A GENERAL THEORY FOR QUANTIFYING BELIEFS*

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Abstract

This paper presents conditions under which a person's beliefs about the occurrence of uncertain events are quantified by a capacity measure, i.e., a nonadditive probability. Additivity of probability is violated in a large number of applications where probabilities are vague or ambiguous due to lack of information.

The key feature of the theory presented in this paper is a separation of the derivation of capacities for events from a specific choice model. This is akin to eliciting a probability distribution for a random variable without committing to a specific decision model. Conditions are given under which Choquet expected utility, the Machina-Schmeidler probabilistically sophisticated model, and subjective expected utility can be derived as special cases of our general model.

Keywords: ambiguity, nonadditive probability, nonexpected utility, measures of belief

1. **INTRODUCTION**

Models of decisions under uncertainty require two inputs: a measure of subjective beliefs about the occurrence of uncertain events, and a measure of tastes about the outcomes that will eventually be realized. In the classical subjective expected utility theory the former is captured by a probability measure and the latter by a utility function.

In this paper we present conditions under which a person's beliefs about the occurrence of uncertain events are quantified by a capacity measure. A capacity measure satisfies monotonicity with respect to set inclusion, but unlike a probability measure it may not necessarily be additive for two disjoint events. The additivity property of probability measures is violated in a large number of applications where probabilities are vague or ambiguous due to lack of information.

In recent work, capacity measures have been derived from some assumptions about a person's decisions that together imply maximization of a Choquet Expected Utility (CEU) form. Such a decision theoretical foundation for capacities makes them observable, thus scientifically well-founded. It is, however, unclear whether CEU is the most appropriate decision model for all situations. In particular, the CEU form is not Fréchet differentiable, which limits its use (see Chew, Karni, & Safra, 1987). Further, in some applications, the information about capacities of events may be useful in itself regardless of the specific decision model that a decision maker may eventually employ. Hence this paper provides a decision-theoretic foundation for capacities in a very general decision model that only imposes a minimal restriction in the form of a dominance axiom. By separating the derivation of capacities for events from a specific choice model such as CEU, our approach provides a greater flexibility for applications and research. We provide conditions under which the Machina-Schmeidler (1990) probabilistically sophisticated model, CEU, and subjective expected utility can be derived as special cases of our general model.

De Finetti (1937) and Savage (1954) provided a hallmark contribution to the theory of decisions under uncertainty by providing assumptions about a person's behavior that imply a measure of subjective beliefs which mimics the properties of mathematical probability. This measure is widely known as subjective probability. In many applications, however, the additivity property of the subjective probability is violated. Starting with Schmeidler (1989, first version 1982), several contributions in recent years (Gilboa, 1987, Wakker, 1989, Nakamura (1990), Sarin & Wakker, 1990) have relaxed the additivity property and proposed a less restrictive capacity measure for quantifying an individual's subjective beliefs. All of these contributions, to our knowledge, propose a CEU form for the preference functional. Our aim here is to derive the capacity measure without requiring that a person maximize CEU in choosing among acts. This aim has a close parallel with that of Machina & Schmeidler (1990), who derived a probability measure without requiring that a person maximize subjective expected utility.

Judgments and preferences that may lead to nonadditive probability have been well-known to economists and psychologists for a long time. Knight (1921) made the distinction between risk and uncertainty based on whether the event probabilities are known or unknown. Keynes (1921) has argued that confidence in probability influences decisions under uncertainty. Schmeidler (1989) has argued that the amount of information available about an event may influence probabilities in such a way that the probabilities are not necessarily additive. Edwards (1954), Ellsberg (1961), and recently Einhorn & Hogarth (1985), have observed that a majority of subjects violates additivity of probability.

Why do such frequent and persistent violations of additivity occur in experimental data? The issue is not fully resolved yet, but it seems that an individual's willingness to bet on an event does not depend solely on the perceived likelihood of the occurrence of the event. It seems to be influenced by the confidence one has about one's judgment. Thus, for example, when one does not know the proportion of yellow or white balls in an urn, one is cautious on betting either on the yellow or the white ball. This cautiousness (often called aversion to ambiguity) leads to a revealed probability of less than 0.5 for betting on either color, thus to nonadditivity of probability. Tversky & Kahneman (1990) advance, as an alternative, the idea of source dependence. A person may prefer to bet on sports-related events rather than political events, even if the events have the same perceived likelihood of occurrence. Such a preference may be observed because of an inherent liking of sports-related events. A model that permits the betting behavior to be influenced by psychological concerns such as regret, confidence, or source dependence seems to be more appropriate for describing peoples' actual behavior. The notion of capacity as advanced by Schmeidler does indeed place less stringent restrictions than probability on a person's betting behavior. It is this notion of capacity that we fully exploit in providing a general model of decisions under uncertainty. Alternative approaches to model confidence in probabilities have been presented in Jaffray (1989) and Nau (1986).

Nonadditive capacities are often rejected prematurely or are viewed with suspicion because they entail a violation of the theory of probability. In a theory of decision making, however, we will show that it is possible to relax the additivity assumption while preserving a consistency in choices. For over two centuries since Bernoulli (1738), the assumption that the utility of money (total cash balance) is the sum of the individual utilities of the cash balances in two bank accounts has been discarded. In a similar vein, in the recent literature the assumption that a measure of subjective belief about the occurrence of either of two disjoint events is the sum of the individual measures of subjective beliefs about the occurrence of each event has been relaxed. Whether or not the restriction to additivity is justified in a special class of applications must be judged by the appeal of the assumptions about the decision maker's preferences that are needed to restrict the capacity to the additive case. In Figure 6.1, we provide a unifying map of alternative assumptions that lead to additive capacities and further down to subjective expected utility.

We begin by presenting some notation and definitions in Section 2. Axioms and the general decision model are presented in Section 3. Since the notion of capacity is central to our development, special properties of capacity functions are discussed in Sections 4 and 5. Section 6 shows that several existing models can be derived as special cases of our general model by imposing additional assumptions. Examples and discussions are contained in Section 7.

2. ELEMENTARY DEFINITIONS

This section presents the notation and some definitions that are subsequently used throughout the paper. There is a set C of *consequences* (payoffs, prizes, outcomes) and a set S of *states of nature*. The states in S are mutually exclusive and collectively exhaustive so that exactly one state is the true state. We shall let A denote a collection of subsets of S. The elements of A are called *events*; capacities will be assigned to them. The setup of this paper allows for a considerable generality, and does not require that A is a σ-algebra. We do however assume that A contains a σ-algebra A^{ua} that should be thought of as containing unambiguous events[1]. The conditions in the main Theorem 3.1 below will ensure that the σ-algebra A^{ua} is sufficiently rich to "calibrate" the ambiguous events from $\mathsf{A} \backslash \mathsf{A}^{ua}$. For the collection $\mathsf{A} \backslash \mathsf{A}^{ua}$ of ambiguous events, considerable flexibility is allowed. For example, A^{ua} may describe events related to the outcome of a roulette wheel, and $\mathsf{A} \backslash \mathsf{A}^{ua}$ may consist of the events: {rain, cloudy, sunshine}. As A need not be a σ-algebra, no unions of the events in $\mathsf{A} \backslash \mathsf{A}^{ua}$, e.g. rain-or-cloudy, as well as no intersections of A with events related to the roulette wheel need to be incorporated. Further comments on the choice of A^{ua} are provided following Theorem 3.1.

F denotes a set of *acts*, i.e., its elements are functions from S to C. A measurability condition will be imposed below. We assume that the elements from F have a finite range, i.e., are *simple*.[2] We do not assume that F contains all (measurable) simple

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[1]It is to be noted that we do not require an a priori definition of unambiguous or ambiguous events. The events in A^{ua} only are to satisfy the axioms below. This will imply that probabilities can be assigned to the events in A^{ua}.

[2]In the setup of this paper there is no difference between simple acts and the "step-acts" as considered in Sarin & Wakker (1990, see Footnote 1 there).

functions from S to $\mathcal{C}$, but $\mathcal{F}$ is assumed to contain the entire set $\mathcal{F}^{ua}$ of simple *unambiguous* acts, i.e., acts f such that $f^{-1}(E)\in\mathcal{A}^{ua}$ for each $E\subset\mathcal{C}$. Act f is constant if, for some $\alpha\in\mathcal{C}$, $f(s)=\alpha$ for all states s. Often a constant act is identified with the resulting consequence. Note that all constant acts are contained in $\mathcal{F}^{ua}$, so are contained in $\mathcal{F}$. Statements of conditions are simplified by defining f_A as the restriction of f to A, f_Ah as the act that assigns consequences f(s) to all $s\in A$, and consequences h(s) to all $s\in S\backslash A$. Given that consequences are identified with constant acts, $f_A\alpha$ designates the act that is identical to f on A and constant α on $S\backslash A$; $\alpha_A\beta$ is similar. Further, for a partition $\{A_1,...,A_m\}$, $\alpha^1_{A_1}...\alpha^m_{A_m}$ denotes the act that assigns consequence α^j to each $s\in A_j$, j=1,...,m. A binary relation $\succcurlyeq$ over $\mathcal{F}$ gives the decision maker's preferences. The notations $\succ$, $\preccurlyeq$, $\prec$, and $\sim$ are as usual. Further, $\succcurlyeq$ is a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all f,g) and transitive.

We define $\succcurlyeq$ on $\mathcal{C}$ from $\succcurlyeq$ on $\mathcal{F}$ through constant acts. We impose the following measurability condition on the acts: For each act $f\in\mathcal{F}$, and each consequence α, the *cumulative event* $\{s\in S: f(s)\succcurlyeq\alpha\}$ is contained in $\mathcal{A}$. Note that this is less restrictive than the usual measurability conditions for σ-algebras $\mathcal{A}$. Cumulative events play a central role in our analysis. Following Savage (1954) (see also de Finetti, 1931, 1937), we define $\succcurlyeq$ on $\mathcal{A}$ from $\succcurlyeq$ on $\mathcal{F}$ through "bets on events": $A\succcurlyeq B$ if there exist consequences $\alpha\succ\beta$ such that $\alpha_A\beta \succcurlyeq \alpha_B\beta$; Postulate P5 below will ensure that such acts exist in $\mathcal{F}$. An event $A\in\mathcal{A}^{ua}$ is *null* if $f_Ah \sim g_Ah$ for all $f,g\in\mathcal{F}^{ua}$; it is *non-null* otherwise.

A function $v : \mathcal{A} \to [0,1]$ is a *capacity* if $v(\varnothing) = 0$, $v(S) = 1$, and v is *monotonic with respect to set-inclusion*, i.e., $A\supset B \Rightarrow v(A) \geq v(B)$. If $\mathcal{A}$ is an algebra, then the capacity v is a (*finitely additive*) *probability measure* if furthermore v is *additive*, i.e., $v(A\cup B) = v(A) + v(B)$ for all disjoint A,B. A capacity v is *convex-ranged* if for every $A\supset C$ and every μ between v(A) and v(C) there exists $A\supset B\supset C$ such that $v(B)=\mu$.

3. **THE MAIN RESULT**

We use Savage's setup to formulate our axioms that lead to a nonadditive capacity measure. Since our main result generalizes Savage (1954), Machina & Schmeidler (1990, hereafter abbreviated M&S), and Sarin & Wakker (1990), we note here some important differences.

Savage’s Postulate P2 (the sure-thing principle) will not be used in our main theorem. This was dropped in M&S as well, but was assumed for unambiguous acts in Sarin & Wakker (1990). Instead we use P2*, which can be seen to be the restriction of the sure-thing principle to unambiguous two-consequence acts and is implied by the P4* condition used by M&S. We drop M&S's P4* and instead use a cumulative dominance condition.

We first state the axioms and then the main theorem. Special cases are discussed in the next section.

POSTULATE P1. Weak ordering.

POSTULATE P2* (*sure-thing principle for unambiguous two-consequence acts*). For all consequences $\alpha \succ \beta$ and unambiguous events A,B,H with $A \cap H = B \cap H = \varnothing$:

$$\alpha_A\beta \succcurlyeq \alpha_B\beta \quad \Leftrightarrow \quad \alpha_{A \cup H}\beta \succcurlyeq \alpha_{B \cup H}\beta.$$

POSTULATE P3. For all events $A \in \mathcal{A}$, acts $f \in \mathcal{F}$, and consequences α,β: $\alpha \succcurlyeq \beta \Rightarrow \alpha_A f \succcurlyeq \beta_A f$ whenever the latter two acts are contained in $\mathcal{F}$. The reversed implication holds as well if $A \in \mathcal{A}^{ua}$, A is nonnull, and $f \in \mathcal{F}^{ua}$.

POSTULATE P4 (*cumulative dominance*). For all acts f,g we have:

$$f \succcurlyeq g \;\; \text{whenever} \;\; \{s \in S: f(s) \succcurlyeq \alpha\} \succcurlyeq \{s \in S: g(s) \succcurlyeq \alpha\} \text{ for all consequences } \alpha.$$[3]

POSTULATE P5 (*nontriviality*). There exist consequences $\alpha \succ \beta$ such that $\alpha_A \beta_{A^c} \in \mathcal{F}$ for all events $A \in \mathcal{A}$.

POSTULATE P6 (*fineness of the unambiguous events*). If $\alpha \in \mathcal{C}$ and, for $f \in \mathcal{F}^{ua}, g \in \mathcal{F}$, $f \succ g$, then there exists a partition $(A_1,...,A_m)$ of S, with all elements in $\mathcal{A}^{ua}$, such that $\alpha_{A_j} f \succ g$ for all j, and the same holds with $\prec$ instead of $\succ$.

Postulates P1-P6 are used to derive a capacity measure v over events in a general decision model that will now be described. For a simple probability distribution over $\mathcal{C}$, the *cumulative distribution function* assigns to each consequence $\alpha \in \mathcal{C}$ the probability of $\{\beta \in \mathcal{C}: \beta \succcurlyeq \alpha\}$. It turns out that the present definitions, with $\succcurlyeq$ instead of $\leq$ as used in distribution functions in probability theory, are more convenient. For an act f and a capacity v, the *cumulative distribution function* $F_{f,v}: \mathcal{C} \rightarrow [0,1]$ is defined by $F_{f,v}: \alpha \mapsto v(\{s \in S: f(s) \succcurlyeq \alpha\})$. If $\alpha^1 \succcurlyeq ... \succcurlyeq \alpha^m$ and $\{\alpha^1, ..., \alpha^m\} \supset \text{range}(f)$, then we may denote $F_{f,v}$ by $(\alpha^1,v_1;...;\alpha^m,v_m)$, where $v_j := v(\{s \in S: f(s) \succcurlyeq \alpha^j\})$ for all j. Note that

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[3]Let us comment on a technical detail in relation to P4 in Sarin & Wakker (1990). That condition needed to involve events that were inverses under act f of so-called "cumulative consequence sets". These events are, for general acts, more general than the cumulative events as defined in this paper. For simple acts as considered in this paper, however, the events are truly identical. Hence we chose now the simplest formulation. In general, the formulation of Sarin & Wakker (1990) should be adopted.

each cumulative distribution function generated by an act and a capacity in this manner can be associated with the simple probability distribution over $\mathcal{C}$ that assigns probability $v_i - v_{i-1}$ to consequence α^i; set here $v_0=0$. Thus, for a fixed capacity v on S, the set of cumulative distribution functions generated by the acts is a subset of the set of cumulative distribution functions generated by simple probability distributions over $\mathcal{C}$. In the main Theorem 3.1 below, the two sets will turn out to be identical. Also note that our notation automatically implies, for $\alpha^{i-1}=\alpha^i$, the equalities $v_{i-1}=v_i$ and $(\alpha^1,v_1;..;\alpha^{i-1},v_{i-1};\alpha^i,v_i;\alpha^{i+1},v_{i+1};..;\alpha^m,v_m) = (\alpha^1,v_1;..;\alpha^{i-1},v_{i-1};\alpha^{i+1},v_{i+1};..;\alpha^m,v_m)$.

A function $\hat{V}$ is a *cumulative distribution functional* if its range is $\mathbb{R}$ and its domain consists of all the cumulative distribution functions generated by simple probability distribution functions over $\mathcal{C}$. Further, a cumulative distribution functional $\hat{V}$ is required to satisfy (strict first-order) stochastic dominance[4] and *mixture continuity*. The latter condition, proposed by M&S, requires, for each pair of cumulative distribution functions F_f and F_g, continuity of $\lambda \mapsto \hat{V}(\lambda F_f+(1-\lambda)F_g)$ on [0,1]. Note that this implies that the range of $\hat{V}$ is convex, i.e., an interval. It is easily verified that stochastic dominance is equivalent to strict monotonicity with respect to the v_i's, and α^i's with positive $v_i - v_{i-1}$ in $(\alpha^1,v_1;...;\alpha^i,v_i;...;\alpha^m,v_m)$, and mixture continuity is equivalent to continuity in the v_i's. A function $V:\mathcal{F}\to\mathbb{R}$ is a *cumulative capacity functional* if it agrees with a cumulative distribution functional, i.e., there exist a capacity v (the capacity *related to* V) and a cumulative distribution functional $\hat{V}$ such that $V(f) = \hat{V}(F_{f,v})$ for all acts f. Under the conditions of the theorem below, the capacity related to $\hat{V}$ will be determined uniquely.

We call a function *ordinal* if it can be replaced by another function if and only if the other function is a continuous strictly increasing transform of the original function; we abstain from the possibly more accurate term continuously ordinal.

THEOREM 3.1. *The following two statements are equivalent:*

(i) **There exists a non-constant cumulative capacity functional V that represents $\succcurlyeq$. On $\mathcal{A}^{ua}$ the capacity v, related to V, is additive and convex-ranged; on $\mathcal{F}^{ua}$ the functional V is mixture-continuous.**

(ii) **Postulates P1,P2*,P3, ..., P6 are satisfied.**

Further, the function V *is ordinal and the capacity v in (i) is unique.* □

PROOF.

CASE 1. Suppose (ii) holds. We derive (i). From Savage (1954) we obtain a unique atomless convex-ranged additive probability measure P on $\mathcal{A}^{ua}$, such that for all $\alpha\succ\beta$ and

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[4]I.e., $\hat{V}(F_{f,v})>\hat{V}(F_{g,v})$ whenever $F_{f,v}\neq F_{g,v}$ and $F_{f,v}\geq F_{g,v}$ on its entire domain.

unambiguous events A,B, $[\alpha_A\beta_{A^c}\succcurlyeq\alpha_B\beta_{B^c} \Leftrightarrow A\succcurlyeq B \Leftrightarrow P(A)\succcurlyeq P(B)]$. Following Sarin & Wakker (1990, Lemma A.1 and below) we get:

$$\text{For each event A there exists an } A^{ua}\in \mathcal{A}^{ua} \text{ such that } A\sim A^{ua}, \tag{3.1}$$

from which the capacity $v(A)=P(A^{ua})$ can be defined; monotonicity of v with respect to set-inclusion is implied mainly by P5 and P3. We consider now two cases.

CASE 1a. There exists a maximal consequence μ and a minimal consequence ν.

Following Sarin & Wakker (1990, Lemma A.2), there exists for each act f a probability $0\le p\le 1$ such that the unambiguous act that assigns probability p to μ and probability $1-p$ to ν is equivalent to f. We set $V(f)=p$. This functional indeed represents $\succcurlyeq$, and is non-constant. We must show that it satisfies the other conditions in the Theorem. By two-fold application of P4 (once with $\succcurlyeq$, once with $\preccurlyeq$), different acts with the same cumulative distribution function are equivalent, so they receive the same V value. Therefore we can define the functional $\hat{V}$ on the set of cumulative distribution functionals of simple acts such that $V(f) = \hat{V}(F_{f,v})$ for all simple acts f. Note that by convex-rangedness of P, the domain of $\hat{V}$ indeed consists of all cumulative distribution functions of simple probability distributions over $\mathcal{C}$. From P4 it follows immediately that $\hat{V}(F_{f,v})\ge\hat{V}(F_{g,v})$ whenever $F_{f,v}\ge F_{g,v}$ on its entire domain. Let us now show strict inequality for $F_{f,v}\ne F_{g,v}$. There exist unambiguous f^{ua},g^{ua} with the same distribution functions as f and g, so we can assume $f^{ua}=f$, $g^{ua}=g$. It is elementarily shown that for $F_{f,v}\ge F_{g,v}$ and $F_{f,v}\ne F_{g,v}$ there exist unambiguous f' and g' with the same distribution functions, but such that $f'(s)\succcurlyeq g'(s)$ for all states s. There must exist a nonnull event to which f' assigns one consequence, and g' one strictly dispreferred consequence. From P3,P4, and transitivity, it straightforwardly follows that $f'\succ g'$. Indeed $\hat{V}(F_{f,v})>\hat{V}(F_{g,v})$, and strict stochastic dominance is satisfied. The derivation of mixture-continuity follows mainly from P6, and is similar to M&S; note that in our setup it suffices to derive the condition on $\mathcal{F}^{ua}$, as the latter set generates all simple distribution functions. Indeed V may be called a cumulative capacity functional.

Uniqueness of the capacity follows from uniqueness of the probability measure P as established by Savage, and (3.1). It is immediate that another function V' represents $\succcurlyeq$ if and only if V' is a strictly increasing transform of V. It is also immediate that any continuous strictly increasing transform of V gives another functional satisfying all conditions of the theorem, including mixture continuity for $\hat{V}$. Finally, that only strictly increasing transformations can be applied that are continuous, follows because the range of V is equal to the range of V when restricted to the unambiguous acts, and this is convex, so an interval, by mixture continuity of $\hat{V}$.

Case 1a is now completed.

CASE 1b. There does not exist a maximal consequence, or there does not exist a minimal consequence. Let $\mu \succ \nu$ be two consequences. If a maximal consequence exists, let μ be that consequence; if a minimal consequence exists, let ν be that consequence. Further μ and ν are arbitrary. Suppose there does not exist a maximal consequence. We first construct a sequence $\mu \prec \mu^1 \prec \mu^2 \prec ...$ such that each consequence is dominated by some μ^j. For any consequence $\alpha \succ \mu$ there exists, similarly to (3.1) and Lemma 13 in Sarin & Wakker (1990), a $0<p_\alpha<1$ such that the act giving α with probability p_α, and ν with probability $1-p_\alpha$, is equivalent to μ. Mainly by P3, $\beta \succ \alpha$ implies $p_\beta<p_\alpha$. Because there does not exist a maximal consequence, there does not exist a minimal p_α. Let $\mu \preccurlyeq \mu^1 \preccurlyeq \mu^2 \preccurlyeq ...$ be such that the associated sequence $p_{\mu^1}, p_{\mu^2}, ...$ tends to the infimum of the p_α's. Then each consequence $\alpha \succ \mu$ is dominated by a μ^j.

Similarly, if there does not exist a minimal consequence, then we construct $\nu \succcurlyeq \nu^1 \succcurlyeq \nu^2 \succcurlyeq ...$ such that each consequence dominates some ν^j.

Below, if μ is maximal, we simply set $\mu^1=\mu^2= ... =\mu$; if ν is minimal, then set $\nu^1=\nu^2= ... =\nu$. By Case 1a, we get a representing functional V^m of the desired kind on the set of acts with consequences in $\{\alpha \in C: \nu^m \preccurlyeq \alpha \preccurlyeq \mu^m\}$. We can further ensure that the range of V^m is a subset of $[-m,m]$, and coincides with the range of V^{m-1} on common domain. Note that uniqueness of the capacity from Case 1a implies that the capacities associated with V^1,V^2, etc., all coincide. Thus all functions V^m can be combined into one function V that incorporates all acts in its domain. The conditions for V are straightforwardly verified; ordinality follows primarily because the range of V is an interval.

Case 1b, thus Case 1, is completed.

CASE 2. Suppose (ii) holds. We derive (i). Postulate (i) ($\succcurlyeq$ is a weak order) is direct; P2* follows from additivity of the capacity v on A^{ua} and strict stochastic dominance; Postulates P3 and P4 follow from strict stochastic dominance; P5 follows from non-constantness of V; finally, P6 follows mainly from mixture continuity of the functional, similarly to M&S.

This completes the proof of Case 2, and thus of the theorem. □

It is assumed in Theorem 3.1 that in order to elicit a capacity v, a sub σ-algebra of unambiguous events, A^{ua}, has been preselected. The choice of a particular A^{ua} however, is left to the decision maker. Example A1 in the appendix shows that there may be several sub σ-algebras that satisfy the postulates for $\mathcal{A}^{ua}$ but result in different capacities. Thus, the interpretation of $\mathcal{A}^{ua}$ as σ-algebra of unambiguous events cannot be justified solely by satisfaction of the Postulates in the main Theorem 3.1. It requires in addition a subjective evaluation of the decision maker that indeed the events of $\mathcal{A}^{ua}$ are unambiguous. Alternatively, the decision maker may omit any such subjective evaluation and simply let the capacity depend on the particular set $\mathcal{A}^{ua}$, where the capacity now is strictly *relative to* $\mathcal{A}^{ua}$. The generality of the representing cumulative capacity model does not allow a unique determination of a σ-algebra $\mathcal{A}^{ua}$. The dependence of a capacity on a chosen $\mathcal{A}^{ua}$ is related to Tversky & Kahneman's idea of source-dependence.

4. SUPERADDITIVITY, SUBADDITIVITY, AND COMPLEMENTARITY OF THE CAPACITY

The method to calibrate capacities through P4 is elementary but has proven to be powerful in Theorem 3.1. Without further restrictions, the capacity function is quite general. In the following sections we will formulate behavioral conditions that restrict capacities in a way that resembles closely the calibration technique. This will lead to transparent results, and again we hope that this transparency is considered a virtue; compare Figure 4.1 below. For simplicity we will assume henceforth that $\mathcal{A}$ is a σ-algebra.

The first property we consider is "superadditivity". A capacity is *superadditive* if, for all events A,B, $v(A)+v(B) \leq v(A\cup B)+v(A\cap B)$. By substituting $A_1=A\cap B$, $A_2=A\backslash B$, $A_3=B\backslash A$, we can rewrite this condition:

$$\text{For all disjoint events } A_1,A_2,A_3$$
$$v(A_1\cup A_2) - v(A_1) \leq v(A_1\cup A_2\cup A_3) - v(A_1\cup A_3). \tag{4.1}$$

That is, the marginal capacity-contribution of event A_2 increases as the set to which it is added increases. *Subadditivity* holds if the inequality (4.1) is reversed. Since we can assess the capacity of an event by comparing it with an unambiguous event, the following approach seems natural to characterize super- and subadditivity. Suppose that for disjoint events A_1,A_2,A_3, there exist disjoint unambigous events B_1,B_2,B_3, such that

$$B_1\sim A_1,\ B_1\cup B_2 \sim A_1\cup A_2,\ B_1\cup B_3 \sim A_1\cup A_3. \tag{4.2}$$

So, B_1 matches A_1, B_2 matches the increment from A_1 to $A_1 \cup A_2$, and B_3 matches the increment from A_1 to $A_1 \cup A_3$. By substitution it follows that (4.1) holds for A_1,A_2,A_3, if and only if

$$B_1 \cup B_2 \cup B_3 \preccurlyeq A_1 \cup A_2 \cup A_3. \tag{4.3}$$

That is, the joint increment of A_2 and A_3 is greater than the joint increment of B_2 and B_3. This idea is formally different, but closely related to, the notion "curved relative to" of Krantz & Tversky (1975). In terms of the capacity, the above conditions say that $v(A_1 \cup A_2) - v(A_1) = P(B_2)$ and $v(A_1 \cup A_2 \cup A_3) - v(A_1 \cup A_3) \geq P(B_2)$, in accordance with the definition of superadditivity. Formula (4.1) with reversed inequality holds if and only if

$$B_1 \cup B_2 \cup B_3 \succcurlyeq A_1 \cup A_2 \cup A_3. \tag{4.4}$$

So (4.3) and (4.4) show how the elicitation procedure of this paper can be used to elicit and characterize super- and subadditivity.

There remains one final complication to be discussed. That concerns the case where no B_1,B_2,B_3 exist to satisfy (4.2); this will occur for example if $v(A_1)<1$, $v(A_1 \cup A_2)=v(A_1 \cup A_3)=v(A_1 \cup A_2 \cup A_3)=1$, which by (4.2) would force $P(B_1)+P(B_2)+P(B_3)$ to be larger than 1, i.e., B_1,B_2,B_3, as in (4.2) do not exist. Formula (4.5) below will show that this can only happen if (4.1) holds with $>$ instead of $\leq$, i.e., if A_1,A_2,A_3 satisfy the subadditivity inequality. Thus, if no B_1,B_2,B_3 exist to satisfy (4.2), then superadditivity of v is necessarily violated. So we define: $\succcurlyeq$ exhibits *superadditivity* if for each triple of disjoint events A_1,A_2,A_3, there exist disjoint unambiguous events B_1,B_2,B_3 satisfying (4.2), and furthermore for all such triples of events, (4.3) holds; $\succcurlyeq$ exhibits *subadditivity* if for each triple of disjoint events A_1,A_2,A_3, and each triple of disjoint unambiguous events B_1,B_2,B_3 satisfying (4.2), (4.4) is satisfied.

LEMMA 4.1. Suppose that the conditions of Theorem 3.1 hold, and that $\mathcal{A}$ is a σ-algebra. Then v is superadditive if and only if $\succcurlyeq$ exhibits superadditivity, and v is subadditive if and only if $\succcurlyeq$ exhibits subadditivity.

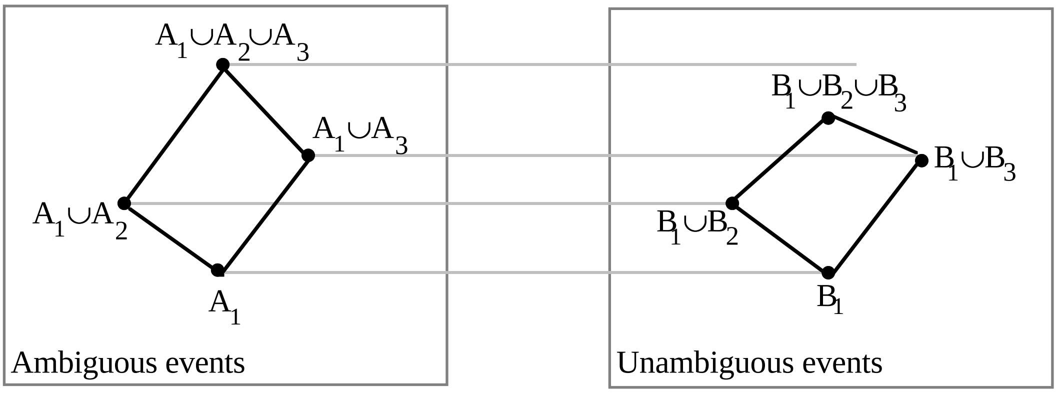

FIGURE 4.1 (*Elicitation of superadditivity*). Events have been revealed as more likely when they are higher. In the ambiguous world there appears to be additional appreciation for the joining of disjoint increments.

PROOF. As a preparation we show:

If $v(A_1 \cup A_2) - v(A_1) \leq v(A_1 \cup A_2 \cup A_3) - v(A_1 \cup A_3)$, then there exist B_1, B_2, B_3 that satisfy (4.2). (4.5)

By (3.1), there exists an unambiguous $B_{123} \sim A_1 \cup A_2 \cup A_3$. By convex-rangedness, we can find $B_{12} \subset B_{123}$ such that $B_{12} \sim A_1 \cup A_2$, and $B_1 \subset B_{12}$ such that $B_1 \sim A_1$. We define $B_2 := B_{12} \backslash B_1$, and $B_3' := B_{123} \backslash B_{12}$. Now

$$v(B_1 \cup B_2) + v(B_1 \cup B_3') - v(B_1) = v(B_1 \cup B_2 \cup B_3') = v(B_{123}) =$$
$$v(A_1 \cup A_2 \cup A_3) \geq v(A_1 \cup A_2) + v(A_1 \cup A_3) - v(A_1),$$

where the inequality follows from the antecedent in (4.5). Since $v(B_1 \cup B_2) = v(A_1 \cup A_2)$ and $v(B_1) = v(A_1)$, we conclude $v(B_1 \cup B_3') \geq v(A_1 \cup A_3)$. By convex-rangedness, there exists $B_1 \cup B_3 \subset B_1 \cup B_3'$ with $B_1 \cup B_3 \sim A_1 \cup A_3$. So (4.2), and thus (4.5), follows.

To show that superadditivity of the capacity implies that $\succcurlyeq$ exhibits superadditivity, suppose A_1, A_2, A_3 are disjoint. By (4.5) there exist B_1, B_2, B_3 satisfying (4.2); take also B_3' as above. Obviously, $v(B_1 \cup B_2 \cup B_3) \leq v(B_1 \cup B_2 \cup B_3') = v(B_{123}) = v(A_1 \cup A_2 \cup A_3)$, and (4.3) follows. Indeed $\succcurlyeq$ exhibits superadditivity.

To show that v is superadditive if $\succcurlyeq$ exhibits superadditivity, suppose A_1, A_2, A_3 are disjoint. By definition, there exist B_1, B_2, B_3 and preferences as in (4.2) and (4.3). By (4.2), $v(A_1 \cup A_2) + v(A_1 \cup A_3) - v(A_1) = v(B_1 \cup B_2) + v(B_1 \cup B_3) - v(B_1) = v(B_1 \cup B_2 \cup B_3)$. By (4.3), the latter is smaller/equal $v(A_1 \cup A_2 \cup A_3)$. The inequality $v(A_1 \cup A_2) + v(A_1 \cup A_3) - v(A_1) \leq v(A_1 \cup A_2 \cup A_3)$ yields superadditivity of v.

Next we turn to subadditivity. Suppose first that v is subadditive. To show that $\succcurlyeq$ exhibits subadditivity, take disjoint A_1,A_2,A_3, and suppose there exist B_1,B_2,B_3 satisfying (4.2); otherwise, we are done immediately. Then $v(B_1 \cup B_2 \cup B_3) = v(B_1 \cup B_2) + v(B_1 \cup B_3) - v(B_1)$ which, by (4.2), is equal to $v(A_1 \cup A_2) + v(A_1 \cup A_3) - v(A_1)$. By subadditivity of v, the latter is smaller/equal $v(A_1 \cup A_2 \cup A_3)$. So $v(B_1 \cup B_2 \cup B_3) \leq v(A_1 \cup A_2 \cup A_3)$, and (4.4) follows; $\succcurlyeq$ exhibits subadditivity.

Finally, we show that the capacity is subadditive if $\succcurlyeq$ exhibits subadditivity. Suppose A_1,A_2,A_3 are disjoint. If no B_1,B_2,B_3 exist that satisfy (4.2), then, by (4.5), the inequality of subadditivity must be satisfied, even strictly. So assume B_1,B_2,B_3 as in (4.2) exist. Then, by (4.4), $v(A_1 \cup A_2 \cup A_3) \leq v(B_1 \cup B_2 \cup B_3) = v(B_1 \cup B_2) + v(B_1 \cup B_3) - v(B_1)$. The latter is, by (4.2), equal to $v(A_1 \cup A_2) + v(A_1 \cup A_3) - v(A_1)$. The inequality $v(A_1 \cup A_2 \cup A_3) \leq v(A_1 \cup A_2) + v(A_1 \cup A_3) - v(A_1)$ implies subadditivity of v. □

Weak superadditivity requires (4.1) only if $A_1 = \varnothing$, i.e., when events A and B are disjoint. Weak subadditivity is defined similarly. Weak superadditivity is characterized by the restriction of (4.2) and (4.3) to empty A_1. Next we turn to the property of "complementarity". Gilboa (1989) argued in favor of this condition. The capacity v satisfies *complementarity* if $v(A) + v(A^c) = 1$ for all events A. The preference relation $\succcurlyeq$ *exhibits complementarity* if for all events A and unambiguous events B~A we have $B^c \sim A^c$. The following lemma is an easy corollary of (3.1), hence the proof is omitted.

LEMMA 4.2. Suppose that the conditions of Theorem 3.1 hold, and that $\mathcal{A}$ is a σ-algebra. Then v satisfies complementarity if and only if $\succcurlyeq$ exhibits complementarity. □

Alternatively, complementarity can be characterized by a variation of condition P4, as follows[5]:

LEMMA 4.3. Suppose that the conditions of Theorem 3.1 hold, and that $\mathcal{A}$ is a σ-algebra. Then v satisfies complementarity if and only if $f \succcurlyeq g$ whenever $\{s \in S: f(s) \preccurlyeq \alpha\} \preccurlyeq \{s \in S: g(s) \preccurlyeq \alpha\}$ for all consequences α.

PROOF. First suppose the variation of P4 holds. Consider, for any $\alpha \succ \beta$, acts of the form α_A,β_{A^c} and α_B,β_{B^c} with B unambiguous, and such that $v(B^c) = v(A^c)$. By twofold application of the variation of P4, the two acts are equivalent. This implies $v(A)=v(B)$. The latter is identical to $1-v(B^c)$, which is equal to $1-v(A^c)$. Complementarity of v follows.

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[5]The condition was proposed by Nehring (1991, personal communication).

Conversely, suppose v satisfies complementarity, and suppose that $v\{s\in S: f(s)\preccurlyeq\alpha\} \leq v\{s\in S: g(s)\preccurlyeq\alpha\}$. Then, by complementarity of v, $v\{s\in S: f(s)\succ\alpha\} \geq v\{s\in S: g(s)\succ\alpha\}$ for all α. This implies, by elementary manipulations, $v\{s\in S: f(s)\succcurlyeq\beta\} \geq v\{s\in S: g(s)\succcurlyeq\beta\}$ for all β (for simple f,g, replace α by a consequence β from the closest more preferred equivalence class from $f(S)\cup g(S)$). By P4, $f\succcurlyeq g$. □

Let us comment more on the above variation, seemingly dual, of P4. Recall that the more likely than relation $\succcurlyeq$ on events was derived from bets "on events": $A\succcurlyeq B$ if, for $\alpha\succ\beta$, $\alpha_A\beta\succcurlyeq\alpha_B\beta$. That is, A and B are cumulative events, describing the receipt of a consequence or anything better. It is natural to adopt this ordering of events in Postulate P4, where cumulative events are also compared. The above, seemingly dual, condition however, applies the more likely than relation as derived for events when they are cumulative to the case where these events have another role, i.e., they are "decumulative" (describing the receipt of a consequence or anything worse). As the above lemma demonstrates, this can only be done in the case of complementarity, where the ordering of events through bets on the events coincides with the ordering of events through bets against the events.

An alternative, and truly dual, characterization could have been obtained in Theorem 3.1, if P4 had been replaced by the condition requiring $f\succcurlyeq g$ whenever $\{s\in S: f(s)\preccurlyeq\alpha\} \preccurlyeq^* \{s\in S: g(s)\preccurlyeq\alpha\}$, where now the ordering $\preccurlyeq^*$ is derived from bets against events: $A\preccurlyeq^* B$ if, for $\alpha\succ\beta$, $\beta_A\alpha\succcurlyeq\beta_B\alpha$. (Here the measurability condition needs to be modified so that for each act all decumulative events are contained in $\mathcal{A}$.) Our version of P4 was chosen because betting on events is more natural than betting against events.

5. ADDITIVITY OF THE CAPACITY: "PROBABILISTICALLY SOPHISTICATED" PREFERENCES

This section considers the special case of additive capacities. We first show ways to obtain an alternative characterization of the result of M&S[6]. They considered the special case of Theorem 3.1 where $\mathcal{A}=\mathcal{A}^{ua}$, and called the characterized preference relations *probabilistically sophisticated*.

OBSERVATION 5.1. If $\mathcal{A}=\mathcal{A}^{ua}$ is taken, then Theorem 3.1 gives a characterization of probabilistically sophisticated preferences. □

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[6]Note that they use the term "non-atomic" instead of our term convex-ranged".

There are two differences between the above observation and the result of M&S. The main difference is that we replaced their "strong comparative probability axiom" P4* by P2* and P4. Example 5.2 demonstrates that (with $\mathcal{A}=\mathcal{A}^{ua}$) our axioms except the structural P6 do not imply the strong comparative probability axiom, so are not stronger. Conversely, P2* is immediately implied by the strong comparative probability axiom (see M&S, end of Subsection 4.1). It can be shown however that in the absence of P6, the strong comparative probability axiom does not imply our P4. Thus, in the absence of P6, the conditions are logically independent. Obviously, in the presence of the other conditions, given $\mathcal{A}=\mathcal{A}^{ua}$, they are logically equivalent because they characterize the same preferences.

EXAMPLE 5.2 (In the absence of P6, the strong comparative probability axiom of M&S is not implied by P2* and P4). Let $S=\{s_1,s_2,s_3\}$, with every subset an event, and every event unambiguous. P is an additive probability measure on S, described by $P(\{s_1\})=\frac{4}{9}$, $P(\{s_2\})=\frac{3}{9}$, $P(\{s_3\})=\frac{2}{9}$. The consequence set $\mathcal{C}$ is $\{0,1,9\}$. Suppose $\succcurlyeq$ maximizes expected value, with one exception: $(0_{s_1},9_{s_2},1_{s_3}) \sim (0_{s_1},1_{s_2},9_{s_3})$. Obviously condition P6 is not satisfied. Conditions P1,P2*,P3,P4,P5 are satisfied. The only violation that might be expected is when the equivalence $(0_{s_1},9_{s_2},1_{s_3}) \sim (0_{s_1},1_{s_2},9_{s_3})$ is involved. However, no violation of the conditions, in particular of P4, occurs. It can be seen that the strong comparative probability axiom of M&S is violated, e.g., by the preferences $(0_{s_1},1_{s_2},9_{s_3}) \succcurlyeq (0_{s_1},9_{s_2},1_{s_3})$ and $(0_{s_1},0_{s_2},1_{s_3}) \prec (0_{s_1},1_{s_2},0_{s_3})$. □

The second difference between Observation 5.1 and the result of M&S is that Observation 5.1 does not require the existence of maximal and minimal consequences. The analysis of Case 1b in the proof of Theorem 3.1 shows that this restrictive assumption could also have been dropped in M&S.

Theorem 3.1 makes further generalizations of the M&S result possible. For example, cases where $\mathcal{A}\neq\mathcal{A}^{ua}$ can be incorporated by including conditions that imply additivity of v. The additivity of the capacity v is easily obtained by (5.1) or (5.2) below.[7] Note that (5.2) is stronger than (5.1), but it may be more intuitive. This generalizes Observation 5.1 and the result of M&S because conditions such as P6 are invoked only for the subset $\mathcal{A}=\mathcal{A}^{ua}$, rather than for all events.

$$\text{For all disjoint events } A \text{ and } A', \text{ and disjoint unambiguous events } B^{ua}\sim A,\ B^{ua'}\sim A', \text{ we have } A\cup A' \sim B^{ua}\cup B^{ua'}. \tag{5.1}$$

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[7]The proof is obtained mainly by (3.1), and is omitted.

P4*. For each partition $\{A_1,\ldots,A_m\}$, and unambiguous partition $\{B_1^{ua},\ldots,B_m^{ua}\}$, of S, we have:

$$A_j \sim B_j^{ua} \text{ for all } j \;\Rightarrow \qquad (5.2)$$
$$\alpha^1_{A_1}\ldots\alpha^m_{A_m} \;\sim\; \alpha^1_{B_1^{ua}}\ldots\alpha^m_{B_m^{ua}}$$
$$\text{for all consequences } \alpha^1,\ldots,\alpha^m.$$

Obviously, Theorem 3.1 and Formula (5.2) can be used to determine additivity of the capacity on subclasses of events, larger than $\mathcal{A}^{ua}$. For instance, if (5.2) is imposed on one fixed partition $\{A_1,\ldots,A_m\}$, or on all partitions from some subalgebra larger than $\mathcal{A}^{ua}$, then the capacity is additive there.

6. SPECIAL FORMS OF CUMULATIVE CAPACITY FUNCTIONALS

In this section special forms of the cumulative capacity functional will be studied; see Figure 6.1. Again, we assume that $\mathcal{A}$ is a σ-algebra. In particular, we shall consider the case of Choquet expected utility and versions of weighted utility and quadratic utility with a nonadditive capacity v, and relate these to the case where v is additive. *Choquet expected utility* (*CEU*) is the special case of a cumulative capacity functional where there exists a *utility* function $u:\mathcal{C}\rightarrow\mathbb{R}$ such that, for $f=(\alpha^1{}_{A_1},\ldots,\alpha^m{}_{A_m})$, $\alpha^1 \succcurlyeq \ldots \succcurlyeq \alpha^m$,

$$V(f) \;=\; u(\alpha^m) + \sum_{i=1}^{m-1}\big(u(\alpha^i)-u(\alpha^{i+1})\big)v(\{A_1\cup\ldots\cup A_i\}). \qquad (6.1)$$

This form is characterized by replacing P2* with Savage’s sure-thing principle P2, restricted to the unambiguous acts; see Sarin & Wakker (1990). In a similar way, by retaining P2* and replacing P2 with P4*, we obtain the M&S model, see Section 5. Finally, by assuming both P2 and P4*, we obtain the subjective expected utility model.

Table 6.1 demonstrates the generality of the cumulative capacity functional in comparison to alternative models. The class of models that permit nonadditive capacities for ambiguous events, but require expected utility maximization for the unambiguous events (Schmeidler, 1989, or Sarin & Wakker, 1990), is consistent with the choices in the third and fourth pair, but is inconsistent with the choices in the first and second pair. The M&S model does the reverse,– it can describe the choices in the first and second pair but fails to describe the pattern in the third and fourth pair. The cumulative capacity functional is consistent with the entire pattern of choices over the four pairs.

In general, we can specialize our cumulative capacity functional to several alternative

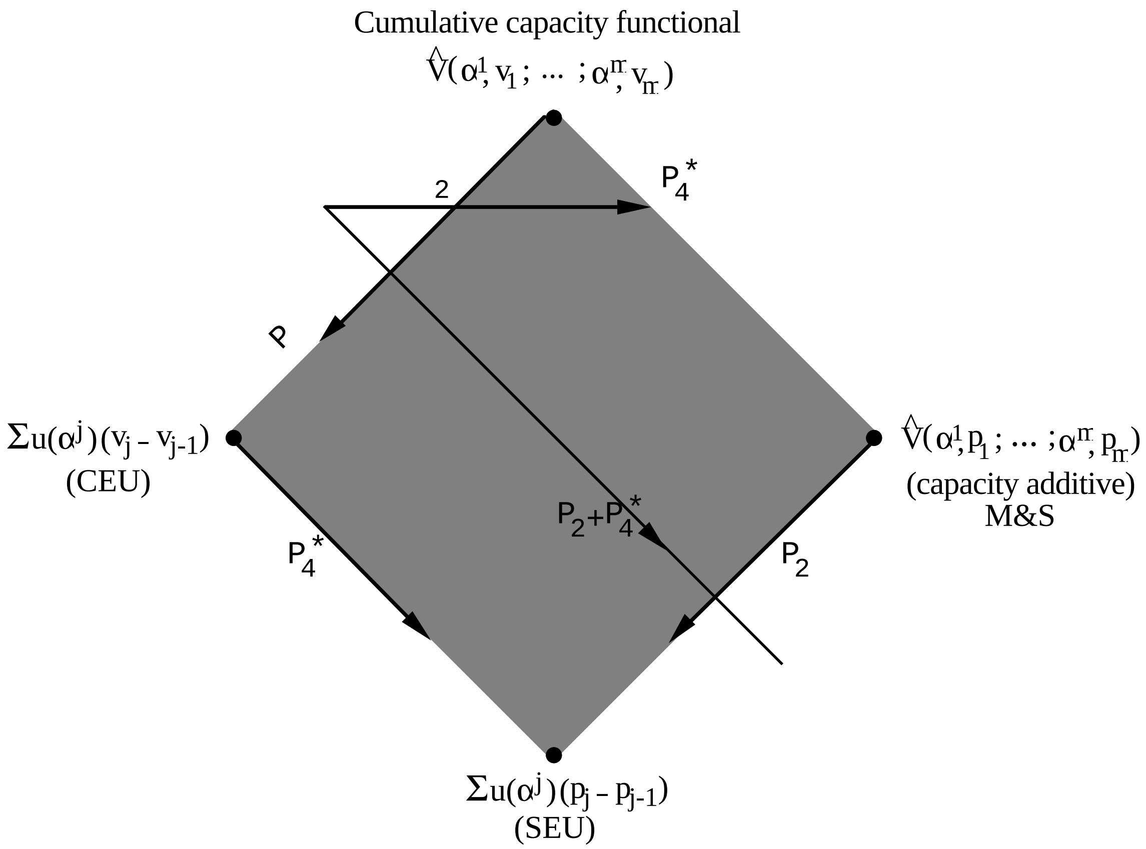

FIGURE 6.1 (*Special cases of cumulative capacity functionals*). Let $f = (\alpha^1_{A_1}, \dots, \alpha^{rr}_{A_{rr}})$, $\alpha^1 \succcurlyeq \dots \succcurlyeq \alpha^{rr}$. Let $v_j = v(A_1, \dots, A_j)$. We write p_j for v_j if v is additive; recall that p_j denotes a "cumulative" probability. P_2 is imposed only on the unambiguous acts.

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		(20)	(25)	(40	)	(15)
		Red	Black	Yellow	Orange	White
*	a_1	3000	3000	3000	3000	3000
	a_2	0	4000	4000	4000	4000
	a_3	0	3000	0	0	0
*	a_4	4000	0	0	0	0
*	a_4	4000	0	0	0	0
	a_5	0	0	4000	0	0
	a_6	4000	0	0	4000	0
*	a_7	0	0	4000	4000	0

TABLE 6.1. A ball will be drawn from an urn. The sum of the number of yellow and orange balls is 40, but the exact number of yellow or orange balls is not known. In each

pair of acts the act marked with an asterisk is the preferred choice for a majority of subjects.

forms. Postulate P4 gives the tool to extend any model for decision making under risk (e.g., rank-dependent utility, weighted utility[8], quadratic utility[9]) that respects transitivity, completeness, stochastic dominance, and mixture continuity[10] to the case of ambiguity with nonadditive capacities. For any ambiguous act one calculates the cumulative distribution function and applies the particular model for decision making under risk to that cumulative distribution function. Thus, we can for instance obtain a version of weighted utility for nonadditive measures of belief. This shows that the cumulative capacity functional is indeed strictly more general than CEU. Axioms that lead to other specialized forms for cumulative capacity functionals deserve to be explored in future research.

7. **DISCUSSIONS**

The flexibility that a nonadditive capacity measure affords to a decision maker can easily be seen by means of a simple example. Suppose an urn is filled with 100 balls. There are 20 red balls, the number of yellow-or-white balls sums to 40, the number of green-or-white balls sums to 40, and the remainder is blue. Obviously, there are as many blue balls as white balls. Colors are abbreviated below by their first letter. Suppose a decision maker expresses a preference for betting on R over Y ($R \succ Y$). Now, a probability measure will require that $R \cup W \succ Y \cup W$ and $R \cup W \cup G \succ Y \cup W \cup G$ and so on. However, if a person likes specificity and dislikes ambiguity, then he may exhibit the pattern $R \succ Y$, $R \cup W \prec Y \cup W$, $R \cup W \cup G \succ Y \cup W \cup G$, $R \cup W \cup G \cup B \prec Y \cup W \cup G \cup B$, which is consistent with a nonadditive capacity but not with a probability measure. Since such preference patterns are commonly observed empirically, the additional flexibility of the capacity measure may be useful in descriptive applications.

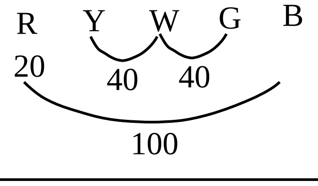

Figure 7.1 (*Numbers of balls in the urn*).

[8]See Chew (1983, Econometrica).

[9]See Chew, Epstein, & Segal (1991).

[10]That is, the models as described by M&S, or, equivalently, Observation 4.4.

One objection could be that a money pump (or Dutch book) can be made out of a person with a nonadditive capacity. Suppose a person is offered a bet: win \$10 if Y; win \$0 otherwise. Assuming the person is a CEU maximizer with linear utility, he pays 10.v(Y) for the bet. Now, he is offered a bet: win \$10 if W, win \$0 otherwise. Supposing he evaluates this bet as before, he pays 10.v(W). Finally he is offered a bet: win \$–10 if W or Y; win \$0 otherwise; he pays $-10.[1-v((W\cup Y)^c)]$. It is easily seen that the three bets jointly yield \$0 no matter what. However, his total payment will be zero only if $v(W) + v(Y) + v((W\cup Y)^c) =1$, which, in general, is equivalent to additivity of the capacity. In this example, if $v(W) + v(Y) + v((W\cup Y)^c) > 1$, then the person can be milked a positive amount. If $v(W) + v(Y) + v((W\cup Y)^c) < 1$, then the person can also be milked a positive amount, in the situation where the person owns the three lotteries, and sells them sequentially, at the prices as described above. It is to be noted that the money pump argument requires that a person evaluates each bet independently. A person who exhibits a nonadditive capacity will, however, evaluate a bet differentially depending on whether it was the only bet he undertook or if he took it in conjunction with other bets. The above money pump argument does not therefore pose a problem for a person with a nonadditive capacity.

Suppose we define a relation "is more likely than" through introspection or some mechanism other than betting behavior. Such a relation may indeed satisfy some axioms that imply an additive function a person may use to order the events by their perceived likelihood of occurrence. Yet, his betting behavior on these events may reveal a nonadditive capacity. The capacity measure may not reflect the degree of likelihood of occurrence of an event. It is centrally focused on predicting peoples' betting behavior. In a race with two rabbits (assuming no ties and that one rabbit must win) it makes no sense to say that the brown rabbit has a 0.4 chance of winning and the white rabbit has a 0.4 chance of winning. A decision maker could, however, strictly prefer to bet on the toss of a fair coin than bet on either the white or the brown rabbit. In fact his betting behavior may well reveal the capacity associated with the event white rabbit wins to be 0.4 and the same for the event brown rabbit wins. Our decision maker could even agree that the chance of winning for either rabbit is 0.5. His betting behavior may be influenced by a low confidence in his judgment or anticipated regret. He may self-blame if, for example, he bets on the brown rabbit and finds it trailing by a long margin. He could lose on the toss of a coin as well. But in this case he attributes his misfortune to bad luck and not to bad judgment. An alternative explanation can be provided by source-dependence, see Tversky & Kahneman (1990). The capacity measure permits one's betting behavior to be influenced by such psychological concerns. In contrast, the probability measure imposes a stringent code of consistency that effectively requires that these widely observed psychological concerns should play no role in peoples' choices. It is conceivable that in the future researchers may agree that the capacity measure provides a better description of peoples' actual behavior while the probability measure is an ideal to be attained upon reflection and introspection.

APPENDIX

EXAMPLE A1. This example builds on the idea of "source-dependence" as explicated in several works by Tversky, see for instance Heath & Tversky (1991) and Tversky & Kahneman (1990). The example shows that the capacity v resulting from Theorem 3.1 depends on the particular σ-algebra $\mathcal{A}^{ua}$. That is, there may be several sub σ-algebras that satisfy the axioms for $\mathcal{A}^{ua}$, and that yield different capacity measures.

Suppose $S = [0,1] \times [0,1] \times \{\text{rain,no rain}\}$, where, for a state (s_1,s_2,s_3), s_1 describes the outcome of a fairly constructed roulette wheel W_1, s_2 describes the outcome of an asymmetric roulette wheel W_2, and s_3 describes whether or not it will rain. Let $\mathcal{A}_1$ be the σ-algebra describing events related to the first, fair, roulette wheel, i.e., events of the form $A_1 \times [0,1] \times \{\text{rain,no rain}\}$ for Borel sets $A_1 \subset [0,1]$. $\mathcal{A}_2$ is the σ-algebra describing events related to the second, asymmetric, roulette wheel, i.e., events of the form $[0,1] \times A_2 \times \{\text{rain,no rain}\}$ for Borel sets $A_2 \subset [0,1]$. Suppose that $\mathcal{C}=[0,M]$, with $u:\mathcal{C} \rightarrow \mathbb{R}$ denoting the identity function. For acts depending solely on events from $\mathcal{A}_1$, the cumulative capacity functional is expected (value =) utility. For acts f depending only on $\mathcal{A}_2$, V is the "rank-dependent form with a quadratic probability transformation function". That is,

$$V(f) = \int_{\mathbb{R}_+} [P(\{s \in S: f(s) \geq \tau\})]^2 d\tau,$$

where P denotes the Lebesgue measure. For example, the act receiving \$1 conditional upon the event $\{(s_1,s_2,s_3) \in S: s_2 \leq \frac{1}{2}\}$ has V value $\int_{[0,1]} [1/2]^2 d\tau + 0 = \frac{1}{4}$, the act receiving \$1 conditional upon the event $\{(s_1,s_2,s_3) \in S: s_1 \leq \frac{1}{4}\}$ has V value $\frac{1}{4}$ as well. We assume both acts are equivalent to receiving \$1 if it rains.

Formally, we can take either $\mathcal{A}_1$ or $\mathcal{A}_2$ as the σ-algebra $\mathcal{A}^{ua}$ of unambiguous events. In both cases all axioms are satisfied, and the additive probability measure will be the Lebesgue measure, representing $\succcurlyeq$ on $\mathcal{A}^{ua}$. Denote the resulting capacities by v_1 and v_2 respectively. In the first case,

$$v_1(\text{rain}) = v_1\big(\{(s_1,s_2) \in S: s_1 \leq \tfrac{1}{4}\}\big) = \frac{1}{4} \quad (= v_1\big(\{(s_1,s_2) \in S: s_2 \leq \tfrac{1}{2}\}\big)),$$

in the second case

$$v_2(\text{rain}) = v_2\left(\{(s_1,s_2)\in S: s_2\leq\frac{1}{2}\}\right) = \frac{1}{2} \quad (= v_2\left(\{(s_1,s_2)\in S: s_1\leq\frac{1}{4}\}\right)).$$

So $v_1\neq v_2$. $P=v_1\neq v_2$ on $\mathcal{A}_1$, $P=v_2\neq v_1$ on $\mathcal{A}_2$. One easily derives $v_1 = (v_2)^2$. In this case v_1 seems the appropriate capacity. This decision, however, is based on subjective evaluations, and cannot be inferred from the axioms in Theorem 3.1 because the axioms are satisfied for both capacities. □

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