is convex. This also holds on any subset $\mathcal{F}'$ of $\mathcal{F}$.* □

PROOF. Consider three acts $x, y, \lambda x + (1-\lambda)y$, contained in comoncones with orderings $\rho_x, \rho_y, \rho_\lambda$, respectively. Then, with the first inequality due to concave utility and the second due to convexity of W (Eq. 11): $RDU(\lambda x + (1-\lambda)y) = \int_S (U\big(\lambda x + (1-\lambda)y)\big)dP_{\rho_\lambda} \geq \int_S (\lambda U(x) + (1-\lambda)U(y))dP_{\rho_\lambda} = \lambda \int_S U(x)dP_{\rho_\lambda} + (1-\lambda)\int_S U(y)dP_{\rho_\lambda} =$

$\lambda EU_{P_{\rho_\lambda}}(x) + (1-\lambda)EU_{P_{\rho_\lambda}}(y) \geq \lambda EU_{P_{\rho_x}}(x) + (1-\lambda)EU_{P_{\rho_y}}(y) = \lambda RDU(x) + (1-\lambda)RDU(y)$. □

Proof of Corollary 4. Necessity follows from Observation 20. Sufficiency follows from applying the main Theorem 3 to any two-dimensional subspace $\{\alpha_E\beta \in \mathcal{F} : \alpha \geq \beta\}$ with E nondegenerate. □

Proof of Corollary 5. Both utilities being strictly increasing and continuous, we define the continuous strictly increasing φ by $U^2(.) = \varphi(U^1(.))$. Deviating from the notation elsewhere in this paper, all outcomes are expressed in U^1 units in this proof. It means that we replace I by $U^1(I)$ which again is a nonpoint interval, that U^1 is linear, and $U^2 = \varphi$. Eq. 4 can be rewritten as $y_j = P^1(A)x_j + (1 - P^1(A))z_j$. Regarding necessity, if φ is concave then, by Corollary 4, $\succcurlyeq^2$ is convex w.r.t. outcome mixing, which, given the equality just rewritten, implies Eq. 5 and, hence, that $\succcurlyeq^2$ is more outcome-risk averse than $\succcurlyeq^1$. The rest of this proof concerns sufficiency. We assume that $\succcurlyeq^2$ is more outcome-risk averse than $\succcurlyeq^1$ and derive concavity of φ.

There exists a nondegenerate event A for $\succcurlyeq^1$, and a nondegenerate event B for $\succcurlyeq^2$. From now on we only consider binary acts $x_{1_A}x_2$ for $\succcurlyeq^1$ and $x_{1_B}x_2$ for $\succcurlyeq^2$, denoting them (x_1, x_2). Define $0 < q = P^1(A) < 1$. Consider any $x = (x_1, x_2) \succcurlyeq^2 (z_1, z_2) = z$. To derive convexity of $\succcurlyeq^2$ on the set of binary acts considered here, we have to show:

$$\forall\ 0 < \lambda < 1 : \lambda x + (1-\lambda)z \succcurlyeq^2 z. \tag{12}$$

We first show it for $\lambda = q$. By continuity we can decrease x_1, x_2 into x_1', x_2' such that $(x_1', x_2') \sim^2 (z_1, z_2)$. Define $y_j = qx_j' + (1-q)z_j$, $j = 1, 2$. With these definitions, Eq. 4 is satisfied with x' instead of x. By Eq. 5, $y \succcurlyeq^2 z$. By monotonicity, $qx + (1-q)z \succcurlyeq^2 y \succcurlyeq^2 z$. Eq. 12 holds for $\lambda = q$. By repeated application and transitivity, the equation follows for a subset of λs dense in $[0, 1]$ and then, by continuity, for all λ. Convexity of $\succcurlyeq^2$ on the two-dimensional set of acts considered here has been proved. Concavity of φ follows from the main Theorem 3. □

Proof of Corollary 6. RDU on a comoncone coincides with EU on that comoncone w.r.t. a finitely additive probability measure P, which is convex. Hence, concavity of U implies convexity of $\succcurlyeq$ by Observation 20. Conversely, assume that $\succcurlyeq$ is convex.

Apply the main Theorem 3 to any two-dimensional subspace $\{\alpha_E\beta \in \mathcal{F} : \alpha \geq \beta\}$ of the comoncone SF' with E nondegenerate, and concavity of U follows. □

The following lemma is the main step in deriving implications of the main Theorem 3 for weighting functions. The inequality in the lemma states that the decision weight of s_1 is nondecreasing in rank. It implies the same inequality for s_2 and is equivalent to convexity of W. Showing this for higher dimensions goes the same way as for two dimensions, which is why this lemma captures the essence.

LEMMA 21 *Assume $n = 2$, RDU, and convexity of $\succcurlyeq$. Then $W(s_1) \leq 1 - W(s_2)$.* □

PROOF. Take an outcome in $int(I)$, 0 wlog, at which U is differentiable. Wlog, $U(0) = 0$. We consider a small positive α tending to 0, with $o(\alpha)$, or o_α for short, the usual notation for a function with $lim_{\alpha\to 0}\frac{o_\alpha}{\alpha} = 0$. In other words, in first-order approximations o_α can be ignored. We write $\pi_1 = W(s_1), {\pi_2}' = W(s_2)$.

Assume $\pi_1 > 0$ and ${\pi_2}' > 0$; otherwise we are immediately done. Then, because of continuity of U on $int(I)$ and differentiability at 0, we can obtain, for all α close to 0, the left indifference in

$$({\pi_2}'\alpha, 0) \sim (0, \pi_1\alpha + o_\alpha) \preccurlyeq (\mu{\pi_2}'\alpha, (1-\mu)(\pi_1\alpha + o_\alpha)). \tag{13}$$

The preference is discussed later. We compare two values: the $\mu, 1-\mu$ mixture of the RDU values (which are the same) of the left two acts and the RDU value of their $\mu, 1-\mu$ mixture, which is the right act. We take $\mu > 0$ so small that the left outcome $\mu{\pi_2}'\alpha$ in the mixture is below the right outcome. Informally, by local linearity, in a first-order approximation the only difference between the two values compared is that for the left value the left outcome ${\pi_2}'\alpha$ receives the highest-outcome decision weight π_1 whereas for the right value it receives the lowest-outcome decision weight $1 - {\pi_2}'$. Convexity of $\succcurlyeq$ implies the preference in Eq. 13, which implies $1 - {\pi_2}' \geq \pi_1$.

Formally, note that different appearances of o_α can designate different functions. Thus we can, for instance, write, for constants k_1 and k_2 independent of α: $k_1 o_\alpha + k_2 o_\alpha = o_\alpha$. The following is most easily first read for linear utility, when all terms o_α are zero. Write $u' = U'(0)$; μ can be chosen independently of α. Here is the comparison of the aforementioned two values: $\mu\pi_1 u'{\pi_2}'\alpha + o_\alpha + (1-\mu){\pi_2}'u'\pi_1\alpha + o_\alpha \leq (1-{\pi_2}')u'\mu{\pi_2}'\alpha +$

$o_\alpha + \pi_2{}'u'(1-\mu)\pi_1\alpha + o_\alpha$. Dividing by $\mu u'\pi_2{}'\alpha$, we obtain $\pi_1 \le 1 - \pi_2{}' + \frac{o_\alpha}{\alpha}$. Now $\pi_1 \le 1 - \pi_2{}'$ follows. □

Proof of the main Corollary 7. Necessity of the preference condition follows from Observation 20. We, therefore, assume convexity of $\succcurlyeq$. Concavity of U follows from considering any nondegenerate event E and applying the main Theorem 3 to the set of acts $(E : x_1, E^c x_2)$ with $x_1 \ge x_2$. We finally derive convexity of W.

Assume A, B, B' as in Eq. 3. Write $C = B' - B$, $R = S - (A \cup B')$. Take $\gamma > \beta \in int(I)$, and consider $\mathcal{F}^* = \{(B : \gamma, C : x_1, A : x_2, R : \beta) \in \mathcal{F} : \gamma \ge x_1 \ge \beta, \gamma \ge x_2 \ge \beta\}$. This space is isomorphic to the space $\mathcal{F}$ of Lemma 21 and the convexity inequality needed here follows from the one of the lemma. Details are as follows. Take outcome space $I^* = [\beta, \gamma]$, $s_1^* = C, s_2^* = A$, and weighting function $W^*(E) = \frac{W(E \cup B) - W(B)}{W(A \cup B \cup C) - W(B)}$. (If the denominator is 0, then the convexity inequality is trivially satisfied.) The inequality $W^*(\{s_1, s_2\}) - W^*(s_1) \ge W^*(s_2)$ in Lemma 21 is the same as the required $W(A \cup B') - W(B') \ge W(A \cup B) - W(B)$. □

Proof of Corollary 8. Statement (i) follows from the main Corollary 7 because convexity of W on every $\mathcal{F}_E$ is equivalent to subadditivity. Statement (ii) follows immediately from the main Theorem 3. □

Appendix C. Proofs and literature for risk (§5)

Proof of Corollary 10. For any fixed probability vector $(p_1, \ldots, p_n)$ we define $S = \{s_1, \ldots, s_n\}$, the probability measure P on S through $P(s_j) = p_j$, and $W = w \circ P$. The second claim in the Corollary now follows from Corollary 6. For the first claim, convexity of w implies convexity of every W just constructed and, hence, outcome-convexity of $\succcurlyeq$ follows from the main Corollary 7 applied to such W.

Now assume outcome-convexity. The above construction and the main Corollary 7 imply concavity of U and convexity of every W so constructed. Take any probabilities p_1, p_2, p_3 adding to less than 1. Convexity of W implies $W\{s_2, s_1\} - W\{s_1\} \leq W\{s_3, s_2, s_1\} - W\{s_3, s_1\}$, that is, $w(p_2 + p_1) - w(p_1) \leq w(p_3 + p_2 + p_1) - w(p_3 + p_1)$. This condition, increasing w differences, is equivalent to convexity of w. □

FURTHER LITERATURE ON AVERSION TO MEAN-PRESERVING SPEADS. The equivalence of outcome-convexity with aversion to mean-preserving spreads (and its variations discussed below) holds only under RDU. In general, there is no logical relation between these conditions. Before discussing further details, we note that outcome-convexity has been studied in the literature only when an underlying state space was specified, but this is equivalent to our definition for simple lotteries. Under compact continuity, outcome-convexity does imply aversion to mean-preserving spreads (Chateauneuf & Lakhnati 2007 Theorem 4.2; Dekel 1989 Proposition 2). If convexity (w.r.t. probabilistic mixing) holds, a condition implied by aversion to mean-preserving spreads under continuity, then by Dekel (1989 Propositions 2 and 3), under weak continuity, aversion to mean-preserving spreads becomes equivalent to convexity.

Bommier, Chassagnon, & Le Grand (2012 Result 3) considered a more-risk-averse than relation weaker than aversion to mean-preserving spreads, with distribution functions crossing once. They showed for linear w (EU) that their condition is equivalent to concavity of U. They also showed for linear utility (see their proof on pp. 1638-1639) that their condition is equivalent to convexity of w. They provided, more generally, comparative results. Chew & Mao (1995, Theorem 2 and Table II) used a yet weaker elementary risk aversion condition, implied by our outcome-convexity, and showed, under RDU, that it holds if and only if w is convex and U is concave. They assumed Gateaux differentiability, which under RDU is equivalent to differentiability of w, and continuity. Hence, under the latter two assumptions, they provided an alternative way to obtain our Corollary 10(i). Ebert (2004, Theorem 2) used a progressive transfer property, equivalent to Chew & Mao's (1995) elementary risk aversion, to characterize concavity of U plus convexity of w. Importantly, he did not need differentiability of U, although he did assume continuity. He considered welfare theory where states are reinterpreted as people and probabilities p_j reflect proportions of a population. He used extra structural richness in allowing for any arbitrary replication of any group ("event") in the population. □

Proof of Corollary 12. This follows from Statement (i) in Corollary 10. □

Proof of Corollary 13. We define states and W as in the proof of Corollary 10. Statement (i) follows from Corollary 10 because convexity of $W = w \circ P$ on every $\mathcal{F}_E$ is equivalent to subadditivity of w. Statement (ii) follows from Corollary 10. □

Appendix D. Further proofs for risk (§6)

Proof of Corollary 15. Fix nonindifferent outcomes $x_{n+1} \prec x_n \prec \cdots \prec x_1$ for some $n \geq 2$ with probabilities $p_1, \ldots, p_{n+1}$, depicted in Figure 1. In this figure, the minimal outcome x_{n+1} is 0, but it may be any other outcome. If a lottery does not take the value x_{n+1}, then $p_{n+1} = 0$. We focus on lotteries $(p_1 : x_1, \ldots, p_{n+1} : x_{n+1})$. This lottery is uniquely determined by $(q_1, \ldots, q_n)$ with $q_j := p_1 + \cdots + p_j$ for all j. Here q_j is the *rank* of outcome x_{j+1}, i .e., the probability of receiving a better-ranked outcome. Writing $I := [0, 1]$, we have transformed the set of lotteries into the set $I^{n\downarrow} := \{(q_1, \ldots, q_n) \in I^n : q_1 \geq \cdots \geq q_n\}$, with every lottery P uniquely corresponding with a $P' \in I^{n\downarrow}$. Our correspondence is compatible with convex combination. That is, a probabilistic mix $\lambda P \oplus (1 - \lambda)Q$ corresponds with $\lambda P' + (1 - \lambda)Q'$. Convexity of $\succcurlyeq$ on lotteries is equivalent to convexity of the corresponding $\succcurlyeq'$ on $I^{n\downarrow}$ in the sense of Corollary 6.

We normalize U such that $U(x_{n+1}) = 0$ and $U(x_1) = 1$. Define $\pi'_j := U(x_j) - U(x_{j+1}), j = 1, \ldots, n$. Then $RDU(q_1, \ldots, q_n) = \sum_{j=1}^{n} \pi'_j w(q_j)$. In other words, we have a representation as in Corollary 6 but with w in the role of U. All conditions of Corollary 6 are satisfied. Convexity of $\succcurlyeq$ is equivalent to convexity of $\succcurlyeq'$ and this is, by Corollary 6, equivalent to concavity of w. □

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