The Deceptive Beauty of Monotonicity, and the Million-Dollar Question: Row-First or Column-First Aggregation?

Chen Li, c.li@ese.eur.nl, Erasmus School of Economics, Erasmus University, 3000 DR Rotterdam, the Netherlands
Kirsten I.M. Rohde, kirsten.rohde@maastrichtuniversity.nl, School of Business and Economics, Maastricht University, Maastricht, The Netherlands
Peter P. Wakker, wakker@ese.eur.nl, Erasmus School of Economics, 3000 DR, Erasmus University Rotterdam, the Netherlands

January 2026

Abstract. This paper proposes a unified framework for optimization over two or more components (e.g., risk and time). We identify a common cause (the "monotonicity problem") underlying many current debates in behavioral decision theory, concerning correlation preference in intertemporal choice, incentive compatibility of the random incentive system, hedging in ambiguity measurements, the judgment aggregation paradox, ex post versus ex ante fairness in welfare, and many others. Further, the monotonicity problem implies that a "middle ground" for single component optimization, used in virtually all behavioral theories, is not available for multi-component optimization. That leads to an unavoidable bifurcation dilemma, where one has to choose one of only two disjoint routes available. Stances taken in the above debates all amount to a choice of one of those two routes. We provide general techniques for properly choosing in this dilemma, thus clarifying and unifying many debates, and obtaining many generalizations and new insights for many fields. For instance, our analysis supports the validity of the random incentive system and of ambiguity measurements despite hedging, criticisms of monotonicity in the Anscombe-Aumann framework of ambiguity, ex post over ex ante fairness, and it favors particular framings over others in experiments.

Keywords: discounted expected utility • multiobjective optimization • hedging under ambiguity • ex ante inequality • random incentive system

1. Introduction

It may come as a surprise that many debates on various topics in the literature, from multiattribute risk aversion to Harsanyi's veil of ignorance or to incentive compatibility of the random incentive system, share a common hidden cause: the "monotonicity problem." This paper identifies that problem and provides diagnoses and solutions. The mathematics underlying the problem has been known for almost a century, since Nataf (1948), and special cases and parts of the problem have been discussed in numerous papers, independently in many fields. But the universality, unavoidability, and acuteness of the underlying monotonicity problem for all behavioral decision models with two or more components of optimization[1] have not been observed before. That is, the problem is more serious and fundamental than has been known before. With the underlying cause identified for these many debates, we can provide diagnoses with a clear roadmap and steering techniques to navigate through the dilemmas, obtaining generalizations and new insights for many fields.

For decisions with a single component, behavioral decision models typically operate in a "middle ground" where classical strong separability is relaxed but basic monotonicity is maintained, e.g., in nonexpected utility for risk or in equity models for welfare. Whereas this "middle ground" has delivered numerous fruitful behavioral models for single components, when it is applied to decision situations with two or more components, puzzling paradoxes appear.

To illustrate the puzzles and prepare for identifying their underlying cause, Sections 2 and 4 present two paradoxes, the separability and the no-separability paradox, with a puzzling appearance and disappearance of separability, respectively. Our central Theorem 7 in Section 5 then shows that the two paradoxes are two sides of the same coin. As it turns out, the "middle ground", so fruitful for only one component, does not exist for two or more components. This is what we call the

[1] For instance, if we optimize over the two components risk and time, or over risk and persons. Components can also concern commodities, production inputs, prices, expert opinions, health attributes, regions, and so on.

monotonicity problem, and it is the abovementioned common cause. Structural changes occur when going from one component to two or more, and new techniques need to be developed.

Section 6 presents the first application of the monotonicity problem to quantitative optimizations. It leads to a "bifurcation dilemma" (Figure 4). We show that many stances in the literature, such as preference for ex post versus ex ante fairness, or for validity of the random incentive system (RIS; defined in Subsection 9.2) versus a hedging confound in ambiguity measurements, amount to choosing one of the only two possible routes in the bifurcation dilemma. Under classical models such as discounted expected utility (DEU), the workhorse of decision analysis (Baucells & Bodily 2024), different orders of "aggregation" (integration) give the same result so that the chosen order is immaterial. However, under behavioral generalizations, the order of aggregation does matter. Most papers implicitly, without any discussion, then choose one order of aggregation. Several papers did signal that this order can matter and discussed discrepancies between the orders. Section 6 shows that this choice of order is a special case of the bifurcation dilemma, and that it is more critical than has been known before.

Section 7 applies our results to the special case of welfare under risk. Section 8 then provides guidelines and steering techniques to solve the monotonicity problem and to make proper choices in the bifurcation dilemma. There we take welfare under risk as running example, but our recommendations and techniques apply to all cases of multiple components. Section 9 further presents new insights for particular cases of multiple components.

This paper is not a standard theory paper. Regarding its central theorem, Theorem 7, *mathematical* generalizations have been provided before (Mongin & Pivato 2015 Proposition 1). The basic *mathematical* result has been known for almost a century (Nataf 1948). However, its managerial implications, of absence of a middle ground for behavioral theories leading to a critical bifurcation dilemma, has not been understood before. We thus unify a wide range of seemingly unrelated debates and provide many new insights for various behavioral subfields. We next explain our novelties in further detail.

Special cases of the monotonicity problem have been discussed in particular contexts before. However, its acuteness, where one has to choose one of two disjoint routes (Corollary 11), and in each route immediately has to give up half of the interactions of potential interest, has not been observed before. Most papers focused on one field of application, such as risk and welfare, and presented results only there. We provide results for many fields.

By pointing out that the same monotonicity problem applies to all fields, we organize pre-existing analyses in different fields into one unified framework. This unification facilitates knowledge transfer across different fields. For example, when discussing the validity of the RIS, it is useful to know that one is essentially considering the same arguments as in discussions of ex ante versus ex post fairness in welfare, of monotonicity in the Anscombe-Aumann framework, of correlation aversion, and so on. No-one working on RIS has mentioned such relations before.

We propose structural techniques to resolve or alleviate the monotonicity problem. Our partial information technique is entirely novel. While the framing and timing techniques have been discussed in isolated contexts, we are the first to synthesize them into a general framework, applicable in all fields. Section 8 presents practical guidelines to navigate the choice in the bifurcation problem. We next give examples of new insights for particular fields.

For welfare our guidelines show that, *other things equal*, Broome's (1991) argument against Harsanyi's utilitarianism is more convincing than Diamond's (1967) competing argument. Diamond and Broome "simply" chose differently between the only two tractable routes available.

For the Anscombe-Aumann framework on ambiguity, we show that the monotonicity condition chose the less plausible of the two possible routes. Our analysis makes clear that existing criticisms of the framework, despite their varied forms, essentially hinge on this critical choice in the bifurcation problem. Our unified new perspective leads to new counterexamples to the Anscombe-Aumann framework in Figures 9 and 10.

Regarding the RIS, previous justifications of its incentive compatibility appealed to "isolation," which is our risk monotonicity. We are the first to show that this isolation concerns one of two possible routes in a bifurcation situation, and is the more plausible one, providing new support for the validity of RIS. Our analyses also clarify that the widespread misunderstanding, that validity of RIS would require full strength of expected utility, stems from the separability paradox in Section 2. We show that similar observations hold for the hedging problem in ambiguity experiments (Section 9.2). Our guidelines demonstrate that hedging is usually not a serious problem and show how it can be minimized. Note, again, that many papers have been written on only hedging for ambiguity, whereas our paper brings novelty (and unity) to that and many other cases. The judgement paradox has never yet been related to any of the above problems.

As for Theorem 1 on DEU, its novelty is not in mathematical generality but in achieving simplicity and appeal. We are the first to state the axiomatization of the important DEU model entirely verbally, thus making the axioms accessible to nonspecialists. As we show later, this theorem is a first signal of the monotonicity problem and of the absence of middle ground.

To focus the main text of this paper, limit its size, and maintain high accessibility, some powerful mathematical generalizations based on our aggregation techniques, generalizing several well-known preference axiomatizations with simplified proofs, are presented in Supplementary Appendix C. Those axiomatizations generalize Anscombe & Aumann (1963), Gul (1992), and Harsanyi (1955).

2. Discounted Expected Utility and the Separability Paradox

This section presents our first paradox, yet within the classical, not yet behavioral, framework.

2.1. Definitions for uncertainty and time

We consider choices between "actstreams," i.e., matrices as in Figure 1. Cells describe money amounts (real numbers). We here deal with two components, uncertainty (states of nature), and time (timepoints). If state of nature s^i obtains then at timepoint t_j one receives money x_j^i. Columns designate acts, i.e., maps from states to outcomes, and rows similarly are outcome streams. An actstream gives a stream yielding acts or, equivalently, an act yielding streams.

FIGURE 1. An actstream

$$
\begin{array}{cc}
 & \begin{array}{ccccc} t_1 & \cdots & t_j & \cdots & t_n \end{array} \\
\begin{array}{c} s^1 \\ \vdots \\ s^i \\ \vdots \\ s^m \end{array} &
\begin{bmatrix} x_1^1 & \cdots & \cdots & \cdots & x_n^1 \\ \vdots & & & & \vdots \\ \vdots & & x_j^i & & \vdots \\ \vdots & & & & \vdots \\ x_1^m & \cdots & \cdots & \cdots & x_n^m \end{bmatrix}
\end{array}
$$

Any outcome stream $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ can be identified with the matrix having that outcome stream in each row, i.e., the degenerate lottery giving that outcome stream with certainty. Any act $(\alpha^1, \dots, \alpha^m) \in \mathbb{R}^m$ can be identified with the matrix having that act in the first column, and outcome 0 elsewhere; i.e, receiving that act at t_1 and nothing after. This way, preferences over acts and streams are derived from preferences over actstreams.

Expected utility (*EU*) holds if there exist positive *probabilities* $p^1, \dots, p^m$ and a *utility function U* ($U: \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing) such that preferences over acts (elements of $\mathbb{R}^m$) are represented by *expected utility*

$$(\alpha^1, \dots, \alpha^m) \mapsto \sum_{i=1}^m p^i \times U(\alpha^i) \tag{1}$$

Discounted utility (*DU*) holds if there exist *discount factors* $0 < d_j$ ($j = 1, \dots, n$) and a utility function U such that preferences over streams are represented by *discounted utility* (*DU*)

$$(\beta_1, \dots, \beta_n) \mapsto \sum_{j=1}^n d_j \times U(\beta_j) \tag{2}$$

Exponential discounting can readily be obtained by adding a preference condition that guarantees the same discount rate over time.

Discounted expected utility (*DEU*) holds if there exist probabilities, discount factors, and a utility function U such that preferences over actstreams are represented by their *discounted expected utility* (*DEU*)

$$\sum_{i=1}^{m} p^i \sum_{j=1}^{n} d_j \times U(x_j^i) \tag{3}$$

(Keeney & Raiffa 1976 Ch. 9). The order of aggregation, row-first or column-first, is immaterial, because Eq. 3 is equivalent to

$$\sum_{j=1}^{n} d_j \sum_{i=1}^{m} p^i \times U(x_j^i) \tag{4}$$

We will usually use Eq. 3. DEU has the following implications:

(1) EU holds for uncertainty preferences.

(2) DU holds for intertemporal preferences.

(3) EU and DU use the same utility function.

Each of these implications has often been criticized on normative grounds. For instance, numerous debates on cardinal utility (Moscati 2018) and on the difference between risky and riskless utility (Abdellaoui, Attema, & Bleichrodt 2010; Dyer & Sarin 1982; Keeney & Raiffa 1976) have been advanced, challenging implication (3). The three implications have also been extensively criticized on empirical grounds; see Starmer (2000) for (1), Attema (2012) for (2), and Abdellaoui et al. (2013) for (3). In Subsections 2.2 and 2.3, we present the axioms needed to axiomatize DEU.

2.2. "Unobjectionable" axioms

AXIOM 1. *Weak ordering*: transitivity and completeness (including reflexivity).

AXIOM 2. Continuity: the usual (Euclidean) continuity on $\mathbb{R}^{m \times n}$.

AXIOM 3. *Outcome monotonicity*: strictly increasing any cell x_j^i strictly improves the actstream.

AXIOM 4. *Act monotonicity*: at any timepoint, replacing the act there by a weakly [strictly] preferred act leads to a weakly [strictly] preferred actstream.

AXIOM 5. *Stream monotonicity*: at any state, replacing the stream there by a weakly [strictly] preferred stream leads to a weakly [strictly] preferred actstream.

2.3. Objectionable axioms

Given the strong separabilities (defined in Section 3) over states and timepoints involved in DEU, which have been so widely falsified empirically, one may expect strong objectionable axioms to be listed in this section. However, there is none! That is, the axioms in Subsection 2.2 suffice to give DEU. This may come as a paradox. How can such seemingly innocuous preference conditions have such strong implications? The paradox is displayed in Theorem 1 in the following Subsection.

2.4. Axiomatization of discounted expected utility

THEOREM 1. The following two statements are equivalent.

(i) Discounted expected utility holds.

(ii) Weak ordering, continuity, and monotonicity with respect to outcomes, acts, and streams hold.

The first paradox of this paper, the *separability paradox*, refers to the question of how such seemingly weak preference conditions (Pivato & Tchouantez 2024: "uncontroversial") can have such strong implications, with strong separabilities over states and timepoints. Because of its simplicity, we claim that Theorem 1 provides the most appealing axiomatization of DEU presently available. As we will show later, Theorem 1 is the first puzzling consequence of the monotonicity problem. Before touching on the crux of that problem, Section 3 introduces a unified framework that will serve to showcase the monotonicity problem.

In many contexts, extensions to infinite components, such as continuum time intervals, are desirable. Supplementary Appendix B shows that they can readily be achieved using standard tools from mathematical measure theory (e.g., Theorem 14).[2] Grabisch, Monet, & Vergopoulos (2023) provided such a result for continua of components that, unlike our results, do not need a continuum of outcomes. The important point to note is that our intuitive axioms, mainly the monotonicities, remain unaffected in this process. Only the technical continuity is modified. Thus, these modifications do not affect the practical and conceptual implications discussed in the main text of this paper.

3. General Definitions and Versions of Separability

We throughout assume that all decisions are made at one fixed timepoint, preceding all timepoints of a time component if the latter is present. The decision timepoint also precedes any information about the resolution of risk or uncertainty if an uncertainty component is present. Thus, if the true state was determined prior to the decision, the decision maker does not know which it is.[3] We also assume that all uncertainty is resolved at one fixed timepoint, i.e., in one stage, prior to any receipt of outcomes. Thus, we also do not deal with multistage complications such as preferences for the timing of the resolution of uncertainty, as in Kreps & Porteus (1978) and Epstein & Zin (1989), and we do not consider the corresponding recursive formulas.

We now fully formalize our analysis, and add one generalization, amounting to state- and time-dependence of the utility function U in Eq. 3. This greatly enhances the applicability of our results.

---

[2] The theorem can also be readily extended to risk. For example, if all s^i have known probabilities $1/m$ implying symmetry (and subjective probabilities p^i=1/m) we obtain all equal-probability distributions, comprising all rational probabilities. Extension to all probability distributions follows from common continuity (Supplementary Appendix B).

[3] Prior resolution of uncertainty is only a matter of perception and never of strategic relevance, and dynamic decision principles and updating play no role in this paper.

Our general framework considers preferences $\succcurlyeq$ over *matrices.* There are two *components*: (1) a finite *row set* $R = \{r^1, \dots, r^m\}$ with its *attributes* being m *rows* and (2) a finite *column set* $C = \{c_1, \dots, c_n\}$ with its *attributes* being n *columns*. Before, the components concerned uncertainty and time, with m attributes (states) and n attributes (timepoints), respectively. For simplicity, we continue to assume that the *outcome space* is $\mathbb{R}$, say monetary. In some of our examples, outcomes may concern nonmonetary goods.[4] We assume $m, n > 1$ fixed. *Rows* in $\mathbb{R}^n$ map C to $\mathbb{R}$ and *columns* in $\mathbb{R}^m$ map R to $\mathbb{R}$. A *matrix* in $\mathbb{R}^{m \times n}$ (called actstream before) maps $R \times C$ to $\mathbb{R}$. It specifies a row $(x_1^i, \dots, x_n^i)$ for each r^i and a column $(x_j^1, \dots, x_j^m)$ for each c_j. A matrix in $\mathbb{R}^{m \times n}$ consists of $m \times n$ *cells* x_j^i.

FIGURE 2. A matrix

$$\begin{array}{cc}
 & \begin{array}{ccccc} c_1 & \cdots & c_j & \cdots & c_n \end{array} \\
\begin{array}{c} r^1 \\ \vdots \\ r^i \\ \vdots \\ r^m \end{array} & \begin{bmatrix} x_1^1 & \cdots & & \cdots & x_n^1 \\ \vdots & & & & \vdots \\ \vdots & & x_j^i & & \vdots \\ \vdots & & & & \vdots \\ x_1^m & \cdots & & \cdots & x_n^m \end{bmatrix}
\end{array}$$

Separability is central in our analysis. A subset of cells is *separable* if preferences over those cells, while keeping the outcomes at all other cells fixed, are independent of the levels where the other cells are kept fixed. We will consider various versions of separability, imposed on various collections of subsets of cells. The strongest version is *strong separability*, imposing separability on all subsets of cells. *Weak separability* means that every single cell is separable.

We will also consider intermediate levels of separability, applied to rows or to columns, as follows. We consider underlying preferences $\succcurlyeq^i$ over rows $(x_1^i, \dots, x_n^i)$ and $\succcurlyeq_j$ over columns

[4] Mathematical extensions of our theorems to connected topological outcome spaces (e.g., convex sets of commodity bundles) are straightforward. However, this paper seeks for conceptual implications and accessibility rather than mathematical generality.

$(x_j^1, \ldots, x_j^m)$ that can be derived from $\succcurlyeq$ over matrices by keeping "outside cells" fixed. This "conditioning" procedure works well if proper separability conditions hold:

DEFINITION 2. *Weak separability of rows*, or *row monotonicity*, holds if each row is separable. *Weak separability of columns*, or *column monotonicity*, holds if each column is separable.

Separability of a subset is equivalent to the possibility to define an underlying preference relation, "conditional" on that subset ($\succcurlyeq^i$ and $\succcurlyeq_j$ above), so that we have monotonicity with respect to that relation. Hence, the relations $\succcurlyeq^i$ and $\succcurlyeq_j$ work well under the separability conditions in Definition 2. Outcome monotonicity is defined as before (strictly increasing any cell strictly improves the matrix), and readily implies weak separability. Definition 2 generalizes act and stream monotonicity by allowing for row and column dependence, indicated by the sub- and superscripts in $\succcurlyeq^i$ and $\succcurlyeq_j$. For consistency with the literature, we maintain the monotonicity terminology.

DEFINITION 3. *Strong separability of rows* holds if every union of rows is separable. *Strong separability of columns* holds if every union of columns is separable.

Weak and strong separability, as default, take cells as "unit," whereas their alternatives in Definitions 2 and 3 specify rows or columns as "unit."

4. The Behavioral Middle Ground and the No-Separability Paradox

This section presents our second paradox for decisions involving two or more components. To illustrate the essence of the paradox, we first take a step back, and look at how researchers deal with separability when modeling decisions involving only one component.

Sono (1945) and Leontief (1947) introduced separability. They did so for consumer theory, where only one component is considered, describing commodities. As in the preceding section, a subset of commodities is separable if preferences over those commodities, while keeping the levels of the other commodities fixed, are independent of the levels where those other commodities are kept fixed. Strong separability means that every subset of commodities is separable, and weak separability means that every single commodity is separable.

It has long been known that strong separability is very restrictive. For three or more commodities, it implies maximization of an additively decomposable function (Gorman 1968). That is, it precludes any interaction between commodities, implying constant marginal rates of substitution. Weak separability is weak. For one component, it is already implied by outcome monotonicity, which is commonly considered nonobjectionable.

Separability turned out to be central in many fields. Thus, for decision under uncertainty, Savage's (1954) famous sure-thing principle (his P2), the watershed between Bayesian and non-Bayesian models (Wu, Zhang, & Gonzalez 2004 p. 401) and the main cornerstone of normative decision analysis, is nothing but a reinvention of strong separability, imposed on a state space.

Classical decision models were first developed for one component. They include expected utility for uncertainty, discounted utility for time, and utilitarianism for welfare. They all assume strong separability. Modern behavioral models relax strong separability to capture the certainty effect in uncertainty, habit formation in time, fairness in welfare, and numerous other interactions violating separability. When relaxing strong separability in classical models, behavioral models do commonly maintain weak separability though, and the stronger outcome monotonicity. That is, they operate in what we call the *middle ground*: models that give up strong separability but maintain weak separability. Thus, when Fishburn (1978) pointed out that separate probability weighting, as partly used in original prospect theory (Kahneman & Tversky 1979), violates outcome monotonicity, Tversky & Kahneman (1992) updated prospect theory, using Quiggin's (1982) invention of rank dependence, to return to the middle ground, which improved empirical performance (Hirshman & Wu 2025).

It is natural to expect behavioral models involving two or more components to also operate in such a middle ground. The following example demonstrates a typical attempt to achieve such middle ground, which will lead to the second paradox of this paper.

EXAMPLE 4 [No-Separability Paradox: A Paradoxical Disappearance of Separability]. We consider actstreams as in Figure 1 and Eq. 3 (DEU). We assume two timepoints with no discounting ($d_1 = d_2 = 1$), two equally likely states ($P(s^1) = P(s^2) = 0.5$), and linear utility ($U(\alpha) = \alpha$). A manager wants to relax strong separability for risky states, i.e., generalize EU for risk, by allowing extra pessimism as in Allais' paradox, through overweighting of the worst outcome. She wants to maintain strong separability for timepoints t_j though. Thus, in Eq. 3, the right summation, DU over columns, is kept, but the left summation, EU, is replaced by a nonexpected utility formula that satisfies stochastic dominance.[5] It may be any nonexpected utility model with overweighting of the worst outcome, such as rank-dependent utility with pessimism. At first sight, the manager seems to have achieved the desired middle ground, with weak but no strong separability of risky states (rows), and strong, so surely weak, separability of timepoints (columns).

FIGURE 3. t_1 is not separable

$$\begin{array}{c c} & \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{array}{c} s^1 \\ s^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \end{array} \succ \begin{array}{c c} & \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{array}{c} s^1 \\ s^2 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{array}{c} s^1 \\ s^2 \end{array} & \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \end{array} \prec \begin{array}{c c} & \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{array}{c} s^1 \\ s^2 \end{array} & \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \end{array}$$

[5] Kochov (2015) and Bastianello & Faro (2023) argued for maintaining stochastic dominance (monotonicity).

However, a closer look reveals a serious problem. The overweighting of the worst state implies the preferences in Figure 3, which implies a puzzling violation of separability of t_1. Contrary to what the manager had hoped for, preferences implied by the proposed model violate strong separability and even weak separability (column monotonicity) of timepoints. Unbeknownst to the manager, she has introduced interactions between timepoints "under the cover." □

Example 4 is paradoxical. The manager only wanted to give up strong separability of states, to work in the middle ground of behavioral theories, and she did not touch the formula of time aggregation. Yet, inadvertently, separability of timepoints was lost, even their weak separability. The model thus fell out of the desired middle ground.

We have presented two paradoxes. In the separability paradox, where the most restrictive decision model DEU was implied by the seemingly most innocuous conditions, separability appeared in a puzzling manner. In the no-separability paradox, where the gentle relaxation of strong separability for one component implied violation of even weak separability of the other, separability disappeared in a puzzling manner. One couldn't help but wonder:

QUESTION 5. How can the behavioral middle ground (no strong separability but still weak separability) be reached, if at all, for two or more components?

This question will be answered in the next section. Many, seemingly unrelated, issues in the behavioral field, presented later, will be clarified by this answer.

5. Resolving the Paradoxes and Answering Question 5

To prepare, we present a theorem that has essentially been known for almost a century. *Additive utility* (*AU*) holds if preferences over matrices are represented by

$$\sum_{i=1}^{m}\sum_{j=1}^{n}V_j^i(x_j^i) \tag{5}$$

for strictly increasing continuous functions $V_j^i(x_j^i)$. AU readily implies strong separability of cells and, consequently, of rows and columns. The following remarkable observation, known as the Theorem of aggregation, is basic to this paper. Its history is discussed after the observation.

OBSERVATION 6 [Theorem of Aggregation] The following two statements are equivalent for $\succcurlyeq$ on $\mathbb{R}^{m\times n}$.

(i) Additive utility holds.

(ii) Weak ordering, continuity, and monotonicity with respect to outcomes, rows, and columns hold.

It is obvious that Statement (i) implies Statement (ii), and even strong separability of cells. For the reversed implication, it is clear that row and column monotonicity preclude particular interactions between cells. (A subset of cells is subject to *interactions* if it is not separable.) However, the interactions that are directly precluded this way are only few. The surprising point of Observation 6 is that, in this setting with multiple components, all interactions are precluded "indirectly" after all, also for the many subsets of cells besides unions of rows or unions of columns. This point was, essentially, Nataf's (1948) finding, although his proof has sometimes been criticized for being inaccessible. Nowadays, the result can readily be obtained as one of the many surprising implications of Gorman's (1968) strong result. Hence, we will not give a separate proof. Our results can easily be extended to more than two components: additive utility holds if and only if Statement (ii) holds with now monotonicity (weak separability) for every component.

With Observation 6 available, the two preceding paradoxes are no more surprising. The no-separability paradox immediately follows: with weak separability of states but no strong separability, not only strong separability of timepoints must be violated, but even their weak separability must be. Whereas the no-separability paradox only showed that unintended interactions *may* arise, Observation

6 shows that the case is yet more critical: such unintended interactions *always* arise. They are unavoidable. Regarding the separability paradox of Theorem 1, with Observation 6 available, the underlying mathematics becomes understandable. Theorem 1 concerned the special case of Observation 6 where $\succcurlyeq^i$ and $\succcurlyeq_j$ were independent of i and j, respectively, implying that the V_j^i in Observation 6 can be taken proportional, and Theorem 1 quickly follows (see appendix). As we explained, its novelty is not in mathematical generality but in accessibility and appeal.

The most shocking implication of Observation 6 is the following theorem, providing a negative answer to Question 5.

THEOREM 7 [Central Theorem; the Monotonicity Problem: Absence of Middle Ground]. A middle ground for behavioral approaches, with strong separability of components abandoned but weak separability ("monotonicity") maintained, is not available for two or more components.

This negative answer provides the common cause underlying many confusions and debates carried out independently in many subfields of behavioral decision theory. Whereas Observation 6 has essentially been known for almost a century, its vast implications for behavioral decision theory, starting with Theorem 7, have not been understood before. The remainder of this paper will discuss these implications further.

This section ends with some terminology. Whenever risk, the most-studied component in the literature and also in this paper, is involved we let it correspond with rows r^i, which then are states with known probabilities. We then refer to row monotonicity as *risk monotonicity*. Further, in general contexts, where rows need not designate streams, we use the terms *uniform row monotonicity* instead of stream monotonicity. *Uniform column monotonicity* similarly generalizes act monotonicity. Uniformity means that all rows (or columns) have the same "conditional" preferences $\succcurlyeq^1 = \cdots = \succcurlyeq^m$, or all columns have the same $\succcurlyeq_1 = \cdots = \succcurlyeq_n$. The difference between DEU and AU, or between uniform and general monotonicity/separability, or between Theorem 1 and Observation 6, never plays a role in any of the conceptual discussions in this paper.

6. A First Application: A Bifurcation Dilemma for Behavioral Quantitative Optimization

For quantitative optimizations with two (or more) components, *recursive* procedures, defined next, are commonly used because they are tractable. They can occur in two ways, i.e., using two orders of aggregation, in the next two definitions.[6]

DEFINITION 8. *Row-monotonic aggregation* holds if there exist *row-functions* R^{i} and a *column-function* $\overline{C}$, all continuous and strictly increasing in each coordinate, such that preferences are represented by

$$\overline{C}\left(R^1(x_1^1, \dots, x_n^1),\ \dots\dots, R^i\left(x_1^i, \dots, x_n^i\right),\ \dots\dots, R^m(x_1^m, \dots, x_n^m)\right) \tag{6}$$

Here one first, for every row r^i, aggregates over the columns $c_1, \dots, c_n$, and one next aggregates the m resulting values into the final value.

DEFINITION 99. *Column-monotonic aggregation* holds if there exist *column-functions* C_{j} and a *row-function* $\overline{R}$, all continuous and strictly increasing in each coordinate, such that preferences are represented by

[6] The popular terms row-first and column-first aggregation, used in our title, are ambiguous. For instance, row-first has as well referred to row-monotonic aggregation (first aggregate within a row) as to column-monotonic aggregation (first aggregate rows (within a column)). Hence, we will not use these terms in our formal analysis.

$$\overline{R}\left(C_1(x_1^1, \dots, x_1^m),\ \dots\dots, C_j\left(x_j^1, \dots, x_j^m\right),\ \dots\dots, C_n(x_n^1, \dots, x_n^m)\right) \tag{7}$$

Now one first, for every column c_j, aggregates over the rows $r^i, \dots, r^m$ and one next aggregates the n resulting values into the final value. Under uniform row monotonicity, we can take all R^i in Eq. 6 the same, i.e., independent of i, and under uniform column monotonicity, we can take all C_j in Eq. 7 the same, independent of j.

A choice between the two procedures is commonly made randomly or implicitly, without any discussion. In classical theories the choice is indeed immaterial. Further, the two procedures do not seem to be very restrictive anyhow – contrary to what we will show below – because they involve many functions that can be chosen independently and with almost no restrictions imposed on them. Similarly, in preference axiomatizations one of the two monotonicities is often imposed without further discussion, as-if self-evident.

The following preparatory observation shows that orders of aggregation, i.e., aggregation monotonicities, are quantitative versions of the corresponding preference monotonicities.

OBSERVATION 10. Given weak ordering, continuity, and outcome monotonicity, column-monotonic aggregation can be used if and only if column monotonicity holds. Row-monotonic aggregation can be used if and only if row monotonicity holds.

We briefly describe the proof. Row- (column-)monotonic aggregation can be derived from the corresponding preference condition by taking constant-equivalent functions for the functions R^i, C_j, R, and C. The rest is straightforward.

The monotonicity problem showed that, for two or more components, row and column monotonicity are more restrictive than has been known before. Therefore, Observation 10 is alarming.

It leads to the following impossibility result. It, again, has vast and paradoxical implications for quantitative behavioral theories, as next sections will show. Figure 4 displays its critical nature.

COROLLARY 11 [Bifurcation for Recursive Optimization]. If one wants to adopt a quantitative behavioral model with interactions (violations of strong separability), and for tractability reasons use a recursive model, then the only two routes available, row-monotonic or column-monotonic aggregation, are mutually exclusive, and one faces a bifurcation dilemma.

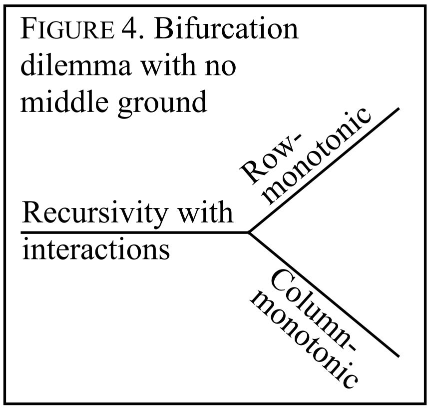

FIGURE 4. Bifurcation dilemma with no middle ground

Corollary 11 implies that in Example 4, to maintain weak (or even strong) separability of timepoints, one has to first aggregate risks for each timepoint and only then aggregate over timepoints. Weak separability of states then is just unachievable (unless one also has strong separability of states). If one starts from aggregating timepoints for each risky state, then separability of timepoints, even weak, can never be achieved anymore, no matter how one generalizes the functionals (unless one reverts to strong separability of states). These observations illustrate that the order of aggregation, immaterial in classical models, becomes critical in behavioral generalizations.

In behavioral approaches, the choice of route and, hence, the order of aggregation in Figure 4 is mostly made implicitly, without any argument given (Andreoni & Sprenger 2012; Machina 2014 Eq. 6 & footnote 11 & p. 3821 l. -3). As we have shown, the choice is critical though and explicit arguments for the route chosen are desirable. Several papers did provide discussions, including Dejarnette et al.

(2020 Section 4), Epper & Fehr-Duda (2024), Marinacci (2015 p. 1026), and Onay & Öncüler (2009). But the critical nature of the issue (Figure 4) has not been observed before. Not only may interactions arise, as has been observed before, but even they are unavoidable, as we show. Our Theorem 7 explains why in the numerous discussions in the literature no-one ever came up with an actual middle ground: it does not exist. The following claim illustrates another new practical implication of Theorem 7. Whereas the division of logical implications over assumptions in the claim remains informal, the claim shows the true face of the monotonicity problem and signals the alarming restrictiveness of recursive optimization procedures, inadvertently precluding many interactions. The "at least" clause below is because of interactions precluded by both monotonicities.

CLAIM 12 [Precluding Many Interactions]. Given weak ordering, continuity, outcome monotonicity, and $\mathrm{m} = \mathrm{n}$, row monotonicity precludes at least half of the possible interactions (violations of strong separability), and so does column monotonicity. Each condition precludes all interactions allowed by the other.

For $m \neq n$, one monotonicity is less restrictive than the other and precludes fewer interactions, but the situation is similarly alarming.

7. A Second Application: Risk and Welfare

This section presents implications of our results for welfare under risk, where the two relevant components concern risky states and persons. We show how several issues in this field are, once again, due to the monotonicity problem. We consider generalizations of Harsanyi's (1955) utilitarianism that amount to choices in the bifurcation dilemma. Harsanyi's utilitarianism adds row-uniformity to AU; i.e., it is a column-dependent generalization of DEU.

EXAMPLE 13 [Welfare and Risk]. Rows r^i refer to risk, i.e., states with known probabilities p^i, and columns c_j refer to persons. For simplicity, we assume $p^i = 1/m$ for all i. Harsanyi's (1955) utilitarian model is AU with $V_k^i = U_k/m$ for all i, k, where U_k is the utility function of person k. Preferences over a matrix are of a benevolent social planner with no stakes of her own. Harsanyi's Pareto principle is column monotonicity, and his expected utility assumption for the social planner implies uniform risk (= row) monotonicity.

Harsanyi's axiomatization was received as a paradox because people were not aware of the underlying cause, the monotonicity problem. A generalization of Harsanyi's axiomatization can be obtained from Observation 6 by adding symmetry of rows 16 (implying uniformity of row preference and EU with equal probabilities). The extension to general, possibly continuous, probabilities follows from Theorem 14 in Supplementary Appendix B. The axiomatization obtained this way is more general than Harsanyi's (1955) axiomatization in considerably weakening his assumption of expected utility for risk. In return, Harsanyi did not need continuity in outcomes, and could handle subdomains of the matrix space.[7] This alternative axiomatization is a mathematical implication of our preceding results, but we do not elaborate on it because we focus on conceptual implications.

Pivato & Tchouantez (2024) provided the most general mathematical results along the above lines known to us. They allowed for nonstandard real numbers and they weakened continuity to solvability. These generalizations are empirically and conceptually preferable but have the drawback of using concepts that are not well-known. The authors further weakened completeness of preference and

[7] Harsanyi did not explicity introduce persons as different attributes, but his domain can be remodeled accordingly, turning it into a subdomain of Anscombe & Aumann's (1963) framework. This way, Anscombe & Aumann's theorem is a corollary of Harsanyi's. For details, see De Meyer & Mongin (1995). Undoubtedly, Harsanyi (1955) devised his result independently without relating it to the preceding Nataf (1948). Mongin & Pivato (2015) also pointed out the relations between these theorems.

allowed for state- and person-dependent utility. They also provided impossibility-result interpretations and they surveyed further literature.

FIGURE 5

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

FIG. 5a. Row monotonicity

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \succ \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \prec \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

FIG. 5b. Row monotonicity violated

Harsanyi's model has often been criticized for ignoring inequality aversion, illustrated in Figures 5 and 6. We assume uniformity, i.e., symmetry of c_1 and c_2 ("anonymity") and also of r^1 and r^2, which have probability 0.5.

Diamond (1967) proposed Figure 5 as a criticism of Harsanyi's utilitarianism. In all matrices, both rows (states) give the good outcome to one of the two persons. Hence, by symmetry (anonymity) all rows are equivalent and by row monotonicity, all matrices are indifferent, and so are they under Harsanyi's utilitarianism (Fig. 5a). Diamond pointed out that, to the contrary, the strict preferences in Fig. 5b are plausible under inequality aversion, nowadays usually interpreted as ex ante inequality. In the dispreferred matrices, one person certainly receives the good outcome and the other person certainly not, so that there is inequality from the ex ante perspective. In the preferred matrices there is equality from the ex ante perspective in the sense that both persons receive the same lottery, $1_{0.5}0$. Diamond emphasized that the sure-thing principle (i.e., strong separability of the states/rows) is violated in Fig. 5b. The preference over the first row is affected by the second here. Expected utility and Harsanyi's utilitarianism are violated. Our monotonicity problem shows that this problem of ex ante fairness is more fundamental than Diamond pointed out: even weak separability of states is violated, which for three or more states is considerably more fundamental than strong separability. Column monotonicity may still hold.

FIGURE 6

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

FIG. 6a. Column monotonicity

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \prec \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \succ \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} r^1 \\ r^2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

FIG. 6b. Column monotonicity violated

Broome (1991 p. 185) proposed Figure 6 as a criticism of Harsanyi's utilitarianism. In all matrices, both persons always receive $1_{0.5}0$. Hence, under column monotonicity ("Pareto principle"), all matrices are indifferent, and so are they under Harsanyi's utilitarianism (Fig. 6a). Broome argued that, to the contrary, the strict preferences in Fig. 6b are plausible under inequality aversion. From the ex post perspective, the dispreferred matrices certainly, under both r^1 and r^2, result in inequality, and the preferred matrices certainly (for every row) result in equality. The preference over the first column is affected by the second here. Column monotonicity, i.e, the Pareto principle, is violated.

Our bifurcation dilemma provides new insights. First, the dilemma of Broome versus Diamond, i.e., of ex post versus ex ante fairness, is more critical than has been observed before. The choice is between the only two tractable approaches available, and they are disjoint (beyond classical utilitarianism). Diamond's choice, i.e. ex ante fairness, is the route downward in Figure 4, and Broome's choice, i.e. ex post fairness, is the route upward. Further, the dilemma is more acute than known before. Diamond's approach not only violates the sure-thing principle, but even risk-monotonicity.

A natural question now arises, as it will do in every behavioral application with two (or more) components: which of the routes in the bifurcation dilemma in Figure 4 is more natural, upward or downward? The next section will discuss the question for general components, using welfare under risk as running example. As follows from Corollary 11, Broome and Diamond provided yet another discussion of the bifurcation dilemma. The next section will, further, argue that, in general, Broome's criticism is more serious than Diamond's, and will provide similar new insights for many other domains.

8. Guidelines and Techniques to Avoid Undesirable Violations of Monotonicity

In behavioral approaches with multiple components we face the bifurcation dilemma, and have to decide on the plausibility of monotonicity/separability of various components. One wants to avoid undesirable violations of monotonicity. This section provides guidelines for choosing in the bicurcation dilemma. For consistency with much literature, we continue to often use the term monotonicity, but sometimes we prefer the more neutral term separability.

We first provide a default ranking of plausibility, assuming other things equal. In general, separability is most plausible for uncertainty and risk because there can be no physical interactions between mutually exclusive events (Broome 1991 end of Section 1.3 and Section 8.3; Dejarnette et al. 2020 p. 632). Whereas Samuelson (1950) first criticized separability for uncertainty, the exclusiveness argument later won him over to accept separability as normative for uncertainty (Samuelson 1952).[8] It

[8] Samuelson (1950 p. 120) wrote, famously: "Let the axioms satisfy themselves." However, Samuelson (1952 p. 672) wrote: "Prior to 1950, I hesitated to go much further. But much brooding over the magic words "mutually-exclusive" convinced me that there was much to be said for a further "strong independence axiom" [strong separability]."

gives a firm normative basis to decision analysis (Wu 1996). Within uncertainty, separability is more convincing for risk than for ambiguous events (Wakker 2010 Section 10.4). Next, interactions are less likely to occur between different persons at different locations than within one person at different timepoints (Baucells & Sarin 2010). Thus, the following

default ranking of separability (or "monotonicity"):

[risk > ambiguity > welfare > time] (8)

results. For time, payment in consumption is more separable than payment in money (Cohen et al. 2020). For commodities or attributes, separability is less plausible than for uncertainty, but can take any remaining place in the ordering depending on the nature of the attributes. The above ordering, while unavoidably informal, is plausible and will have many implications for many fields, as will be shown. Thus, in our running example of welfare under risk, ex post fairness as propagated by Broome (maintaining risk monotonicity) is, other things equal, more plausible than the ex ante fairness propagated by Diamond. This is a further new insight into the Broome-Diamond dilemma. Similarly, with time and risk as components, risk-monotonic aggregation is most plausible (Abdellaoui et al. 2019), allowing for nonneutrality towards intertemporal correlations. In applications there can be many other arguments though, that can overrule Eq. 8 and lead to deviations.

Researchers often add a component, not for its own interest but as an auxiliary tool to facilitate the analysis of other components. It then is desirable that separability holds with respect to that extra component. The ranking in Eq. 8 explains, therefore, why risk is most popular as auxiliary component. It is the main tool in decision analysis (Keeney & Raiffa 1976). Harsanyi (1955) used risk as auxiliary component for welfare (our Example 13). Other examples include Anscombe & Aumann (1963) for ambiguity. Theorem 1 and Observation 6 explain why strong results could be obtained this way.

Empirically, it is also plausible that decision makers mostly adopt one of the two routes in the bifurcation dilemma. Again, for tractability reasons, but now from the psychological perspective of a cognitively limited decision maker rather than from the modeling perspective of a researcher.

Nevertheless, some interactions and spillover effects due to the presence of other attributes and stimuli can still be expected. Hence, empirically, people will be close to one of the two routes in Figure 4, but with small deviations.

We next discuss further aspects of stimuli and their framings that can impact the plausibility of separability, and the resulting route in the bifurcation dilemma of Figure 4, possibly leading to deviations of the default ranking of Eq. 8. We explain some steering techniques based on such aspects. These techniques can be used to avoid undesirable violations. For example, spillover effects in preference measurements, hedging effects in ambiguity measurements, and particular forms of inequality aversion, can be confounding and undesirable in experiments and in applications. We will introduce three *steering techniques* for the running example of welfare under risk here. They can be used for general multicomponent optimizations. The next section gives further examples, applications, and references.

The first steering technique is the *framing technique*. In general, a two-stage display of matrices will enhance one of the two kinds of monotonicity. Thus, Fig. 7a enhances row monotonicity and Fig. 7b enhances column monotonicity. Framing can also be done verbally. In Figure 1, the framing "For each i, at state s^i you receive stream $(x_1^i, \dots, x_n^i)$" enhances row-monotonicity, similarly to Fig. 7a. The framing "For each j, at timepoint t_j you receive act$(x_j^1, \dots, x_j^n)$" enhances column-monotonicity, similarly to Fig. 7b.

FIGURE 7. Two-stage displays of matrices

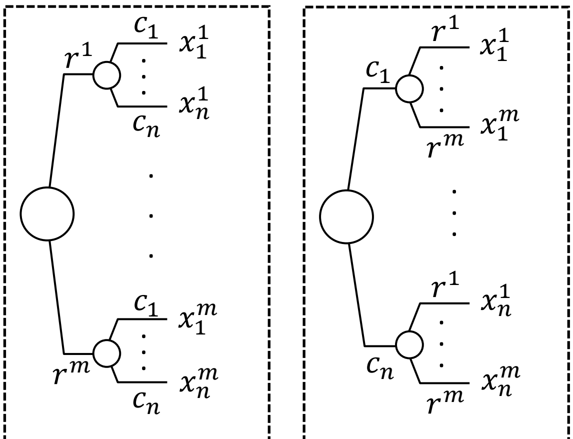

FIG. 7a. Visual display enhancing row-monotonic aggregation

FIG. 7b. Visual display enhancing column-monotonic aggregation

Regarding our running example of welfare under risk, in Figure 5, if a social planner wants the dispreferred matrices in Fig. 5b to be accepted by the public for some good extraneous reason, then the framing of Fig. 7a (with c_js designating persons) is best suited to enhance the row monotonicity of Fig. 5a. In Figure 6, if a social planner wants the dispreferred matrices in Fig. 6b to be accepted, the framing of Fig. 7b (with c_js designating persons) is best suited to enhance the column monotonicity of Fig. 6a.

The second steering technique, the *timing technique*, can be used if risk or uncertainty is involved, and concerns the perceived timing of the resolution of uncertainty—early, before decision time, or late, after decision time. Early resolution of uncertainty enhances a perception as in Fig. 7a (with the r^js uncertain events) and row-monotonic aggregation. In Figure 5, it leads to Fig. 5a. It enhances ex post fairness, focusing on affairs after resolution of uncertainty. Late resolution of uncertainty enhances a perception as in Fig. 7b and column-monotonic aggregation. In Figure 6, it leads to Fig. 6a. It enhances ex ante fairness, providing an argument counter to the default ranking of Eq. 8. Thus, the perception of fairness can be steered by choosing prior or late resolution. We stress that this paper only considers situations where, if resolution takes place before the decision time, then the decision maker knows that the uncertainty has been resolved, but does not know how it has been resolved. It is, therefore, of no strategic relevance here and only concerns perception. The timing technique has been widely discussed and tested in the welfare literature and other fields (Section 9.3). In particular, Onay & Öncüler (2009) used the two different framings in Figure 7, with risk and time as components, to generate early and late resolution. They thus combined the framing and timing techniques. Our analysis gives a theoretical justification of such experimental procedures.

The third technique, the *partial-info technique*, provides only partial information to the decision maker or to persons involved in welfare. For example, in Figures 5 and 6, the two persons c_1, c_2 may not be informed about the outcomes that the other person receives. This enhances separability of the columns and, hence, column-monotonic aggregation. There then is less room for inequality aversion because the persons themselves cannot perceive it.

One can also avoid the monotonicity problem by only considering particular subdomains of matrices. Our analysis as yet made the idealized assumption, common in decision theory and preference axiomatizations, that we deal with a full domain containing all matrices in $\mathbb{R}^{m\times n}$. This assumption is essential for our theorems. Some studies on risk and time only considered actstreams with one nonzero outcome (Baucells & Heukamp 2012), or matrices where nonzero outcomes appear only at one timepoint, in which case the order of aggregation is immaterial under many behavioral models. Halevy (2008) considered a restricted (comonotonic) domain where both orders of aggregation can hold simultaneously for behavioral theories. Alon & Gayer (2016) imposed the Pareto principle only if agreement on probabilities and utilities. For principled discussions of decision principles this escape route, of restricting the domain, is not very convincing. If conditions deemed fully appropriate cannot survive extension to all possibilities, then this remains a point of concern. Subdomains arise in several applications in the next section and will be further discussed there.

9. Further Applications

This section presents several further applications. We elaborate on details in two more applications in our area of expertise, ambiguity, in Subsections 9.1 and 9.2, and briefly mention many others in Subsection 9.3. In all examples in this section, rows r^i model risky events.

9.1. Third Application: Monotonicity in the Anscombe-Aumann Framework for Ambiguity

The monotonicity problem also occurs in the well-known Anscombe-Aumann (AA) framework for ambiguity, where the traditional approach involves an implicit choice in the bifurcation dilemma of Figure 4. We again apply our techniques to shed new light on the relevant issues. A criticism of the traditional approach will result.

We use Fishburn's (1970 Section 13.1) two-stage AA framework that has become the standard today. In Figure 2, *roulette events* (rows) $r^1, \ldots, r^m$ partition the universal event and have known probabilities. *Horse events* (columns) $c_1, \ldots, c_n$ also partition the universal event but are ambiguous. The AA framework adopts uniform column monotonicity, called *horse monotonicity* here, using the same EU functional (for C_j in Eq. 7) for each column. It implies that only the marginal distributions given every horse c_j matter. We thus have correlation neutrality between columns. This implication is characteristic of the modern AA framework. A notation: by $\alpha_p\beta$ we denote a lottery, i.e., probability distribution, yielding α with probability p and β with probability $1-p$.

We first assume a full domain where all matrices are available, as for instance in Machina (2014) who assumed simultaneity of the horse and roulette events. Figure 8 displays ambiguity aversion as commonly assumed in the literature. The rows have 0.5 probability each. The indifference in the figure follows from the AA assumptions: each horse yields lottery $(1_{0.5}0)$, and in this sense there is no ambiguity. The strict preferences reflect ambiguity aversion as commonly taken in the AA approach. For instance, the left matrix, a gamble with known probability, is preferred to the right upper matrix, a gamble with unknown probability. The two strict preferences reveal a violation of risk (row) monotonicity: preferences over the first row are affected by the second row, and rows interact.

Having committed to horse monotonicity, the common AA framework necessarily has to give up risk monotonicity (and conditioning on risky events) to capture the behavioral nonneutrality towards ambiguity. This follows from the monotonicity problem. However, by the default ranking of Eq. 8, other things equal, risk (row) monotonicity is more plausible than column monotonicity. The common AA framework thus chose the less plausible route in the bifurcation dilemma of Figure 4. Jaffray (1992, personal communication) emphasized the implausibility of horse monotonicity and recommended risk monotonicity for ambiguity, adopting it in all his works (e.g., Jaffray 1989). Eichberger & Pasichnichenko (2021) and Monet & Vergopoulos (2024) followed Jaffray's approach in this regard. Again, our analysis gives a theoretical justification for these approaches.

The early Keeney & Raiffa (1976) provided a rich toolbox for risk monotonicity with interactions between columns. By the timing technique, for risk monotonicity it works best to let the resolution of the roulette events precede those of the horse events, rather than come after as commonly assumed in the AA framework. Oechssler & Roomets (2021) used Fig. 7b in their experiment, which enhances horse monotonicity. but nevertheless found much risk monotonicity, providing strong empirical evidence *against* horse monotonicity.

FIGURE 8. Violation of risk monotonicity in the Anscombe-Aumann framework

$$\begin{array}{cc} & \begin{array}{cc} c^1 & c^2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{array}{cc} c^1 & c^2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \succ \begin{array}{cc} & \begin{array}{cc} c^1 & c^2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} c^1 & c^2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array} \prec \begin{array}{cc} & \begin{array}{cc} c^1 & c^2 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

The partial info technique is also used in most current implementations of the AA framework. If some horse s_j wins the race, subjects will only be informed about the resolution of the corresponding lottery, and not what the lotteries for the other horses would have given. This works best if the domain of matrices considered is also restricted by assuming the lotteries for different horses to be stochastically independent.[9] Then the matrices in Figure 8 can no more be used and formally we have escaped from the monotonicity problem there, avoiding violations of risk monotonicity. Theorem 7 still shows that correlations between horses cannot be added without violating risk monotonicity (or sacrificing one of the other conditions), which remains a worrisome issue especially for normative

[9] Equivalently, they can be taken as mutually unspecified, e.g., by taking them as conditional on a horse ("statewise randomization"; Ke & Zhang 2020). Compare Figures 9 and 10 below. The essence is that they are mutually uninformative. Thus, subjects may only be informed about the outcome realized for the winning horse and the roulette resolution there. These points do not impact the conceptual issues discussed here.

purposes. In particular, we cannot add risk prior to, or simultaneously with the horse race and have EU there, because this would, by horse monotonicity and Observation 6, automatically preclude any nonneutral ambiguity attitude. A new insight resulting from our analysis is that already the basic risk monotonicity, rather than EU, is critical here.

Although the modern version of the AA framework can formally escape from the monotonicity problem and the "counterexample" of Figure 8 by the partial info technique, the underlying problem, that weak separability of horse events does not fit well with their ambiguity, remains. We illustrate this problem through another implication in Figure 9, a variation of Figure 8 that uses only stimuli within the restricted domain assumed by restrictive versions of the AA framework, and with partial info fully effective. To clarify the latter claim, Figure 10 displays the same choices as in Figure 9 but now using the matrix notation of this paper, where for each matrix the two columns are stochastically independent. In Figure 9, we take outcome α such that $\alpha \sim 8_{0.5}0$, i.e., it gives the indifference in Fig. 9a. Under expected value maximization, $\alpha = 4$, but in general it depends on the risk attitude. All columns in Figure 9 are indifferent. By AA's horse monotonicity, all matrices should be indifferent. However, under ambiguity aversion the strict preferences in the figure are plausible: for the dispreferred matrices all outcomes are ambiguous whereas for the other matrices none is.

FIGURE 9. Violation of horse monotonicity due to ambiguity aversion

$$\begin{array}{cc} c_1 & c_2 \\ \end{array} \qquad \begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\begin{bmatrix} 8_{0.5}0 & 8_{0.5}0 \end{bmatrix} \sim \begin{bmatrix} \alpha & \alpha \end{bmatrix}$$

FIG. 9a

$$\begin{array}{cc} c_1 & c_2 \\ \end{array} \qquad \begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\begin{bmatrix} 8_{0.5}0 & \alpha \end{bmatrix} \prec \begin{bmatrix} \alpha & \alpha \end{bmatrix}$$

FIG. 9b

$$\begin{array}{cc} c_1 & c_2 \\ \end{array} \qquad \begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\begin{bmatrix} 8_{0.5}0 & 8_{0.5}0 \end{bmatrix} \succ \begin{bmatrix} \alpha & 8_{0.5}0 \end{bmatrix}$$

FIG. 9c

FIGURE 10. Violation of horse monotonicity due to ambiguity aversion using matrices. $P(r^j) = 0.25$ for all j.

$$\begin{array}{c} \\ r^1 \\ r^2 \\ r^3 \\ r^4 \end{array}\!\!\begin{array}{c} c_1 \quad c_2 \\ \begin{bmatrix} 8 & 8 \\ 8 & 0 \\ 0 & 8 \\ 0 & 0 \end{bmatrix} \end{array} \sim \begin{array}{c} \\ r^1 \\ r^2 \\ r^3 \\ r^4 \end{array}\!\!\begin{array}{c} c_1 \quad c_2 \\ \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \\ \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \end{array}$$

FIG. 10a

$$\begin{array}{c} \\ r^1 \\ r^2 \\ r^3 \\ r^4 \end{array}\!\!\begin{array}{c} c_1 \quad c_2 \\ \begin{bmatrix} 8 & \alpha \\ 8 & \alpha \\ 0 & \alpha \\ 0 & \alpha \end{bmatrix} \end{array} \prec \begin{array}{c} \\ r^1 \\ r^2 \\ r^3 \\ r^4 \end{array}\!\!\begin{array}{c} c_1 \quad c_2 \\ \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \\ \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \end{array}$$

FIG. 10b

$$\begin{array}{c} \\ r^1 \\ r^2 \\ r^3 \\ r^4 \end{array}\!\!\begin{array}{c} c_1 \quad c_2 \\ \begin{bmatrix} 8 & 8 \\ 8 & 0 \\ 0 & 8 \\ 0 & 0 \end{bmatrix} \end{array} \succ \begin{array}{c} \\ r^1 \\ r^2 \\ r^3 \\ r^4 \end{array}\!\!\begin{array}{c} c_1 \quad c_2 \\ \begin{bmatrix} \alpha & 8 \\ \alpha & 0 \\ \alpha & 8 \\ \alpha & 0 \end{bmatrix} \end{array}$$

FIG. 10c

Many authors discussed horse monotonicity in the AA framework, both theoretically and empirically. Besides those cited before, they include Machina (2014 p. 3835 3rd bulleted point), Oechssler & Roomets (2021), and Schneider & Schonger (2018). More general discussions of the role of the timing of uncertainty include Berger & Eeckhoudt (2021), Kochov (2015), and Oechssler, Rau, & Roomets (2019). We have presented the debate on monotonicity for AA in its most basic form, showed that the monotonicity problem is underlying, and showed that the issue is more critical than has been known before, as it, again, amounts to a choice in the bifurcation dilemma.

9.2. Fourth Application: Validity of the Random Incentive System and Hedging for Ambiguity

Our next application concerns incentive compatibility of the random incentive system (*RIS*) in experiments (Jacquement & l'Haridon 2018 Section 5.2.3). Let each row of our matrices specify one of m choice situations in an experiment. We assume that in each choice situation a subject chooses an option that is an n-dimensional object. It may be a commodity bundle, an outcome stream, a welfare allocation over n persons, an act assigning outcomes to n states of nature, and so on. In the RIS, one choice situation (row) will be randomly selected for real implementation. Say each has a probability of $1/m$ of being the one implemented for real. At the end of the experiment, the subject receives the option chosen in the implemented choice situation. The other choice situations will not be implemented. During the experiment, the subject does not know which of the m choice situations will be implemented for real. Matrices are strategies, specifying a choice for each of the m choice situations in the experiment.

It trivially follows that incentive compatibility of the RIS is equivalent to risk (row) monotonicity where the underlying preferences are the true preferences in the choice situations. In this context, row monotonicity has been known as isolation (Cohen, Jaffray, & Said 1987, appendix). Trivially, row-monotonic aggregation together with EU for risk then is also sufficient for incentive compatibility, where the assumption of EU is just redundant. This redundancy was pointed out by Bardsley et al.

(2010 p. 269) and several others. Nevertheless, there have been misunderstandings about this point in the literature, where several papers claimed that full-force EU was needed for incentive compatibility. Adding extra assumptions of column monotonicity, uniformity of monotonicities, and a full domain would, by Theorem 1, indeed give EU for risk. But, again, these extra assumptions are not needed for incentive compatibility of RIS. Unawareness of the monotonicity problem, i.e., the restrictiveness of those added extra assumptions explains the misunderstandings in the literature.

As explained before, our theorems assumed full domains and complete preferences, whereas in many applications only subdomains are relevant or available. Nevertheless, these subdomains are often rich enough for our results to provide new insights. This point is further illustrated in the following application. We there show that the monotonicity problem is the underlying cause for the hedging problem in ambiguity measurements, and then how our techniques of Section 8 can be used to avoid this hedging problem.

We assume two Ellsberg urns: a known urn K containing 50 White and 50 Black balls, and an unknown ambiguous urn A containing 100 balls, each White or Black, but in unknown proportions. From both urns a ball will be drawn at random. W_K denotes the event that the ball drawn from urn K is white, and B_K, W_A, and B_A are similar. These events are uncertain to subjects during the experiment. $(B_A\colon 101)$ denotes a gamble yielding €101 if the ball drawn from urn A is black and nothing otherwise. Other gambles are denoted similarly. Imagine that an experiment concerns $m = 2$ choice situations for a given subject. The first, r^1, reveals the preference $(B_A\colon 101) \succcurlyeq (B_K\colon 100)$; the second, r^2, reveals the preference $(W_A\colon 101) \succcurlyeq (W_K\colon 100)$. The RIS randomly selects r^1 or r^2 for real implementation, say each with probability 0.5. Which is implemented for real is unknown to subjects during the experiment. Can we conclude that there is virtually no ambiguity aversion? Hedging, explained below, has often been advanced as a confound invalidating this conclusion.

Three components can be distinguished: the color from K, the color from A, and the selection from $\{r^1, r^2\}$. It is convenient here to combine the first two components into one. We thus define four

c_j as in Figure 11. The figure illustrates the two observed preferences. In the usual RIS, we do not directly consider choices between strategies for the whole experiment (matrices). However, the two aforementioned experimental preferences do, indirectly, reveal the preference between the two matrices in Figure 11.

If we assume column monotonicity, then the preferences in Figure 11 follow from nothing other than stochastic dominance: all columns of the preferred matrix stochastically dominate those of the dispreferred matrix ($101_{0.5}0 > 100_{0.5}0$). In the left matrix, the outcomes under r_2 provide a kind of hedge against those under r_1, and so they do in the right matrix, which explains the term hedging. The preferences then do not speak to ambiguity attitudes in any sense. It has often been observed that, under column monotonicity and ambiguity nonneutrality, validity of the RIS *may* be violated in this way. The monotonicity problem provides the new insight that the case is yet more critical: there then *necessarily* exist choice situations where validity of the RIS is violated.

We here have another case of the bifurcation dilemma. Hedging occurs if the lower route with column monotonicity is chosen in Figure 4[10], whereas validity of RIS holds if the upper route is chosen. Of course, besides hedging, any other reason to violate risk monotonicity discussed before can invalidate the ambiguity measurement here.

FIGURE 11. Hedging in ambiguity

$$
\begin{array}{cc}
 & \begin{array}{cccc} B_K\cap B_A & B_K\cap W_A & W_K\cap B_A & W_K\cap W_A \\ = & = & = & = \\ c_1 & c_2 & c_3 & c_4 \end{array} \\
\begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 101 & 0 & 101 & 0 \\ 0 & 101 & 0 & 101 \end{bmatrix}
\end{array}
\succcurlyeq
\begin{array}{cc}
 & \begin{array}{cccc} B_K\cap B_A & B_K\cap W_A & W_K\cap B_A & W_K\cap W_A \\ = & = & = & = \\ c_1 & c_2 & c_3 & c_4 \end{array} \\
\begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 100 & 100 & 0 & 0 \\ 0 & 0 & 100 & 100 \end{bmatrix}
\end{array}
$$

Discussions of hedging under ambiguity include Bade (2015) and Oechssler, Rau, & Roomets (2019). Baillon, Halevy, & Li (2022) provided the first empirical demonstration that hedging can

---

[10] The hedging in the above example involved event complementarity, an extreme case of hedging (Hartmann 2023).

really occur, and reviewed further literature. As our guidelines of Section 8 show, their presentation of stimuli in fact maximized violations of risk (row) monotonicity, so as to best demonstrate the potential severity of the basic problem. The techniques of Section 8 provide tools for establishing validity of the RIS and, in particular, for avoiding hedging in ambiguity measurements. Johnson et al.'s (2021) Prince, an implementation of RIS to maximize validity, in fact used such techniques informally. Regarding the timing technique, Prince selects the real choice situation prior to the experiment rather than after as usually done, enhancing the desired row monotonicity. Johnson et al. also used framing (e.g., as in Fig. 7b) and partial-info techniques as best as possible. Again, our results provide theoretical justifications.

Regarding the partial-info technique for general RIS, when facing a choice situation in an experiment, subjects are usually not yet informed about the choice situations that come after, precluding all "backward" interactions. To reduce "forward" interactions, each choice situation may be presented on a different page or screen, so that subjects can only know about preceding choice situations from memory. In general, full understanding of strategies in an experiment is humanly impossible. Validity of the RIS can therefore be expected to be good in general (Bardsley et al. 2010 Section 7.5: "behavioral incentive compatibility"). Our analysis shows that the hedging confound for ambiguity measurements will usually not be a serious problem in practice. Aydogan, Berger, & Theroude (2024) and König-Kersting, Kops, & Trautmann (2023) indeed found no hedging.

Some interactions between different choice situations in RIS can nevertheless occur, e.g. due to spillover effects, contrast effects, learning, and so on. Johnson et al.'s (2021) Prince seeks to minimize those.

9.3. Further Applications

There are numerous cases where aggregation over two or more components is central, besides the cases considered before. The monotonicity problem, the bifurcation dilemma, and the guidelines and techniques of Section 8 can be applied to all those cases, underscoring the unity and common underlying cause, and providing new insights similarly as above. We briefly mention some of such cases. A complete review is impossible given the vastness of the relevant literature.

Violations of column monotonicity as in Fig. 6b can be due to correlation aversion. This has been extensively studied in many domains, including multiattribute utility theory (multivariate risk aversion: Attema, l'Haridon, & van de Kuilen 2019; Richard 1975; Tsetlin & Winkler 2009), intertemporal choice (Epper & Fehr-Duda 2024; Lanier et al. 2024; Rohde & Yu 2024), and consumer theory with t_1 and t_2 commodities with diminishing marginal rates of substitution. For temporal ambiguity (unknown probabilities for r^1 and r^2 in Fig. 6b), Kochov (2015) emphasized the plausibility of row monotonicity and correlation preference.

Epstein & Halevy (2019) considered an interesting case: Both rows and columns refer to events with known probabilities, but their correlation is ambiguous. Then ambiguity aversion gives both Figures 5b and 6b with reversed preferences. The monotonicity problem implies that neither Eq. 6 nor 7 can be used to accommodate this phenomenon, so that finding a tractable model (e.g., a tractable subcase of their Section 5) will not be easy.

Andreoni & Sprenger (2012) considered actstreams for risk, with probabilities of the states given. They did not explicitly state or discuss the order of aggregation (Eq. 6 or 7) assumed in their behavioral analyses, i.e., the route in the bifurcation dilemma chosen, which complicates their interpretations. Cheung (2015), Epper & Fehr-Duda (2015), and Miao & Zhong (2015), accordingly, criticized them. Andreoni & Sprenger's Section III.A reports violations of our uniform risk (stream/row) monotonicity, which they interpret as violations of prospect theory. Our monotonicity problem provides the new insight that such a violation is more fundamental: it entails a violation of virtually any existing risk theory if one does not want to abandon column monotonicity.[11] Similar

[11] An exception is the utility of gambling theory (Diecidue, Schmidt, & Wakker 2004; Fishburn 1980). This theory in itself comprises a violation of uniform risk monotonicity and, thus, can accommodate these violations. However, this theory is not very tractable or suited for applications.

violations of uniform risk monotonicity were found before, by Abdellaoui, Diecidue, & Öncüler (2011), Bleichrodt & Pinto (2009), Cettolin & Riedl (2017), and others. Findings that the present bias weakens if risk comes in (as in Epper & Fehr-Duda 2024 and Keren & Roelofsma 1995), entail such a violation. Similarly, papers have shown that delaying risks moderates the certainty effect, implying a violation of Axiom 4 in Subsection 2.2 (act monotonicity) and, indirectly, of column monotonicity (Abdellaoui, Diecidue, & Öncüler 2011; Baucells & Heukamp 2010; Epper & Fehr-Duda 2024; Kemel & Paraschiv 2023).

In judgment aggregation (Grossi & Pigozzi 2014), matrices concern cases to be decided on, rows concern propositions, and columns concern judges. The majority rule can then give different results under row-first aggregation than under column-first. This has been known as the judgment aggregation paradox, and it is similar to our Corollary 11. Proposition-wise independence then is our row independence and matter-wise independence is our column independence. In social discounting, the aggregation concerns persons and time. The issues discussed in this paper are new (to our best knowledge) in price index theory, with commodities and locations as two component, for instance, and prices as matrix entries (Renneboog & Spaenjers 2013), and, undoubtedly, many other fields.

Besides the references mentioned before, numerous papers examined the timing technique, theoretically and empirically. Again, we bring a unification of these analyses. We mention some papers. For time and risk, see Dejarnette et al. (2020 Section 4) and Onay & Öncüler (2009). For welfare and risk (where timing of resolution of uncertainty is only one way to impact an ex post or ex ante viewpoint), see Cappelen et al. (2013), Miao & Zhong (2018), and Rohde & Rohde (2015). For two-fold uncertainty as in the AA framework, see Baillon, Halevy, & Li (2022), Ke & Zhang (2020), Kochov (2015), and Oechssler & Roomets (2021). Also see Machina (2014 footnote 11). Berger & Emmerling (2020) examined the overall effect of inequality aversion in separate components under different orders of aggregation. They provided a unifying framework of their results for the case of several kinds of components.

10. Conclusion

We presented a unified framework for optimization over two (or more) components. A paradoxical appearance of separability, in the simplest axiomatization of discounted expected utility as yet (Theorem 1), and a paradoxical disappearance of separability (Example 4), could be resolved by Theorem 7. In a mathematical sense, this theorem is a small variation of Nataf's (1948) century-old theorem on micro-macro aggregation. However, its vast implications for modern behavioral approaches have not been observed before. First, the theorem signals the monotonicity problem. Although the possibility of such a problem occurring has been observed before, its universality and unavoidability have not. It leads to a general bifurcation dilemma.

We next considered many debates in many behavioral fields, scattered over numerous papers, that are all special cases of the monotonicity problem, with stances taken in the literature amounting to a choice of one of the two routes available in the bifurcation dilemma. Our general guidelines and techniques provide new insights and solutions to these debates. Whereas studies as yet brought new insights for only one case of multiple components, our paper has bridged many models and phenomena in the behavioral field and has provided new insights for many fields.

Appendix. Proofs

As explained in the introduction and in Supplementary Appendix A, Observation 6 follows from Mongin & Pivato (2015 Proposition 1). We next prove Theorem 1. Statement (i) readily implies Statement (ii). We assume Statement (ii), and derive Statement (i). By Observation 6, we obtain an AU representation. We derive proportionality of the V_j^i in the AU representation. We can let all V_j^i take value 0 at 0. The AU representation is a state- and time-dependent version of DEU. Gorman's uniqueness result is at this state- and time-dependent stage: the functions V_j^i, all "grounded" at 0, can jointly be replaced by $\lambda \times V_j^i$ for any $\lambda > 0$, independent of i and j, and by no other functions.

By act monotonicity, the n arrays $(V_j^1, \dots, V_j^m)$ through their sum all represent the same preference relation over acts ("column"). Hence, by standard uniqueness, these n arrays of functions, grounded at 0, are proportional to each other. That is, each $(V_j^1, \dots, V_j^m)$ is d_j times $(V_1^1, \dots, V_1^m)$ for positive $d_2, \dots, d_n$, where we set $d_1 = 1$. Similarly, by stream monotonicity, the m arrays $(V_1^i, \dots, V_n^i)$ though their sum all represent the same preference relation over streams ("rows"), and each is q^i times $(V_1^1, \dots, V_n^1)$ for positive $q^2, \dots, q^m$ with $q^1 = 1$. We can normalize the q^is to sum to 1, and denote them p^i. All V_j^is are proportional to each other and to one function that can be denoted U. For U we can take V_1^1 or any other V_j^i.

For completeness, we give the uniqueness results of Theorem 1. By Gorman's aforementioned uniqueness result at the state- and time-dependent stage, now at this state- and time-independent stage we have: U is unique up to a positive factor (scale), and the d_js are unique up to one other common positive factor. Because of normalization, the p^js are unique. We can relax the requirement $U(0) = 0$ and add any constant, after which U is also unique up to location.

References

Abdellaoui, Mohammed, Arthur Attema,& Han Bleichrodt (2010) "Intertemporal Tradeoffs for Gains and Losses: An Experimental Measurement of Discounted Utility," *Economic Journal* 120, 845–866.

Abdellaoui, Mohammed, Han Bleichrodt, Olivier L'Haridon, & Corina Paraschiv (2013) "Is there One Unifying Concept of Utility? An Experimental Comparison of Utility under Risk and Utility over Time," *Management Science* 59, 2153–2169.

Abdellaoui, Mohammed, Enrico Diecidue, & Ayse Öncüler (2011) "Risk Preferences at Different Time Periods: An Experimental Investigation," *Management Science* 57, 975–987.

Abdellaoui, Mohammed, Emmanuel Kemel, Amma Panin, & Ferdinand M. Vieider (2019) “Measuring Time and Risk Preferences in an Integrated Framework,” *Games and Economic Behavior* 115, 459–469.

Alon, Shiri & Gabrielle Gayer (2016) “Utilitarian Preferences with Multiple Priors,” *Econometrica* 84, 1181–1201.

Andreoni, James & Charles Sprenger (2012) “Risk Preferences Are not Time Preferences,” *American Economic Review* 102, 3357–3376.

Anscombe, Frank J. & Robert J. Aumann (1963) “A Definition of Subjective Probability,” *Annals of Mathematical Statistics* 34, 199–205.

Attema, Arthur E. (2012) “Developments in Time Preference and Their Implications for Medical Decision Making,” *Journal of the Operational Research Society* 63, 1388–1399.

Attema, Arthur E., Olivier L’Haridon, & Gijs van de Kuilen (2019) “Measuring Multivariate Risk Preferences in the Health Domain, *Journal of Health Economics* 64, 15–24.

Aydogan, Ilke, Loic Berger, & Vincent Theroude (2024) “Pay All Subjects or Pay only Some? An Experiment on Decision-Making under Risk and Ambiguity,” *Journal of Economic Psychology* 104, 102757.

Bade, Sophie (2015) “Randomization Devices and the Elicitation of Ambiguity-Averse Preferences,” *Journal of Economic Theory* 159, 221–235.

Baillon, Aurélien, Yoram Halevy, & Chen Li (2022) “Randomize at Your Own Risk: On the Observability of Ambiguity Aversion,” *Econometrica* 90, 1085–1107.

Bardsley, Nicholas,Robin Cubitt, Graham Loomes,Peter Moffat, Chris Starmer,& Robert Sugden (2010) “*Experimental Economics; Rethinking the Rules*.” Princeton University Press, Princeton, NJ.

Bastianello, Lorenzo & José Heleno Faro (2023) “Choquet Expected Discounted Utility,” *Economic Theory* 75, 1071–1098.

Baucells, Manel & Samuel E. Bodily (2024) "The Discount Rate for Investment Analysis Applying Expected Utility," *Decision Analysis* 21, 125–141.

Baucells, Manel & Franz H. Heukamp (2010) "Common Ratio Using Delay," *Theory and Decision* 68, 149–158.

Baucells, Manel & Franz H. Heukamp (2012) "Probability and Time Tradeoff," *Management Science* 58, 831–842.

Baucells, Manel & Rakesh K. Sarin (2010) "Predicting Utility under Satiation and Habit Formation," *Management Science* 56, 286–301.

Berger, Loïc & Louis Eeckhoudt (2021) "Risk, Ambiguity, and the Value of Diversification," *Management Science* 67(3):1639–1647.

Berger, Loïc & Johannes Emmerling (2020) "Welfare as Equity Equivalents," *Journal of Economic Surveys* 34, 727–752.

Bleichrodt, Han & José Luis Pinto (2009) "New Evidence of Preference Reversals in Health Utility Measurement," *Health Economics* 18, 713–726.

Broome, John R. (1991) "*Weighing Goods.*" Basil Blackwell, Oxford, UK.

Cappelen, Alexander W., James Konow, Erik O. Sorensen, & Bertil Tungodden (2013) "Just Luck: An Experimental Study of Risk-Taking and Fairness," *American Economic Review* 103, 1398–1413.

Cettolin, Elena & Arno Riedl (2017) "Justice under Uncertainty," *Management Science* 63, 3739–3759.

Cheung, Stephen (2015) "Risk Preferences Are not Time Preferences: On the Elicitation of Time Preference under Conditions of Risk: Comment (#11),"*American Economic Review* 105, 2242–2260.

Cohen, Jonathan, Keith Marzilli Ericson, David Laibson, & John Myles White (2020) "Measuring Time Preferences," *Journal of Economic Literature* 2020, 58, 299–347.

Cohen, Michèle, Jean-Yves Jaffray, & Tanios Said (1987) "Experimental Comparisons of Individual Behavior under Risk and under Uncertainty for Gains and for Losses," *Organizational Behavior and Human Decision Processes* 39, 1–22.

De Meyer, Bernard & Philippe Mongin (1995) “A Note on Affine Aggregation,” *Economics Letters* 47, 177–183.

DeJarnette, Patrick, David Dillenberger, Daniel Gottlieb, & Pietro Ortoleva (2020) “Time Lotteries and Stochastic Impatience,” *Econometrica* 88, 619–656.

Diamond, Peter A. (1967) “Cardinal Welfare, Individual Ethics, and Interpersonal Comparison of Utility: Comment,” *Journal of Political Economy* 75, 765–766.

Diecidue, Enrico, Ulrich Schmidt, & Peter P. Wakker (2004) “The Utility of Gambling Reconsidered,” *Journal of Risk and Uncertainty* 29, 241–259.

Dyer, James & Rakesh K. Sarin (1982) “Relative Risk Aversion,” *Management Science* 28, 875–886.

Eichberger, Jürgen & Illia Pasichnichenko (2021) “Decision-Making with Partial Information,” *Journal of Economic Theory* 198, 105369.

Epper, Thomas & Helga Fehr-Duda (2015) “Risk Preferences Are not Time Preferences: Balancing on a Budget Line: Comment (#12)” *American Economic Review* 105, 2261–2271.

Epper, Thomas & Helga Fehr-Duda (2024) “Risk in Time: The Intertwined Nature of Risk Taking and Time Discounting,” *Journal of the European Eonomic Assoiation* 22, 310–354.

Epstein, Larry G. & Yoram Halevy (2019) “Ambiguous Correlation,” *Review of Economic Studies* 86, 668–693.

Epstein, Larry G. & Stanley E. Zin (1989) “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica* 57, 937–969.

Fishburn, Peter C. (1970) “*Utility Theory for Decision Making.*” Wiley, New York.

Fishburn, Peter C. (1978) “On Handa’s “New Theory of Cardinal Utility” and the Maximization of Expected Return,” *Journal of Political Economy* 86, 321–324.

Fishburn, Peter C. (1980) “A Simple Model for the Utility of Gambling,” *Psychometrika* 45, 435–448.

Gorman, William (1968) “The Structure of Utility Functions,” *Review of Economic Studies* 35, 367–390.

Grabisch, Michel, Benjamin Monet, & Vassili Vergopoulos (2023) "Subjective Expected Utility through Stochastic Independence," *Economic Theory* 76, 723–757.

Grossi, Davide & Gabriella Pigozzi (2014) "*Judgment Aggregation: A Primer.*" Morgan & Claypool, San Rafael, CA, USA.

Gul, Faruk (1992) "Savage's Theorem with a Finite Number of States," *Journal of Economic Theory* 57, 99–110. ("Erratum," 1993, *Journal of Economic Theory* 61, 184.)

Halevy, Yoram (2008) "Strotz Meets Allais: Diminishing Impatience and the Certainty Effect," *American Economic Review* 98, 1145–1162.

Harsanyi, John C. (1955) "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy* 63, 309–321.

Hartmann, Lorenz (2023) "Strength of Preference over Complementary Pairs Aaxiomatizes Alpha-MEU Preferences," *Journal of Economic Theory* 213, 105719.

Hirshman, Samuel D. & George Wu (2025) "Tests of Rank-Dependent Probability Weighting," working paper.

Jacquement, Nicolas & Olivier l'Haridon (2018) "*Experimental Economics: Method and Applications.*" Cambridge University Press, Cambridge.

Jaffray, Jean-Yves (1989) "Linear Utility Theory for Belief Functions," *Operations Research Letters* 8, 107–112.

Johnson, Cathleen, Aurélien Baillon, Han Bleichrodt, Zhihua Li, Dennie van Dolder, & Peter P. Wakker (2021) "Prince: An Improved Method for Measuring Incentivized Preferences," *Journal of Risk and Uncertainty* 62, 1–28.

Kahneman, Daniel & Amos Tversky (1979) "Prospect Theory: An Analysis of Decision under Risk," *Econometrica* 47, 263–291.

Ke, Shaowei & Qi Zhang (2020) "Randomization and Ambiguity Aversion," *Econometrica* 88, 1159–1195.

Keeney, Ralph L. & Howard Raiffa (1976) "*Decisions with Multiple Objectives.*" Wiley, New York (2nd edn. 1993, Cambridge University Press, Cambridge).

Kemel, Emmanuel & Corina Paraschiv (2023) "Risking the Future? Measuring Risk Attitudes towards Delayed Consequences," *Journal of Economic Behavior and Organization* 208, 325–344.

Keren, Gideon B. & Peter H.M.P. Roelofsma (1995) "Immediacy and Certainty in Intertemporal Choice," *Organizational Behavior and Human Decision Processes* 63, 287–297.

Kochov, Asen (2015) "Time and No Lotteries: An Axiomatization of Maxmin Expected Utility," *Econometrica* 83, 239–262.

König-Kersting, Christian, Christopher Kops, & Stefan T. Trautmann (2023) "A Test of (Weak) Certainty Independence," *Journal of Economic Theory* 209, 105623.

Kreps, David M. & Evan L. Porteus (1978) "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica* 46, 185–200.

Lanier, Joshua, Bin Miao, John Quah, & Songfa Zhong (2024) "Intertemporal Consumption with Risk: A Revealed Preference Analysis," *Review of Economics and Statistics* 106, 1319–1333.

Leontief, Wassily W. (1947) "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," *Bulletin of the American Mathematical Society* 53, 343–350.

Machina, Mark J. (2014) "Ambiguity Aversion with Three or More Outcomes," *American Economic Review* 104, 3814–3840.

Marinacci, Massimo (2015) "Model Uncertainty," *Journal of the European Economic Association* 13, 1022–1100.

Miao, Bin & Songfa Zhong (2015) "Risk Preferences Are not Time Preferences: Separating Risk and Time Preference: Comment (#13)," *American Economic Review* 105, 2272–2286.

Miao, Bin & Songfa Zhong (2018) "Probabilistic Social Preference: How Machina's Mom Randomizes Her Choice," *Economic Theory* 65, 1–24.

Monet, Benjamin & Vassili Vergopoulos (2024) “Ambiguity, Randomization and the Timing of Resolution of Uncertainty,” *Economic Theory* 78, 1021–1045.

Mongin, Philippe & Marcus Pivato (2015) “Ranking Multidimensional Alternatives and Uncertain Prospects,” *Journal of Economic Theory* 157, 146–171.

Moscati, Ivan (2018) “*Measuring Utility: From the Marginal Revolution to Behavioral Economics.”* Oxford University Press, Oxford, UK.

Nataf, André (1948) “Sur la Possibilité de Construction de Certain Macromodèles,” *Econometrica* 16, 232–244.

Oechssler, Jörg, Hannes Rau, & Alex Roomets (2019) “Hedging, Ambiguity, and the Reversal of Order Axiom,” *Games and Economic Behavior* 117, 380–387.

Oechssler, Jörg & Alex Roomets (2021) “Savage vs. Anscombe-Aumann: An Experimental Investigation of Ambiguity Frameworks,” *Theory and Decision* 90, 405–416.

Onay, Selçuk & Ayse Öncüler (2009) “How Do We Evaluate Future Gambles? Experimental Evidence on Path Dependency in Risky Intertemporal Choice,” *Journal of Behavioral Decision Making* 22, 280–300.

Pivato, Marcus & Élise Flore Tchouantez (2024) “Bayesian Social Aggregation with Non-Archimedean Utilities and Probabilities,” *Economic Theory* 77, 561–595.

Quiggin, John (1982) “A Theory of Anticipated Utility,” *Journal of Economic Behaviour and Organization* 3, 323–343.

Renneboog, Luc & Christophe Spaenjers (2013) “Buying Beauty: On Prices and Returns in the Art Market,” *Management Science* 59, 36–53.

Richard, Scott F. (1975) “Multivariate Risk Aversion, Utility Independence, and Separable Utility Functions,” *Management Science* 22, 12–21.

Rohde, Ingrid M. T. & Kirsten I. M. Rohde (2015) “Managing Social Risks – Tradeoffs between Risks and Inequalities,” *Journal of Risk and Uncertainty* 51, 103–124.

Rohde, Kirsten I.M. & Xiao Yu (2024) “Intertemporal Correlation Aversion — A Model-Free Measurement,” *Management Science* 70, 3493–3509.

Samuelson, Paul A. (1950) “Probability and the Attempts to Measure Utility,” *Economic Review* 1, 117–126.

Samuelson, Paul A. (1952) “Probability, Utility, and the Independence Axiom,” *Econometrica* 20, 670–679.

Savage, Leonard J. (1954) “*The Foundations of Statistics*.” Wiley, New York. (2nd edn. 1972, Dover Publications, New York.)

Schneider, Florian & Martin Schonger (2018) “An Experimental Test of the Anscombe-Aumann Monotonicity Axiom,” *Management Science* 65, 1667–1677.

Sono, Masazo (1945) “The Effect of Price Changes on the Demand and Supply of Separable Goods” (in Japanese), *Kokumin Keisai Zasshi* 74, 1–51.

Starmer, Chris (2000) “Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk,” *Journal of Economic Literature* 38, 332–382.

Tsetlin, Ilia & Robert L. Winkler (2009) “Multiattribute Utility Satisfying a Preference for Combining Good with Bad,” *Management Science* 55, 1942–1952.

Tversky, Amos & Daniel Kahneman (1992) “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty* 5, 297–323.

Wakker, Peter P. (2010) “*Prospect Theory: For Risk and Ambiguity.”* Cambridge University Press, Cambridge, UK.

Wu, George (1996) “The Strengths and Limitations of Expected Utility Theory,” *Medical Decision Making* 16, 9–10.

Wu, George, Jiao Zhang, & Richard Gonzalez (2004), “Decision under Risk.” *In* Derek J. Koehler & Nigel Harvey (eds.), *Blackwell Handbook of Judgment and Decision Making*, 399–423, Blackwell Publishing, Oxford, UK.

Supplementary Appendix of "The Deceptive Beauty of Monotonicity, and the Million-Dollar Question: Row-First or Column-First Aggregation?"

CHEN LI, KIRSTEN I.M. ROHDE, & PETER P. WAKKER

January, 2026

SUPPLEMENTARY APPENDIX A. PRECEDING MATHEMATICAL RESULT

The mathematics underlying our results has been known longtime. We do not bring mathematical novelties in the main text. This appendix discusses preceding literature. Observation 6 has been known since Nataf (1948). There, rows described producers and columns described production inputs. Nataf presented[12] Observation 6 to show when macro (column-first, or column-monotonic, defined below) aggregation of production inputs can be equivalent to micro (row-first, or row-monotonic) aggregation: only if there is not any interaction[13]. van Daal & Merkies (1988) provided an early historical account. This production example further illustrates the wide applicability of our framework.

Mongin & Pivato (2015 Proposition 1) provided the mathematically most general versions of Nataf's (1948) result, implying our Observation 6. (Hence, we gave no proof of it.) In the mathematical theory of functional equations, these results have been known as generalized bisymmetry equations.[14] See Maksa (1999), who also pointed out their relatedness to aggregation. The special case of proportional representations in our Theorem 1 is equivalent to mathematical theorems on multisymmetry functional equations, explained by Münnich, Maksa, & Mokken

---

[12] He heavily used differentiability and his proof is not easily accessible.

[13] Formally, we use the suggestive term interaction to indicate preference relations that violate strong separability. In general, not only rows and columns, but every subset of cells can be nonseparable, i.e., be impacted by (interacting with) any other subset of cells.

[14] They search for functions allowing identity of Eqs. 6 and 7 in the main text.

(2000). Mongin & Pivato (2015 Theorem 1) is the most general result of this kind. See Zuber (2016) for related results and literature with Anscombe-Aumann outcome sets. Whereas all aforementioned results and virtually all results cited or given in this paper heavily use continuity in outcomes, Grabisch, Monet, & Vergopoulos (2023) give a version of our Theorem 1 that uses a continuum state space, rather than outcome space.

Thus, the mathematics underlying our results has been known longtime. As for Theorem 1 on DEU, its novelty is not in mathematical generality but in simplicity and appeal. The preference conditions there can be stated verbally and are accessible to nonspecialists more than any preceding axiomatization of DEU. Although many authors, several cited later, used advanced implications of the preceding results in various decision theories, their basic impact for empirical and theoretical work, specified in Section 5 and applied in the rest of this paper, has not been presented before.

SUPPLEMENTARY APPENDIX B. EXTENSION TO INFINITE-DIMENSIONAL MATRICES

Extensions to matrices with infinitely many rows and/or columns are often of interest. This holds mainly for Theorem 1. Infinite-dimensional extensions of Observation 6 are less common because they involve uncommon functionals, generalizing integrals. Wakker & Zank (1999) examined them. We focus on Theorem 1 henceforth, interpreting rows as states and columns as timepoints, but using the general notation of Figure 2.

Equal-likely states in Figure 2 can capture all simple lotteries with rational probabilities (McCarthy, Mikkola, & Thomas 2020). Mixture-closedness or continuous distributions require a continuum of r^i. Such extensions can be obtained by standard techniques from mathematical measure theory. Theorem 14 provides a typical example. It is explained next.

We continue to assume n columns $c_1, \dots, c_n$ with $n \geq 2$ fixed. A row continues to be an element of $\mathbb{R}^n$. In the main text, we considered the special case of risk where each r^i had probability $1/m$, so that matrices could be identified with some simple probability distributions over columns. We now consider more general probability distributions over rows, such as the space of all simple probability distributions or all bounded ones. To this effect, instead of $R = \{r^1, \ldots, r^m\}$, we now assume $R = [0,1)$, endowed with the uniform distribution P and instead of finite-dimensional matrices as before, we now consider functions from $R \times \{c_1, \ldots, c_n\}$ to the reals. We continue to call such functions matrices. Preferences will be over matrices. We make the assumption characteristic of decision under risk: functions on $R \times \{c_1, \ldots, c_n\}$ that generate the same probability distribution over rows are indifferent.

Using obvious notation, a simple probability distribution over rows can be denoted $(p^1{:}\, r^1, \ldots, p^k{:}\, r^k)$, with k variable, and all probabilities positive. We identify it with a matrix that assigns row r^i to each set R^i, where $(R^1, \ldots, R^k)$ partitions $[0,1)$ and $P(R^i) = p^i$ for each i. It, thus, is like the matrix in Figure 2, with R^i for r^i for each i, and $m = k$. It will be sufficient to impose our intuitive axioms only on such simple finite-dimensional matrices. *Row* and *column monotonicity* are now defined to hold for all simple matrices.[15] For each fixed $(R^1, \ldots, R^k)$, Theorem 1 then gives a DEU representation. Normalizing $U(0) = 0$, $U(1) = 1$, these DEU representations agree on common domain by standard uniqueness results, giving a probability measure P' on $[0,1)$ that at this stage might be thought to possibly differ from P and even be only finitely additive. However, partitions $(R^1, \ldots, R^k)$ with $P(R^i) = 1/k$, by symmetry, imply $P'(R^i) = 1/k = P(R^i)$. The unions of such R^i show that P' agrees with P on all $R \subset [0,1)$ with rational P probability. By monotonicity w.r.t. set inclusion, P' and P are identical. We have obtained a DEU representation for all simple matrices.

The extension of our theorems to all bounded matrices now follows using standard techniques from mathematical measure theory. Monotonicity with respect to rows and columns, but also with respect to outcomes, is imposed only on simple matrices. Thus, null events are avoided and strict preferences are properly implied. We reinforce outcome monotonicity to infinite dimensions by adding *pointwise monotonicity*: a matrix is weakly preferred if all its cells weakly dominate. This condition is as unobjectionable for infinite dimensions as it is for finitely many. Every bounded matrix is now "sandwiched" more and more tightly by pointwise dominating and dominated simple matrices. This determines a unique DEU value, such that strict

---

[15] Bear in mind that we assume strictly positive probabilities, avoiding null events as required for outcome monotonicity.

inequality of DEU values implies strict preference (using transitivity). Next, we reinforce continuity into supnorm continuity, ensuring existence of constant equivalents. Then equality of DEU values, again using transitivity, implies indifference and, hence, we have a DEU representation. We have shown the following result.

THEOREM 14. Assume that: (a) matrices map $[0,1) \times \{c_1, \ldots, c_n\}$ to the reals and are measurable; (b) preferences are over matrices; (c) decision under risk holds with respect to the uniform distribution on $[0,1)$. That is, our domain of matrices is equivalent to probability distributions over "rows" in $\mathbb{R}^n$. On the domain of simple matrices/distributions, and also on the domain of all bounded matrices/distributions, discounted expected utility holds if and only if weak ordering, supnorm continuity, pointwise monotonicity, and monotonicity with respect to outcomes, rows, and columns hold.

Extension to unbounded matrices and connected topological outcome spaces (including all convex sets of commodity bundles) can be obtained by Wakker's (1993) truncation continuity. The total subjective weight of space R is still assumed bounded here. Unbounded subjective weight of R may occur, for instance, if R reflects time rather than uncertainty, or populations of variable size. Then further continuity conditions have to be invoked, discussed for instance by Asheim et al. (2010), Banerjee & Mitra (2007), Christensen (2022), Drugeon & Huy (2022), Marinacci (1998), and Pivato (2022). For extensions to infinitely many columns, besides infinitely many rows, our extension techniques are similarly applied to columns.

Theorem 14 can be used for all interpretations of columns. If they refer to ambiguous events (horses), versions of the AA framework result. Here it is usually assumed that only marginal distributions conditional on horses matter, which can be added as a preference condition. Then our structure becomes isomorphic to the set of maps from $\{c_1, \ldots, c_n\}$ to probability distributions over $\mathbb{R}$. Correlations between different c_j then play no role.

SUPPLEMENTARY APPENDIX C. THEORETICAL APPLICATIONS OF NATAF'S AGGREGATION RESULT TO PREFERENCE AXIOMATIZATIONS

We briefly sketch some further theoretical applications to preference axiomatizations, in addition to Theorem 1 in the main text. We first assume that both rows and columns refer to events. Thus, $\{r^1, \ldots, r^m\}$ and $\{c_1, \ldots, c_n\}$ are two partitions of the universal event. In Figure 1, the intersection event $r^i \cap c_j$ gives outcome x_j^i. Outcome monotonicity implies that none of those intersections is empty or null. Uniform row and column monotonicity can be interpreted as versions of stochastic independence: being informed about one partition does not affect preferences over the other. Theorem 1 then gives an appealing axiomatization of subjective expected utility, alternative to Savage (1954). Pfanzagl (1968; Section 12.5) presented this result using the stochastic independence interpretation for $m = n = 2$. Mongin (2020) and Ceron & Vergopoulos (2021) independently generalized it to general m, n.

We next continue to assume that rows and columns refer to events, but we further assume decision under risk for the r^i, with probability $1/m$ for each r^i. We first consider the case where the c_js may have unknown probabilities. Theorem 1 gives expected utility for risk (evaluating each column). Our equally-likely case can cover all simple rational-probability distributions. Supplementary Appendix B shows how more general probability distributions can be incorporated, and that subjective probabilities over rows must be equal to the objective probabilities over rows. Theorem 1 also gives expected utility for the horse events c_j and, thus, provides an alternative axiomatization of the original expected utility model of AA, using the two-stage framework that has become standard today. AA referred to standard mixture independence to axiomatize expected utility for risk, and also assumed horse monotonicity. In our approach, their mixture independence is weakened to risk monotonicity. For our monotonicities the event, say row, to be conditioned on always only involves one outcome per column, whereas for von Neumann-Morgenstern mixture independence (or Savage's sure-thing principle) such events to be conditioned on must be allowed to involve any number of rows, i.e. any number of outcomes per column. The symmetry of our two monotonicity conditions and, thus, of the treatment of risk and uncertainty, adds to the appeal of our alternative Theorem.

As a price to pay, we need continuous utility whereas AA and Fishburn (1970) allowed for complete generality in this regard.

If we interpret the c_js as persons rather than events, Theorem 1 becomes an alternative to Harsanyi's (1955) welfare result based on the veil of ignorance. His Pareto principle is column monotonicity. Like AA, he refers to mixture independence to obtain EU, and we similarly generalize here. In Theorem 1 there is no middle ground: if the social welfare function is ordinal in the individual utilities then it must be cardinal, leading to a linear sum. This is the essence of Harsanyi's result. Grant et al. (2010) provided generalizations that relaxed the independence and monotonicity conditions in Harsanyi's result.

We, finally, present an implication where only one component is available at the outset, but we construct a second kind for auxiliary purposes. Gul (1992) considered a finite state space $\{r^1, \ldots, r^m\}$. *Acts* $(x^1, \ldots, x^m)$ map states to $\mathbb{R}$. Gul's preference relation on acts, denoted $\succcurlyeq'$ here, satisfies weak ordering, continuity, and outcome monotonicity, implying that all states are nonnull. One fixed event A (nontrivial subset of the state space) plays a special role explained later (reminiscent of Ramsey's (1931) ethically neutral event). We define the function $\overline{C}$ on acts as the certainty equivalent ("constant equivalent") function, and $R^1(y_1, y_2) = \cdots = R^m(y_1, y_2)$ as the certainty equivalent function of acts $(A{:}\, y_1, A^c{:}\, y_2)$, using obvious notation.

We take matrices as in Figure 2 with $n = 2$, $c_1 = A, c_2 = A^c$. We define our preference relation $\succcurlyeq$ over matrices as represented by Eq. 6. Thus, row monotonicity holds (Observation 10) and it is uniform because all R^j's are the same. The act $\left(R^1(x_1^1, x_2^1),\ \ldots\ldots, R^m(x_1^m, x_2^m)\right)$ can be identified with the equivalence class of corresponding matrices with entries $x_1^{j\prime}$ and $x_2^{j\prime}$ such that $R^j\left(x_1^{j\prime}, x_2^{j\prime}\right) = R^j\left(x_1^j, x_2^j\right)$ for all j. Uniform column monotonicity for $\succcurlyeq$ over matrices in Figure 2 is equivalent to Gul's Assumption 2 for $\succcurlyeq'$ on acts, a condition called *act independence* nowadays (Chew & Karni 1994). Thus, we obtain as a corollary of Theorem 1:

THEOREM 15. Under the assumptions of this Appendix, the following four statements are equivalent:

(i) Expected utility holds for $\succcurlyeq'$ over acts.

(ii) Discounted expected utility holds for $\succcurlyeq$ over matrices.

(iii) Uniform column monotonicity holds for $\succcurlyeq$ over matrices.

(iv) Act independence holds for $\succcurlyeq'$ over acts.

In the above result, standard uniqueness results for DEU imply that the "discount weight" d_1 of the left column, after normalization, is the probability of event A resulting from the row probabilities. The conditions in Statements (iii) and (iv) are appealing because they mimic mixture independence for risk to the context of uncertainty.

Gul's axiomatization of subjective expected utility through act independence thus follows as a corollary of our Theorem 1. Our result is more general because Gul required the event A to satisfy a symmetry condition implying that it has subjective probability 0.5, which we do not need. Chew & Karni (1994) also provided this generalization. Our verbal proof, involving the Appendix in the main text and the preceding paragraphs, is considerably shorter and more accessible than that in Gul (1992 pp. 104-109) or Chew & Karni (1994). It is remarkable that Gul (1992) can be obtained as, essentially, a corollary of Nataf (1948).

Some other axiomatizations of expected utility used generalizations of bisymmetry axioms that are all more restrictive than Gul's Assumption 2: they also consider more than two columns and many events A (Köbberling & Wakker 2003 Theorem 16). Hence, they also follow as corollaries of our Theorems 1 and 15. Such results include Krantz et al. (1971, Theorem 6.9.10 which assumes $m = n = 2$), Pfanzagl (1959 pp. 287–288 which assumes $m = n = 2$), and Münnich, Maksa, & Mokken (2000 Theorem 2).

REFERENCES FOR SUPPLEMENTARY APPENDIX

Asheim, Geir B., Walter Bossert, Yves Sprumont & Kotaro Suzumura (2010) "Infinite-Horizon Choice Functions," *Economic Theory* 43, 1–21.

Banerjee, Kuntal & Tapan Mitra (2007) "On the Continuity of Ethical Social Welfare Orders on Infinite Utility Streams," *Social Choice and Welfare* 30, 1–12.

Ceron, Federica & Vassili Vergopoulos (2021) "On Stochastic Independence under Ambiguity," *Economic Theory* 71, 925–960.

Chew, Soo Hong & Edi Karni (1994) "Choquet Expected Utility with a Finite State Space: Commutativity and Act-Independence," *Journal of Economic Theory* 62, 469–479.

Christensen, Timothy M. (2022) “Existence and Uniqueness of Recursive Utilities without Boundedness,” *Journal of Economic Theory* 200, 105413.

Drugeon, Jean-Pierre & Thai Ha Huy (2022) “A not so Myopic Axiomatization of Discounting,” *Economic Theory* 73:349–376.

Fishburn, Peter C. (1970) “*Utility Theory for Decision Making.*” Wiley, New York.

Grabisch, Michel, Benjamin Monet, & Vassili Vergopoulos (2023) “Subjective Expected Utility through Stochastic Independence,” *Economic Theory* 76, 723–757.

Grant, Simon, Atsushi Kajii, Ben Polak, & Zvi Safra (2010) “Generalized Utilitarianism and Harsanyi’s Impartial Oberver Theorem,” *Econometrica* 78, 1939–1971.

Gul, Faruk (1992) “Savage’s Theorem with a Finite Number of States,” *Journal of Economic Theory* 57, 99–110. (“Erratum,” 1993, *Journal of Economic Theory* 61, 184.)

Harsanyi, John C. (1955) “Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility,” *Journal of Political Economy* 63, 309–321.

Köbberling, Veronika & Peter P. Wakker (2003) “Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique,” *Mathematics of Operations Research* 28, 395–423.

Krantz, David H., R. Duncan Luce, Patrick Suppes, & Amos Tversky (1971), “*Foundations of Measurement, Vol. I (Additive and Polynomial Representations).”* Academic Press, New York. (2nd edn. 2007, Dover Publications, New York.)

Maksa, Gyula (1999) “Solution of Generalized Bisymmetry Type Equation without Surjectivity Assumptions,” *Aequationes Mathematicae* 57, 50–74.

Marinacci, Massimo (1998) “An Axiomatic Approach to Strong Patience and Time Invariance,” *Journal of Economic Theory* 83, 105–144.

McCarthy, David, Kalle Mikkola, & Teruji Thomas (2020) “Utilitarianism with and without Expected Utility,” *Journal of Mathematical Economics* 87, 77–113.

Mongin, Philippe (2020) “Bayesian Decision Theory and Stochastic Independence,” *Philosophy of Science* 87, 152–178.

Mongin, Philippe & Marcus Pivato (2015) “Ranking Multidimensional Alternatives and Uncertain Prospects,” *Journal of Economic Theory* 157, 146–171.

Münnich, Àkos, Gyula Maksa, & Robert J. Mokken (2000) "n-Variable Bisection," *Journal of Mathematical Psychology* 44, 569–581.

Nataf, André (1948) "Sur la Possibilité de Construction de Certain Macromodèles," *Econometrica* 16, 232–244.

Pfanzagl, Johann (1959) "A General Theory of Measurement—Applications to Utility," *Naval Research Logistics Quarterly* 6, 283–294.

Pfanzagl, Johann (1968) "*Theory of Measurement.*" Physica-Verlag, Vienna.

Pivato, Marcus (2022) "A Characterization of Cesaro Average Utility," *Journal of Economic Theory* 201 105440.

Ramsey, Frank P. (1931) "Truth and Probability." *In* Richard B. Braithwaite (ed.), *The Foundations of Mathematics and other Logical Essays,* 156–198, Routledge and Kegan Paul, London.

Reprinted in Henry E. Kyburg Jr. & Howard E. Smokler (1964, eds.) *Studies in Subjective Probability*, 61–92, Wiley, New York. (2nd edn. 1980, Krieger, New York.)

Savage, Leonard J. (1954) "*The Foundations of Statistics.*" Wiley, New York. (2nd edn. 1972, Dover Publications, New York.)

van Daal, Jan & Arnold H.Q.M. Merkies (1988) "The Problem of Aggregation of Individual Economic Relations; Consistency and Representativity in a Historical Perspective." *In* Wolfgang Eichhorn (ed.) *Measurement in Economics* (Theory and Applications of Economic Indices), 607–637, Physica-Verlag, Heidelberg.

Wakker, Peter P. (1993) "Unbounded Utility for Savage's "Foundations of Statistics," and other Models," *Mathematics of Operations Research* 18, 446–485.

Wakker, Peter P. & Horst Zank (1999) "State Dependent Expected Utility for Savage's State Space; Or: Bayesian Statistics without Prior Probabilities," *Mathematics of Operations Research* 24, 8–34.

Zuber, Stéphane (2016) "Harsanyi's Theorem without the Sure-Thing Principle: On the Consistent Aggregation of Monotonic Bernoullian and Archimedean Preferences," *Journal of Mathematical Economics* 63, 78–83.