COMONOTONIC ADDITIVITY FOR CHOQUET INTEGRALS

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Elucidation added August 2011: After completing this version I discovered (when I asked David Schmeidler in 1990) that the proof of comonotonic additivity of the Choquet integral had been available in the literature after all. So my paper was not the first to provide this basic but important point, contrary to what I thought in 1989. The proof had been provided by Schmeidler (1986 p. 258, Remark 4). I had overlooked this because Schmeidler (1986 p. 255, bottom) wrote that Dellacherie (1970) provided the proof in general but this is not true. (Also Schmeidler did not give the derivation in the proof of his theorem, but in a remark following the proof.) This way I was misled to think that the proof had not been available in the literature yet. Schmeidler was overmodest and he is the first to have proved this result in general. I then dropped this paper and withdrew it in a letter of March 1, 1990 from the journal of Mathematical Economics where I had submitted it.

Later, Dieter Denneberg kindly told me that he liked the proof in my paper. He cited my paper in Denneberg, Dieter (1994) "*Non-Additive Measure and Integral.*" Kluwer Academic Publishers, Dordrecht. Denneberg states comonotonic additivity of the Choquet integral in Proposition 5.1.vi (p. 65) and provides a proof. Probably because of Denneberg's citation, people have sometimes asked for a copy of my paper. Hence I provide it here. I obviously claim no novelty. This ends my addition of August 2011 and now follows the 1989 paper.

Abstract

It is shown that Choquet integrals satisfy comonotonic additivity. While this result has often been used in the literature, no derivation in full generality had been provided yet.

KEYWORDS: nonadditive measure, Choquet integral, comonotonicity, fuzzy measure

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SECTION 1. INTRODUCTION

Choquet (1953-4) introduced what is nowadays known as the Choquet integral, i.e., a way to integrate w.r.t. nonadditive measures. Recently the interest for the Choquet integral has been revived, for one reason because of its usefulness for decision theory where it accomodates with the modern desire to deviate from expected utility (see for instance Schmeidler, 1989, Gilboa, 1987, and Wakker, 1989a), for another reason because of its usefulness for fuzzy set theory where there is great interest in ways to integrate w.r.t. 'fuzzy measures' (see Wakker, 1989c and Murofushi & Sugeno, 1989). The soundness of the Choquet integral was finally established by Anger (1977, Theorem 3) and Schmeidler (1986). They showed that some natural conditions characterize the integral. The main characterizing condition is 'comonotonic additivity'. Surprisingly enough, while the fact tht this condition is implied has been used many times in the literature, and has been derived for many special cases, no derivation in full generality is yet available. This note fills in that gap. The most general result so far, Dellacherie (1970), was written in French, so that an additional use of this note may be the making accessible of Dellacherie's result, somewhat generalized, to a wider audience.

SECTION 2. THE THEOREM

By Ω we denote a nonempty set, by $\mathcal{A}$ an algebra of subsets of Ω. $\overline{\Re}$ denotes $\{-\infty, \infty\} \cup \Re$. **Often functions will without explicit mention be assumed 'measurable' from Ω to $\overline{\Re}$. Measurable** means that inverses of intervals are in $\mathcal{A}$. A function $v : \mathcal{A} \to [0, 1]$ is a **capacity** (or 'fuzzy measure') if $v(\emptyset) = 0$, $v(\Omega) = 1$, and v is **monotonic w.r.t. set-inclusion**, i.e., $A \supset B \Rightarrow v(A) \geq v(B)$. In literature the 'normalization' $v(\Omega) = 1$ is sometimes omitted. Often the following two conditions are added: v is **increasing-set-continuous** if $[A_j \uparrow A \Rightarrow v(A_j) \uparrow v(A)]$, v is **decreasing-set-continuous** if $[A_j \downarrow A \Rightarrow v(A_j) \downarrow v(A)]$; v is **set-continuous** if it is increasing- and decreasing-set-continuous. Obviously for probability measures set-continuity is the same as σ-additivity. Capacities generalize probability measures by weakening additivity to monotonicity, and appear in many applications. For a capacity v, and a function $f : \Omega \to \overline{\Re}$, the **Choquet integral** of f (w.r.t. v), denoted $\int_\Omega f dv$, or $\int f dv$, is

$$\int_{\Re+} v(\{s \in \Omega : f(s) \geq \tau\})\, d\tau + \int_{\Re_-} [v(\{s \in \Omega : f(s) \geq \tau\}) - 1]\, d\tau\ . \tag{1}$$

As usual, the Choquet integral is left undefined if in (1) $\infty - \infty$ would result. In Wakker (1989b, Chapter VI) elucidations and illustrations are given.

A set C of functions is **comonotonic** if:

$$\nexists f, g \in C, \omega, \omega' \in \Omega : [f(\omega > f(\omega'),\ g(\omega) < g(\omega')].$$

Dellacherie was the first to observe the importance of the following condition for the Choquet integral. Intuitions for this condition in decision-theoretic applications can be inferred from Yaari (1987, p. 104) or Wakker (1989c, Example 1 up to Definition 4) or Wakker (1989d, section 4).

In the theorem below we shall make use of the fact that the Choquet integral w.r.t. the finitely additive probability measure P coincides with the 'S-integral' as in section 4.5 of Bhaskara Rao & Bhaskara Rao (1983), which is derived from simple functions through upper and lower sums. It extends, for the special case with $\Re$ as range and with bounded positive measures, the integral of section I.III.2 of Dunford & Schwarz (1958) (i.e., the 'D-integral' of section 4.4 of Bhaskara Rao & Bhaskara Rao, 1983), to the case where the integral can be infinite.

One way to interpret the theorem below is to say that, when its domain is restricted to a comonotonic subset, the Choquet integral is a usual additive integral.

THEOREM 1 *If C is a comonotonic set of functions then there exists a finitely additive probability measure P on $\mathcal{A}$ so that the Choquet integral on C coincides with the integral w.r.t. P.*

PROOF. Consider the family of 'half-bounded closed intervals' (all in $\mathcal{A}$) $\{\omega \in \Omega : h(\omega) \geq \alpha\}$, for any $h \in C$, $\alpha \in \Re$. By comonotonicity this family is 'nested', i.e., for each pair of such intervals, one is a subset of the other. On this family we define $P \equiv v$. By routine methods from measure theory, P can in a unique manner be extended to an additive probability measure on the algebra generated by the family. Since P coincides with v on the family just defined, according to the definition of the Choquet integral the integrals w.r.t. v and P coincide on C. □

As a direct consequence of the above theorem we get:

THEOREM 2 *If two functions f, g are comonotonic, then $\int (f+g)dv = \int f dv + \int g dv$ whenever the right-hand side is defined (so is not $\infty - \infty$).* □

Schmeidler (1986) refers to Dellacherie (1970) for the above result, and mentions a correctable mistake in Dellacherie's exposition. (One detail is that in Dellacherie's Proposition the right-hand side must be defined.) Dellacherie however obtained the above result only for positive functions under the assumption that v satisfies increasing-set-continuity, and that every subset of Ω is in $\mathcal{A}$. Note that the latter two conditions together are very restrictive, and for instance exclude the usual case where $\Omega = [0, 1]$ and v extends the Lebesgue measure. Also our proof is shorter than Dellacherie's, by using a standard procedure from measure theory. Yaari (1987) derives the result under some continuity and boundedness restrictions, where further the capacity is assumed to be a nondecreasing transform of an additive probability measure. Anger (1977) derived under topological assumptions a somewhat weaker property than comonotonic additivity, called 'weak additivity'.

A comment on measurability is in order. When working with (non-σ-)algebras it may happen that for measurable f, g the sum $f + g$ is not measurable. Obviously the above Theorem, as well as the Corollary below, only applies to the case where all involved functions are measurable. A same restriction can be added to Schmeidler (1986, 1989).

Finally we give the following corollary, which is a direct consequence of the above theorem and positive homogeneity:

COROLLARY 3 *If two functions f, g are comonotonic, then for all $\sigma, \tau \geq 0$, $\int(\sigma f + \tau g)dv = \sigma \int f dv + \tau \int g dv$, whenever all involved integrals are defined.* $\square$

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