BELIEF HEDGES: APPLYING AMBIGUITY MEASUREMENTS TO ALL EVENTS AND ALL AMBIGUITY MODELS[1]

AURÉLIEN BAILLON[a], HAN BLEICHRODT[a], CHEN LI[a], &PETER P. WAKKER[a]
a: Erasmus School of Economics, Erasmus University Rotterdam, Rotterdam, the Netherlands, Baillon@ese.eur.nl, Bleichrodt@ese.eur.nl, c.li@ese.eur.nl; Wakker@ese.eur.nl

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When measuring ambiguity attitudes one should control for subjective beliefs, but those are usually not directly observable. Hence, measurements focused on artificial events (secretized urns or researcher-specified probability intervals), where beliefs could be inferred from symmetry conditions. For application-relevant events, however, such symmetries are rarely available. This paper solves this problem. We show that ambiguity attitudes then can still be identified by using belief hedges—collections of events that protect against unknown beliefs. We define two indexes of ambiguity attitudes that can be used under all ambiguity models popular today. This solves a second problem: there are (too) many ambiguity models today, leaving practitioners at a loss to choose one. Our indexes are compatible with virtually all existing indexes and ambiguity orderings wherever those are defined, showing that they properly capture the general ambiguity concepts. We ensure that the indexes are directly observable from revealed preferences, and we axiomatize them.

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[1] A first draft of this paper was entitled: "Balanced Design: The Key to Measuring Ambiguity Attitudes when Beliefs Are Unknown."

1. INTRODUCTION

Since Keynes (1921) and Knight (1921), it has been understood that ambiguity (unknown probabilities) is important, and since Ellsberg (1961) it has been understood that we need fundamentally new decision models for it. Gilboa (1987), Gilboa and Schmeidler (1989), and Schmeidler (1989) introduced such models, with many to follow. When measuring ambiguity attitudes, two problems arise. First, a control for likelihood beliefs is needed, but those beliefs are usually subjective and not directly observable. Second, there are over a dozen of ambiguity models today[2], and it will take more time for the field to converge on the best ones. This abundance poses a problem for practitioners, being which ambiguity model to use. We provide a solution to both problems.

As for the first problem, measurements of ambiguity attitudes have so far focused on artificial events concerning Ellsberg urns with secret compositions or experimenter-specified probability intervals. Then a symmetry of events is available that provides the required controls for subjective beliefs. Unfortunately, such symmetries are usually absent for natural (application-relevant, real-life) events, and for those events ambiguity attitudes can be different. Many authors have argued for the importance of studying natural events.[3] We introduce belief hedges, which allow for direct measurements of ambiguity attitudes for natural events.

Belief hedges provide the desired control for beliefs through symmetry at the level of the *sets of* events considered, instead of at the level of the events themselves. They protect against unknown probabilities similarly as financial hedges protect against unknown outcomes: financial hedges concern *sets of* investments rather than single investments. Belief hedges are easy to implement. Therefore, whereas for more than half a century researchers had to resort to secretized urns when measuring ambiguity, they now can directly refer to the application-relevant events, increasing

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[2] Theoretical surveys include Etner, Jeleva, and Tallon (2012), Gilboa and Marinacci (2016), Marinacci (2015), and Machina and Siniscalchi (2014). Trautmann and van de Kuilen (2015) provided an empirical review.

[3] See Camerer and Weber (1992 p. 361), Ellsberg (2011 p. 223), Heath and Tversky (1991 p. 6), l'Haridon et al. (2018), and Trautmann and van de Kuilen (2015 p. 94).

both validity and the motivation of subjects and clients. For instance, to study climate change decisions, measurements better refer to climate events than to secretized urns.

Previous studies that measured ambiguity attitudes for natural events did so indirectly, by complex data fittings using non-linear multi-parameter ambiguity models. Their approaches could only estimate ambiguity attitudes jointly with risk attitudes (utility, probability weighting) and subjective beliefs (Abdellaoui et al. 2011; Brenner and Izhakian 2018; Gallant, Jahan-Parvar, and Liu 2017). The estimations depended heavily on parametric modeling assumptions regarding all decision components, including risk attitudes, and could not solve the second problem mentioned before: the present abundance of ambiguity models.

Using belief hedges, we can define two new indexes of ambiguity attitudes that avoid the two aforementioned problems. They do not require subjective beliefs to be known and, further, they do not commit to any ambiguity model—let be risk model. In this sense, they are model-free and generally applicable. We show that belief hedges are not only sufficient but also necessary for the indexes to properly measure ambiguity attitudes. We then discuss the interpretation of the indexes for many existing ambiguity models, allowing readers to choose their favorite one. This is how we solve the second problem.

In an experiment, Baillon et al. (2018) used a design to measure ambiguity attitudes for natural events that, as we show, is the simplest special case of belief hedges. Their empirical findings were plausible, supporting our approach. Their design was, however, limited. It could only be used for simple experiments with three events, and not for bigger experiments or for existing data sets such as financial market data. Belief hedges provide the required flexibility for general empirical applicability. A detailed discussion of Baillon et al. (2018) is in §7.

To demonstrate that our indexes are model-free and generally applicable, our paper will be organized in an unusual manner. As yet, papers on ambiguity, including those introducing indexes, assumed one ambiguity model, and confined their analyses and justifications to this model. The first part of this paper, to the contrary, does not assume any ambiguity model yet when introducing the two indexes. This absence underscores that our indexes do not involve parametric fittings. They can simply and directly be derived from choice data.[4] Following basic definitions (§2.1), §2.2 introduces belief hedges for ambiguity aversion, and a corresponding aversion index. This section, while elementary, already conveys the main novelty of belief hedges, explaining why resorts to artificial urns are no longer needed. Section 2.3 presents the same result for a second index of ambiguity, capturing insensitivity. This index is more complex but empirically useful. In the absence of a decision model, we can only provide intuitive plausibility arguments at this stage. We use econometric concepts to do so.

Section 2.4 shows theoretically that our two indexes properly reflect aversion and insensitivity, and that the indexes concern distinct, orthogonal, components in the variance in our data. Section 3 gives tractable examples of belief hedges and a preference foundation of our indexes. It shows that belief hedges provide practitioners the flexibility and feasibility to tackle complex empirical problems.

The second part of our paper, starting in §4, considers ambiguity models. It first presents an extension of our indexes to many outcomes. In general, our indexes are outcome dependent. They are useful tools to examine this dependence in models with outcome-dependent ambiguity attitudes, such as the smooth model (Klibanoff, Marinacci, and Mukerji 2005). Section 5 considers the special case of outcome independence, which includes ambiguity models such as biseparable utility, rank-dependent/Choquet expected utility, and multiple priors. Indexes have been proposed before, and argued for, in several ambiguity models. Section 6 shows that, if such models are assumed, then our indexes agree with the indexes proposed there. However, our indexes remain valid for other ambiguity models. In this sense, they generalize and unify many existing indexes. For example, Dow and Werlang (1992) and Schmeidler (1989) proposed an index for ambiguity aversion using their nonadditive weighting function in Choquet expected utility, and assuming expected utility for risk. Under their assumptions, their weighting functions coincide with our matching probabilities and their index coincides with our aversion index. Our index remains valid, though, if expected utility for risk is violated, and also for other ambiguity models.

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[4] This feature allows for interactive elicitations as common in consultancies (Keeney and Raiffa 1976rrr).

Our indexes also agree with many qualitative orderings of ambiguity attitudes proposed in the literature (Appendix D.1), such as being more ambiguity averse, extending these orderings to complete orderings. Because of the compatibility of our indexes with existing indexes and orderings, many theoretical arguments advanced in the literature for various indexes and orderings support our indexes. This gives a broad theoretical support to the intuitive claims made in the first part of the paper. It shows that our indexes capture the proper general components of ambiguity attitudes.

Like the aforementioned index of Dow and Werlang and Schmeidler, most indexes in the literature used theoretical constructs[5] so that they were not directly observable, and they needed expected utility for risk. We generally avoid those restrictions and generalize and unify the existing indexes in this sense. Our analysis reveals an interesting connection between the econometric concepts used in the first part of the paper and the concepts from the ambiguity literature examined in the second part of the paper. Sections 7-8 present a discussion and conclusion.

2. BELIEF HEDGES

This section defines belief hedges and provides theoretical justifications. In many respects, we treat ambiguity, i.e., uncertainty about probabilities and beliefs, in ways that traditional analyses have treated uncertainty about outcomes.

2.1. Basic Definitions

S denotes a *state space*, finite or infinite. Its subsets are *events*. X denotes a set of *outcomes*, finite or infinite. Outcomes can be money amounts, health states, and so on. *Acts* map S to X and are finite-valued.[6] Act $\gamma_E\beta$ assigns outcome γ to event E

[5] Theoretical constructs, such as the nonadditive weighting function of Dow and Werlang (1992) and Schmeidler (1989), are not directly observable, but derive their empirical meaning indirectly in combination with other theoretical constructs (such as utility), and only within some assumed model (Cozic and Hill 2015).

[6] We can endow S with an algebra of events containing all singletons $\{s\}$, and consider only measurable acts. Then nothing in this paper changes.

and outcome β to all other states. We further assume that *lotteries* $\gamma_p\beta$ (receiving outcome γ with probability p and β with probability $1-p$) are available. We use the term *prospect* for both acts and lotteries. A *preference relation* $\succcurlyeq$ is given over prospects, with the usual notation $\preccurlyeq$, $\succ$, $\prec$, and $\sim$. We assume weak ordering throughout (completeness and transitivity). As usual, we identify constant acts and degenerate lotteries with outcomes. This implies $\gamma_S\beta = \gamma_1\beta$, and $\succcurlyeq$ now also applies to outcomes. An event is *null* if its outcomes never affect preference. *Monotonicity* means: (i) weakly improving an outcome of a prospect weakly improves the prospect; (ii) strictly improving an outcome of an act on a nonnull event strictly improves the act; (iii) strictly improving an outcome of a lottery with positive probability strictly improves the lottery.

A *measurement design* $\mathcal{H}$ is a finite collection of events. It describes the events that will be used to formally define, or experimentally measure, the ambiguity indexes. A central question in our analysis will be which designs are suited for this purpose. In most of this paper (except §3) $\mathcal{H}$ is fixed, and then dependencies on it need not be expressed in notation. By $\{E_1, \ldots, E_n\}$, the *design atoms*, or *atoms* for short, we denote the smallest nonempty intersections of events in $\mathcal{H}$. Belief hedging, defined later, will imply that the atoms partition S, covering all states. The E_js are the "atoms" of the smallest (finite) algebra of events generated by $\mathcal{H}$. For $E \in \mathcal{H}$, $|E|$ denotes the number of atoms contained in E. The Greek nu (ν) denotes the *normalized event size*, or *event size* for short, with $\nu(E) = \frac{|E|}{n}$; thus, $\nu(S) = 1$. Section 3 analyzes to what extent our indexes depend on the design (and, then, the design atoms and ν) chosen. Under empirically plausible assumptions the indexes are largely independent. Limitations will be discussed.

Throughout this paper, statistics refer to $\mathcal{H}$, with the following notation for functions $f, g: \mathcal{H} \to \Re$: $E(f) = \overline{f} = \frac{\sum_{E\in\mathcal{H}} f(E)}{|\mathcal{H}|}$: average; $Var(f)$: variance; $Cov(f,g)$: covariance of f and g. Thus, all these statistics are to be taken over $\mathcal{H}$. Details are in Appendix A. We define the *sensitivity* of f with respect to g as $\frac{Cov(f,g)}{Var(g)}$. It is a first-order approximation of how much f will change on average if g changes by one unit.

2.2. Belief Hedge for Ambiguity Aversion

Sections 2 and 3 introduce and analyze our concepts under minimal assumptions, where we need not commit to any ambiguity model. Thus, these sections assume:

ASSUMPTION 1 [two outcomes]. $X = \{\gamma, \theta\}$, with $\gamma \succ \theta$. □

Dimmock, Kouwenberg, and Wakker (2016 Theorem 3.1) showed that matching probabilities (defined next) are well suited to analyze ambiguity attitudes because, under many ambiguity models, they capture everything relevant to ambiguity attitudes. There is no need to measure risk attitudes, utilities, probability weighting, and so on. Our indexes will also be based on them. We thus assume that a *matching probability* $m(E)$ exists for every event E, defined by

$$\gamma_E\theta \sim \gamma_{m(E)}\theta\ . \qquad (1)$$

Monotonicity implies that m is unique. Incentive-compatible measurements of m are discussed by Karni (2009). Ambiguity reflects how $m(\cdot)$ deviates from a probability measure. For example, ambiguity aversion will imply $m(E) + m(E^c) < 1$, violating additivity.

We summarize the structural assumptions for the entire paper:

ASSUMPTION 2 [Structural Assumption]. $\succcurlyeq$ is a monotonic weak order over acts (finite-valued measurable maps from S to X) and lotteries (two-valued probability distributions over X). Each event has a matching probability. □

In this and the next section we work with a general framework where, besides Assumption 1, only Assumption 2 is made. Hence, the results here hold for virtually all ambiguity models considered today. This is why we call these results model-free.

DEFINITION 3. $\succcurlyeq$ is *ambiguity neutral* if m is a subjective probability measure. □

Ambiguity neutrality effectively means that subjective beliefs are treated the same way as objective beliefs. Observation 16 will show that the condition for two fixed outcomes as defined here implies general ambiguity neutrality under many ambiguity models.

To measure ambiguity aversion, several papers used differences $P(E) - m(E)$, where P denotes subjective probabilities reflecting ambiguity-neutral beliefs, called *a-neutral probabilities*.[7] These differences reflect an ambiguity premium, i.e., willingness to pay—in probability (belief) units— to avoid ambiguity. The bigger the aversion, the bigger this premium (Kahn and Sarin 1988; Viscusi and Magat 1992; Dimmock, Kouwenberg, and Wakker 2016). Ideally, with a number of observations $P(E) - m(E)$ available, we would like to define our aversion index as the average level of differences

$$\overline{P - m}\ .$$

The aforementioned references considered Ellsberg urns, where P could be derived from symmetry assumptions. For natural events the problem is that we do not know the a-neutral P. Our solution is simple: we ensure a fixed and known average level of P:

$$\overline{P} = \frac{1}{2} \text{ for all } P. \tag{2}$$

Necessary and sufficient for Eq. 2 is:

DEFINITION 4. $\mathcal{H}$ is *level-hedged*, or *l-hedged* for short, if:

each state s appears in exactly half of the events in $\mathcal{H}$. (3)

Equivalent for the atoms E_i is that each appears in exactly half of the elements of $\mathcal{H}$ (can be seen via any $s \in E_i$). This implies $\overline{P} = 1/2$, first for all degenerate probability measures assigning probability 1 to E_i, and then also for all their convex combinations, i.e., for all P. The condition is for instance satisfied if $\mathcal{H}$ is complementation closed. We multiply $\overline{P - m} = \frac{1}{2} - \overline{m}$ by 2 for normalization:

DEFINITION 5. If l-hedging (Eq. 3) holds, then the *index of ambiguity aversion* is

$$b = 1 - 2\overline{m}\ . \tag{4}$$

The analysis in this section was simple from an algebraic perspective. Yet, because these results had never been noted before, for more than half a century researchers

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[7] They can be interpreted as the beliefs of the ambiguity neutral twin of the decision maker.

used secretized urns rather than application-relevant events to measure ambiguity aversion. As we have shown now, there is no more need to rely on Ellsbergian informational symmetry of the events once we have ensured l-belief hedges through the measurement design $\mathcal{H}$.

2.3. Belief Hedge for Insensitivity

Theoretically and normatively motivated ambiguity models have focused on ambiguity aversion, a motivational component of ambiguity attitude.[8] However, recent empirical studies have found richer phenomena (Kocher, Lahno, and Trautmann 2018). Whereas for likely events there indeed is strong ambiguity aversion, it gets weaker for events of moderate likelihood, and for low likelihood events it reverses. Then ambiguity aversion even predicts in the wrong direction (Trautmann and van de Kuilen 2015).[9] Araujo et al. (2018) showed that equilibria still often exist for such more general preferences.

The aforementioned likelihood dependence shows a general drift towards fifty-fifty, with insufficient discriminatory power and insufficient responsiveness toward belief changes in the middle region. It is a similar kind of insensitivity as exhibited by inverse-S probability weighting for risk, where weights in the middle are also moved toward fifty-fifty (Wu and Gonzalez 1999), but now it concerns ambiguity attitudes. We incorporate this effect as a second component of ambiguity attitude, and interpret it to be cognitive, an interpretation supported by Anantanasuwong et al.'s (2018) empirical study of a representative sample of financial investors. This component reflects lack of understanding of ambiguity, which comes prior to any aversion or seeking. The decision maker takes ambiguous events (too much) as one blur. We use the term a(mbiguity-generated) insensitivity to refer to the insensitivity generated by ambiguity.

This section explains our measurement of the second component, again independently from beliefs, which is similar to the preceding section but

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[8] A notable exception is Gul and Pesendorfer's (2015 p. 467) Hurwicz expected utility, which explicitly allows for a-insensitivity (our term).

[9] These phenomena are reflected for losses, leading to a four-fold pattern. Overall, for losses there is more ambiguity seeking than aversion. Reflection can readily be accommodated by reflecting our parameters for losses or using dual functionals there. We focus on gains in this paper.

mathematically more involved. Ideally, we would like to use the most common measure of responsiveness of m with respect to P, being the sensitivity

$$\frac{Cov(m,P)}{Var(P)}. \tag{5}$$

This index is used as the slope in regressions (Hill, Griffiths, and Lim 2008, Eq. 2.7), and as β in the CAPM model in finance (Hull 2017). It captures the average derivative of m with respect to P (in our domain of nonextreme events), i.e., the average change in m if P changes by one unit.

In the ε-contamination model, a tractable subclass of α-maxmin multiple priors models, our insensitivity index exactly coincides with the size of the set of priors (§6). In general, the larger the set of priors (perception of ambiguity), the more events are treated alike, as one blur, corresponding to lower discriminatory power. That is, the sensitivity is also lower. In extreme case where the set of priors contains all priors, all nontrivial events E are treated the same way, with all $\beta_E\alpha$ indifferent and with, indeed, maximal insensitivity. Section 6 gives further details and provides similar results for many other popular ambiguity theories, where insensitivity is often interpreted as perception of ambiguity. Measures of uncertainty perception have been used in explaining and predicting economic growth and recession (Shiller et al. 1996, Baker et al. 2016). Our insensitivity index shows a way to measure perception of uncertainty based on revealed preferences.

To measure our insensitivity index, the problem we face is, again, that the a-neutral P is unknown. Our solution is to ensure that we can always replace P by the event size ν (as if P were uniform over atoms), irrespective of what P really is. The intuition is to ensure that the event size properly reflects the average probability P over events of the same size, in the sense that they perfectly co-vary with each other. The following condition is necessary and sufficient for this purpose.

DEFINITION 6. $\mathcal{H}$ is *v(ariation)-hedged* if:

$$\textstyle\sum_{E\ni s}\nu(E) \text{ is the same for each fixed state } s. \tag{6}$$

We provide a sketch of the proof here. The complete proof is in Appendix A. The *v-hedge* condition ensures that each state s and, hence, each atom E_i (through any $s \in E_i$) appears equally often in big events and, accordingly, in small events. This implies that each degenerate probability measure assigning probability 1 to E_i has the same covariance w.r.t. event size ν (proved in Eq. 22 below) and, hence, the same sensitivity. This then also holds for any of their convex combinations, comprising all probability measures P on the atoms, including ν itself. Accordingly, this sensitivity is 1. If, thus, every P "perfectly" covaries with event size ν on average then, under plausible econometric assumptions (see proof of Eq. 7 in Appendix A), the sensitivity of m w.r.t. event size is a good first-order approximation of this sensitivity w.r.t. P. That is, we obtain

$$\frac{Cov(m,P)}{Var(P)} \approx \frac{Cov(m,\nu)}{Var(\nu)}. \tag{7}$$

For Eq. 7, we need one more assumption, to avoid degeneracy:

ASSUMPTION 7 [nondegeneracy]. $\mathcal{H}$ does not contain $\emptyset$ or S. All atoms E_j are nonnull. ν is not constant on $\mathcal{H}$. □

Regarding the first part of the assumption, insensitivity concerns intermediate events away from the extremes and, hence, we exclude the extreme events.[10] This entails no loss of information because the m values of null events and their complements are 0 and 1, respectively, by monotonicity. As for the second part of the Assumption, null events do not affect preference and, hence, can be made to disappear from the atoms by joining them with a nonnull atom (with the obvious adaptation of $\mathcal{H}$). This second part further serves to stay away from extreme events. For the final part, because we derive insensitivity from variations in event size, we need event size to be nonconstant—also after excluding $\emptyset$ and S. This implies $n \geq 3$. The condition also ensures that $Var(\nu)$ is positive, so that the index a below is well-defined. $\mathcal{H}$ is a *belief hedge*, or *hedge* for short, if both l-hedging and v-hedging hold.

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[10] In the terminology of Wakker (2010 §7.7), we focus on the insensitivity region. Boundary restrictions can be used to define this region. Because our theorems are valid irrespective of what those regions are, we do not discuss them in this paper. For applications, we recommend not using events in the measurement design with a-neutral probabilities below 0.05 or above 0.95.

DEFINITION 8. If Assumption 7 and belief hedging (Eqs. 3 and 6) hold, then the *index of a(mbiguity-generated) insensitivity* is

$$a = 1 - \frac{Cov(m,v)}{Var(v)}. \tag{8}$$

Eq. 9 below gives a simple special case of Eq. 8 that can readily be calculated using paper and pencil.

Throughout the rest of the paper we assume that Assumptions 2 and 7 hold, explicitly in theorems and implicitly elsewhere. We, finally, return to our aversion index. The beginning of this section indicated that the prevailing empirical finding for unlikely events is ambiguity seeking. Hence, had we mainly used unlikely events in $\mathcal{H}$, then the average $\overline{P-m}$ would have been small or even negative, underestimating ambiguity aversion. Using mainly likely events in $\mathcal{H}$ would overestimate ambiguity aversion. L-hedging has avoided such biases by taking average event-size 1/2.

2.4. Theoretical Justifications of Belief Hedges

We first show formally that our indexes classify ambiguity neutrality and, accordingly, ambiguity aversion/seeking and (in)sensitivity properly, and that they have been properly normalized, facilitating comparisons across studies (Dekel and Lipman 2010 p. 271). The following theorem also shows that belief hedges are not only sufficient, but also necessary, for our purposes.

THEOREM 9. Under Assumptions 1, 2, and 7, $b = a = 0$ for all ambiguity neutral decision makers if and only if the measurement design $\mathcal{H}$ is a belief hedge. Then the supremum value of b and a is 1.[11] □

Psychologically, we interpret aversion and insensitivity as two distinct components of ambiguity attitudes. This interpretation is supported by an orthogonality—in the usual Euclidean sense as used for instance in statistical analyses

[11] Monotonicity excludes constancy of m. If we relax this condition then the supremum values can occur as maxima: $b = 1$ if m is constant 0 and $a = 1$ for any constant m.

of variance—of the indexes. Anantanasuwong et al. (2018) found this orthogonality empirically. It is reassuring if theoretical implications are consistent with the story behind the theory (Dekel and Lipman 2010, p. 259).

THEOREM 10. Under Assumptions 1, 2, and 7, the indexes a and b capture orthogonal components of the variance in the data. □

3. WHICH DESIGN TO USE AND A PREFERENCE FOUNDATION

Baillon et al.'s (2018) experiment assumed three nonnull atoms $\{E_1, E_2, E_3\}$ and a full design $\mathcal{H} = \{E_1, E_2, E_3, E_1 \cup E_2, E_1 \cup E_3, E_2 \cup E_3\}$ denoted $\mathcal{H}(E_1, E_2, E_3)$. This $\mathcal{H}$ indeed is the simplest special case of a belief hedge. We write $\overline{m_s} = \frac{m(E_1)+m(E_2)+m(E_3)}{3}$ and $\overline{m_c} = \frac{m(E_1 \cup E_2)+m(E_1 \cup E_3)+(E_2 \cup E_3)}{3}$. Substitution (Online Appendix) gives, under Assumptions 1, 2, and 7:

$$b = 1 - \overline{m_c} - \overline{m_s} \text{ and } a = 3\left(\frac{1}{3} - (\overline{m_c} - \overline{m_s})\right). \quad (9)$$

We next give some other tractable examples of belief hedges. $\mathcal{H}$ is a belief hedge if for every $i < n$, every state (a) appears equally often in an event of size i; and (b) it does so with overall frequency $\frac{1}{2}$. This includes all cases where l-hedging holds and $\mathcal{H}$ satisfies the following symmetry with respect to the presence of atoms: for all $i \neq j$ and all $E \in \mathcal{H}$: if $E_i \subset E, E_j \not\subset E$, then also $(E \cup E_j - E_i) \in \mathcal{H}$. This is satisfied if $\mathcal{H}$ contains all unions of E_js except S and $\emptyset$, as with $\mathcal{H}\{E_1, E_2, E_3\}$. These examples show that belief hedging leaves much flexibility. We can always get by with six events, as in $\mathcal{H}(E_1, E_2, E_3)$. Simplicity goes at the cost of reliability though, and the richer $\mathcal{H}$ is, the more reliable the indexes will be. For a finite state space, we can involve all states by taking all single-state events and their complements. The size of such $\mathcal{H}$ then grows linearly with the size of the state space.

In general, different designs need not give the same indexes, as the following example shows.

EXAMPLE 11. Consider an Ellsberg urn with 90 numbered balls, the first 30 red, the last 60 black or yellow in unknown proportion. For E_1 (red), E_2 (non-red and odd),

E_3 (non-red and even), the corresponding design $\mathcal{H}(E_1, E_2, E_3)$ will suggest ambiguity neutrality with $b = a = 0$. However, for E_1 (red), E_2 (black), E_3 (yellow), the corresponding design $\mathcal{H}(E_1, E_2, E_3)$ will give deviations from neutrality. □

Ambiguity is too rich a domain to expect that one ambiguity attitude for a decision maker can cover all events. There can be many kinds of (source) preferences and (lacks of) understanding of uncertainty beyond risk, where emotions and confusions play a role beyond the degree to which probabilities are known or unknown (Tversky and Fox 1995). Ambiguity attitudes depend on sources of uncertainty similarly as utility functions depend on commodities (Cappelli et al. 2018). Our indexes can serve as tools to examine such emotions, and this concerns a large and important topic for future research. We now investigate when different designs do give the same indexes.

DEFINITION 12. The indexes *perfectly fit* if every measurement design $\mathcal{H}$ gives the same indexes. □

The following property characterizes perfect fit: m is *neo-additive* if there exist a probability measure P on S, $0 \leq \sigma \leq 1$, and $0 \leq \tau \leq 1 - \sigma$ such that

$P(E) = 0 \;\Rightarrow\; m(E) = 0;$

$0 < P(E) < 1 \;\Rightarrow\; m(E) = \tau + \sigma P(E);$

$P(E) = 1 \;\Rightarrow\; m(E) = 1.$ (10)

We call W *neo-additive* if the three implications in Eq. 10 hold with W instead of m. To avoid some open mathematical problems[12], we assume monotonicity, which implies $\sigma > 0$. Under Assumptions 1, 2, and 7, and neo-additivity of m (Eq. 10), substitution (Online Appendix) gives:

$$b = 1 - 2\tau - \sigma \quad \text{and} \quad a = 1 - \sigma. \tag{11}$$

We, finally, present a preference axiomatization of perfect fit. As is common in axiomatizations, we assume complete information about preferences. That is, we

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[12] Such open problems in Chateauneuf, Eichberger, and Grant (2007) concern nonnecessity of null event consistency in their Theorem 5.2 and inconsistency between null events in bets on and bets against events under their maximal pessimism.

consider all measurement designs and use m for all events.[13] And, as common in axiomatizations, we assume a continuum domain, through the following conditions of Villegas (1964). We say that m is *fine*[14] if for each nonnull event A there exists an event $B \subset A$ such that $m(A) > m(B) > 0$. For any P, P is *fine* if the same holds for P instead of m. *Event-continuity* holds if, whenever a nested sequence $A_1 \supset A_2 \supset A_3 \cdots$ converges to $\emptyset$ and $m(B) > 0$, there exists a J such that $m(A_j) < m(B)$ for all $j \geq J$, and whenever a nested sequence $B_1 \subset B_2 \subset B_3 \cdots$ converges to B and $m(B) > m(A)$, there exists a J such that $m(B_j) > m(A)$ for all $j \geq J$.

THEOREM 13[15]. Under Assumptions 1, 2, and 7, the following two statements are equivalent:

(i) m is neo-additive and the corresponding probability measure P is fine (atomless) and countably additive.

(ii) Our indexes perfectly fit and m is fine and event-continuous. □

By monotonicity, m in (i) is strictly increasing in P; i.e., $\sigma > 0$. The literature has documented several appealing properties of the neo-additive model (Eichberger, Grant, and Lefort 2012 p. 238 penultimate paragraph). Theorem 13 provides another one. In particular, it shows a new way to test the neo-additive model.

The result is empirically reassuring because the neo-additive model performs well empirically which, together with its tractability, makes it popular. In particular, it captures the empirically prevailing four-fold pattern of ambiguity aversion (Trautmann and van de Kuilen 2015). If the fit is not perfect, then our indexes can still serve as pragmatic estimates. This can be compared with linear regressions, which are commonly used to provide good pragmatic estimations even though relations are known not to be perfectly linear. And, similarly, CRRA (constant

[13] So far we, in fact, only used m on one fixed measurement design.

[14] We avoid the common term atomless because the term atom is used for another purpose in this paper. In the presence of the assumed event-continuity, our condition is equivalent to Savage's (1954) fineness.

[15] To avoid some Banach-Kuratowski-Ulam impossibility results, it is useful to add measure-theoretic structure in this theorem, where the set of events should be a σ-algebra.

relative risk aversion) indexes are used even though utility need not be perfectly CRRA and, even, expected utility may be violated. Limitations are known and reckoned with then. For linear regressions there should be no extreme outliers and for CRRA risk aversion no outcomes of different orders of magnitude should be involved. We next discuss similar limitations of our indexes.

Our pair of indexes surely cannot fit all data well if our source of uncertainty involves different uncertainty mechanisms, as in Example 11, where the source in this sense is not "uniform." In such situations, however, no (pair of) indexes can fit all the data. Likewise, one CRRA index can never perfectly fit data if outcomes concern several commodities with intrinsically different utility curvatures. This problem is avoided if the source of uncertainty is uniform (formalized by Abdellaoui et al. 2011; our Eq. 28). Within uniform sources, our indexes may still not work well if very unlikely events are incorporated into the measurement design. Such events are known to involve many irregularities (Kahneman and Tversky 1979; Sutter and Poitras 2010 p. 183). We recommend avoiding them in applications, e.g., by imposing boundary conditions (Tversky and Wakker 1995; see Wakker's 2010 insensitivity region).

If there is reason to believe that data deviate much from the neo-additive model, then our indexes can become very dependent on the partition chosen, limiting their usefulness. But, as explained before, data usually do not deviate much from the neoadditive model. The theoretical analyses in the second part of this paper show that under most ambiguity theories, our indexes capture ambiguity attitudes perfectly. If practitioners reckon with the limitations just discussed, then our indexes can serve well to capture ambiguity attitudes.

Whereas, for instance, expected utility with CRRA utility is a one-parameter model, ambiguity models involve many parameters, including utility functions (upon which we impose no restrictions), nonexpected utility parameters of risk attitudes such as probability weighting, subjective beliefs, and then the parameters of interest in this paper, capturing ambiguity. Whereas prior studies needed to jointly estimate all model parameters in order to obtain ambiguity measurement, we only need a pair of indexes that are directly observable from six indifferences. Hence, we greatly simplify the empirical measurement by making many high-dimensional unknowns (utility, probability weighting, beliefs) drop from the equations. This is why a few indifferences suffice to give insights into a high-dimensional parametric model, without requiring complex data fittings.

4. EXTENSION TO MANY OUTCOMES AND OUTCOME-DEPENDENT AMBIGUITY MODELS

The following sections drop Assumption 1 (two outcomes) and consider general outcome sets X. The results obtained before can trivially be extended as follows, without commitment to any ambiguity model.

OBSERVATION 14. All results of §§2 and 3, including Theorems 9, 10, 13, and Eq. 9 remain valid if we drop Assumption 1 but fix two outcomes $\gamma \succ \theta$ for the matching probabilities m (Eq. 1) and the indexes b, a. □

In several models, ambiguity attitudes depend on the outcomes considered (Chew et al. 2008; Gul and Pesendorfer 2014; He 2018; Nau 2006; Neilson 2010; Olszewski 2007). Then the indexes in Observation 14 will depend on the outcomes γ, θ chosen, and can be used to investigate this dependence. For example, constant ambiguity aversion w.r.t. absolute utility increases (Grant and Polak 2013), or w.r.t. proportional utility increases (Chateauneuf and Faro 2009), or these conditions w.r.t. wealth changes (Cerreia-Vioglio, Maccheroni, and Marinacci 2016), are inherited by matching probabilities and our indexes. These conditions can, therefore, be tested using our indexes.

The rest of this section analyzes our indexes in more detail for the most popular outcome-dependent ambiguity model, the smooth model (Klibanoff, Marinacci, and Mukerji 2005). Our analysis is similar to Izhakian and Brenner (2011) who provided local ambiguity premiums expressed in monetary units. Our premiums are expressed in probability units. Details and complete definitions are in Appendix C. We first show there, for some given event E and our ambiguity premium:

$$p - m(E) = \frac{1}{2}\sigma^2 A(p) + o(\sigma^2) \ . \tag{12}$$

Here p is the a-neutral probability of E, $A = -\frac{\varphi''}{\varphi'}$ is the Arrow-Pratt index of ambiguity aversion (Klibanoff, Marinacci, and Mukerji 2005 p. 1865), σ^2 is the variance of the second-order uncertainty μ about p, and $o(\sigma^2)$ expresses first-order approximation as σ^2 vanishes. Eq. 12 is Pratt's (1964) Eq. 5, but with probabilities replacing monetary outcomes. It illustrates once more that uncertainty about probabilities is treated in this paper in the same way as uncertainty about outcomes was treated in traditional analyses. Our aversion index b, which is Eq. 12 averaged over $\mathcal{H}$ and multiplied by 2 for normalization (see §2.2), is:

$$b = \overline{\sigma^2 A(p)} + o(\sigma^2) \ . \qquad (13)$$

It is the product of what is sometimes interpreted as ambiguity perception (σ^2) and a relative index per perceived unit, $A(p)$. A similar decomposition occurs in Observation 18 below, where it is discussed further. Eq. 13 makes the average of this product, involving the not directly observable p, A, and σ^2, directly observable because b is.

We obtain as insensitivity index (see Appendix C):

$$a = \frac{1}{2}\frac{Cov(\sigma^2 A(p), \nu)}{Var(\nu)} + o(\sigma^2) \ . \qquad (14)$$

It captures how the aversion premium (Eq. 12) increases with event size ν, which indeed reflects insensitivity. This degree of ambiguity (perception), depending on variance of the event probability, is similar to Izhakian's (2017) measure in his variation of the smooth model that uses Choquet expected utility rather than expected utility in the second stage.

The ambiguity attitude analyzed here depends on the outcome interval $[\theta, \gamma]$ considered, as is typical of the smooth model (e.g., Klibanoff, Marinacci, and Mukerji 2005 Proposition 4). We obtain outcome independence, as we will show, for the special case of $\varphi(x) = -e^{-\rho x}$ (Klibanoff, Marinacci, and Mukerji 2005 Proposition 2). This case is of special interest because it concerns the intersection with the variational model (Maccheroni, Marinacci, and Rustichini 2006). This intersection is exactly the multiplier preference model of Hansen and Sargent (2001). In this special case, we have outcome independence with $A \equiv \rho$, and

$$b = \rho\overline{\sigma^2} + o(\sigma^2) \ ; \qquad (15)$$

$$a = \frac{1}{2}\rho\frac{Cov(\sigma^2, \nu)}{Var(\nu)} + o(\sigma^2) \ . \qquad (16)$$

As for Eq. 16, because, for any event, σ^2 is the same for that event and its complement, a will not covary much with ν. It will even disappear if $\mathcal{H}$ is complementation-closed. It shows that this class of smooth and variational preferences does not allow for much insensitivity, which is consistent with constancy of ambiguity aversion. Outcome independence is central in the next section.

5. EXTENSION TO OUTCOME-INDEPENDENT AMBIGUITY MODELS

This section continues to consider general outcome sets X, dropping Assumption 1. Many models assume that ambiguity attitudes are outcome independent.[16] Then so will our indexes be. Those models are all special cases of the following one. For simplicity, we assume the existence of a worst outcome.[17] *Uniseparable utility* holds if there exists a worst outcome θ ($\forall\gamma \in X{:}\, \gamma \succcurlyeq \theta;\ \exists\gamma \succ \theta$) such that

$$\gamma_E\theta \rightarrow W(E)U(\gamma) \text{ and } \gamma_p\theta \rightarrow w(p)U(\gamma) \tag{17}$$

represents preferences for prospects with at most one outcome γ other than θ. For monetary gain outcomes, typically $\theta = 0$; in the health domain, often θ = death. Under prospect theory, θ is the reference outcome. U is the nonconstant *utility function*; we scale $U(\theta) = 0$. Uniseparable utility rules out state dependence of utility (Karni and Schmeidler 2016), and, through W, unforeseen contingencies (Dekel, Lipman, and Rustichini 1998). W is a nonadditive (*event) weighting function*; i.e., $W(\emptyset) = 0$, $W(S) = 1$, and W is *set-monotonic* ($A \supset B$ then $W(A) \geq W(B)$). Further, $w{:}\,[0,1] \rightarrow [0,1]$ is a *(probability) weighting function*, with $w(0) = 0$, $w(1) = 1$, and w strictly increasing. *Expected utility* implies (a) W is additive (i.e., W is a subjective probability measure) and (b) w is the identity. *Expected utility under risk* only implies (b). Under uniseparable utility, we can redefine m in the following outcome-independent manner:

DEFINITION 15. $m(E) = p$ if $\gamma_E\theta \sim \gamma_p\theta$ for some $\gamma \succ \theta$. □

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[16] They include biseparable utility (Ghirardato and Marinacci 2001) and, thus, Choquet expected utility or rank-dependent utility, prospect theory for gains, maxmin EU, and the α-maxmin model (Ghirardato, Maccheroni, and Marinacci 2004). Further included are separate-outcome weighting models ($\gamma_E\beta \rightarrow W(E)U(\gamma) + W(E^c)U(\beta)$; Einhorn and Hogarth 1985), Chateauneuf and Faro's (2009) confidence representation with worst outcome θ, Izhakian's (2017) uncertain probability model, and Lehrer and Teper's (2015) event-separable representation.

[17] We focus on gains so that sign-dependence, as in prospect theory for ambiguity, plays no role.

By Eq. 17, Definition 15 is equivalent to $\gamma_A\theta \sim \gamma_p\theta$ for all $\gamma \succ \theta$ (and it is equivalent to $W(A) = w(p)$). It, thus, extends our preceding definition (Eq. 1) to more than two outcomes. We could have increased outcome independence by imposing Savage's (1954) P4 full force, which amounts to allowing θ to vary, as in biseparable utility, at the cost of some generality.

For the sake of easy reference, we provide the following trivial reformulation of the results derived in preceding sections, adapted to general X and uniseparable utility.

OBSERVATION 16. All results of §§3-4 remain valid, including Theorems 9, 10, 13, and Eq. 9 if we replace Assumption 1 by uniseparable utility and use Definition 15 instead of Eq. 1. Ambiguity neutrality (Definition 3) implies $W(.) = w(P(.))$ for a subjective probability measure P (= m). □

The observation shows how our indexes and results can be applied to, basically, all event-driven ambiguity models. Observation 16 also shows that ambiguity neutrality in Definition 3, restricted to two fixed outcomes, agrees with common definitions. Ambiguity neutrality comprises both probabilistic sophistication (Machina and Schmeidler 1992) and indifference between subjective and objective probabilities (Dean and Ortoleva 2017 Footnote 31).

6. GENERALIZING AND UNIFYING EXISTING AMBIGUITY INDEXES

This section applies our indexes to a number of outcome-independent ambiguity models. We assume that $\mathcal{H}$ is a belief hedge throughout. Many papers in the literature considered partial more-ambiguity-averse-than orderings. Our aversion index is an extension, i.e., completion, of virtually all of these (Appendix D.1). We next turn to quantitative indexes and show that our indexes generalize many preceding ones. In each case, if the assumptions of the ambiguity model considered hold, then our indexes agree with their indexes, but our indexes remain valid more generally.

We first discuss indexes measured in experiments. Baillon and Bleichrodt (2015) considered a domain $\mathcal{H}(E_1, E_2, E_3)$ as in our Eq. 9, and measured five event-specific indexes. They did not provide controls for beliefs. Our indexes show how their indexes can be aggregated to provide that control, capturing both aversion and insensitivity.[18] Our indexes also generalize the indexes of aversion and insensitivity/perception that Abdellaoui et al. (2011), Dimmock, Kouwenberg, and Wakker (2016), and Dimmock et al. (2016) measured experimentally (Appendix D.2).

We next turn to indexes in theoretical studies. Our aversion index generalizes the aversion index

$$1 - W(E) - W(E^c) \qquad (18)$$

of Dow and Werlang (1992) and Schmeidler (1989, example on pp. 571-572 & p. 574) in Choquet expected utility (or rank-dependent utility); see Observation 29. It has been commonly used in theoretical studies (Ghirardato and Marinacci 2002 Proposition 22; Klibanoff, Marinacci, and Mukerji 2005 Definition 7). Eq. 18 can be taken as a first example of a belief hedge. A similar idea underlies source preference (Tversky and Fox 1995). MacCrimmon and Larsson (1979 p. 381-384) provided an early test of this condition. Because Eq. 18 uses the theoretical construct of W, and assumes expected utility for risk, it could not be readily implemented empirically. Our aversion index shows how to make it observable.

Chateauneuf, Eichberger, and Grant's (2007) *neo-additive model* concerns the special case where W is neo-additive. The authors' interpretations strongly suggest expected utility for risk, and we assume it.[19] Then $m = W$ is neo-additive and Eq. 11 gives our indexes. Chateauneuf, Eichberger, and Grant (p. 544 top) interpret a (we use our notation) as lack of confidence (or distrust) in the a-neutral probability P, and $\frac{b}{2a} + \frac{1}{2}$ as an index of pessimism. Ignoring the irrelevant term $\frac{1}{2}$, their pessimism index is our aversion per unit of distrust in P, which is a relative analog of our absolute index. We compare such relative and absolute versions after Observation 18 below. Under Choquet expected utility with expected utility for risk, our Theorem 13 provides an alternative axiomatization of Chateauneuf, Eichberger, and Grant's (2007) neo-additive model. Their model is in the intersection of Choquet expected utility and multiple priors models, to which we turn next.

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[18] Using their notation: $b = \overline{BC}$ and $a = \overline{(LA + UA)}/3$.

[19] Without this assumption, their indexes do not solely capture ambiguity attitudes but also risk attitudes.

In multiple prior models, C denotes a convex set of probability distributions over S. $P^*(E) = \sup_{P \in C} P(E)$ denotes *upper probabilities* and $P_*(E) = \inf_{P \in C} P(E)$ denotes *lower probabilities*. In the α-*maxmin* model (Ghiradato, Maccheroni, and Marinacci 2004) preferences maximize, for $\gamma \succcurlyeq \beta$:

$$\gamma_E \beta \to W(E)U(\gamma) + \big(1 - W(E)\big)U(\beta)$$

with $W(E) = \alpha P_*(E) + (1-\alpha)P^*(E)$ $(0 \leq \alpha \leq 1)$. For risky events one fixed probability is assumed, i.e., expected utility is assumed for risk. Maxmin expected utility is the special case of $\alpha = 1$ (Gilboa and Schmeidler's 1989; Alon and Gayer 2016). The literature has focused on the ambiguity aversion/seeking component, taking $2\alpha - 1$ (or equivalent linear rescalings) as index.

THEOREM 17. Assume Assumptions 2 and 7, complementation-closed of $\mathcal{H}$, and the α-maxmin model. Then our aversion index is

$$b = (2\alpha - 1)\big(\overline{P^* - P_*}\big)\,. \tag{19}$$

□

Here $\big(\overline{P^* - P_*}\big)$ is the average discrepancy between upper and lower probabilities of events, which is sometimes interpreted as ambiguity perception—or as the Dempster-Shafer plausibility-belief gap (Gul and Pesendorfer, 2014, Corollary 2). $2\alpha - 1$ reflects aversion to ambiguity per perceived unit, and is 0 under ambiguity neutrality. In applications, sets of priors and P^* and P_* are usually assumed exogenously given because no tractable method is known to derive C, the set of priors, empirically from preferences. Then the indexes $2\alpha - 1$ and b are informationally equivalent, and it depends on the context which one is most convenient. Further discussion follows after Observation 18 below.

A quantitative index of ambiguity perception, related to insensitivity, has been explicitly proposed for a tractable subclass of α-maxmin, ε-α-maxmin. Here $C = \{(1-\varepsilon)Q + \varepsilon T\}$ with a fixed baseline probability Q, a fixed $\varepsilon \in [0,1]$, and the variable T any probability measure (Dimmock et al. 2015; axiomatized by Chateauneuf, Eichberger, and Grant 2007). It is a subclass of the ε-contamination model (Basili, Chateauneuf, and Scianna 2018; Ellsberg (1961 pp. 663-669) that has been used in many fields. Dimmock et al. (2015) showed:

OBSERVATION 18. Assume Assumptions 2 and 7 and ε-α-maxmin. Then the ambiguity perception ε equals a-insensitivity a (ε $= a$), and the aversion parameter $2\alpha - 1$ is a rescaling of the aversion index b ($b = (2\alpha - 1)\varepsilon$). □

Observation 18 is consistent with Theorem 17 because $\left(\overline{P^* - P_*}\right) = \varepsilon$. The parameter pairs a, b and α,ε can readily be transformed into each other and carry the same information. One restriction is that the α maxmin model, unlike our approach, requires $a = \varepsilon \geq 0$. Eichberger, Grant, and Lefort (2012 p. 245) discussed theoretical pros and cons and pointed out the desirability of experimental investigations. Baillon et al.'s (2018) experiments found $a = \varepsilon < 0$ for many subjects (Appendix D.2).

In Observation 18, our index b is the product of what is often interpreted as ambiguity perception, and aversion per unit of perception. Which pair of indexes (b, a or $2\alpha - 1, \varepsilon$) is more convenient depends on the intended application. Index b is most relevant for determining ambiguity premiums.[20] In normatively oriented studies, ambiguity per perceived unit has often been taken as a person-dependent but source-independent index. Descriptively, source and person (in)dependence of various components are topics for future empirical research. Similar comments apply to the alternative indexes in Eqs. 13 and 19, and those in Dimmock et al. (2016).

One contribution of our results to multiple prior theories is that we show how to make ambiguity aversion and the perceived level of ambiguity directly observable, without the need to measure utility U or the set of priors C. This result was obtained before by Dimmock et al. (2015) for Ellsberg-urn events when assuming symmetries of beliefs.

7. DISCUSSION

Whereas our indexes can be used beyond Ellsberg urns, it remains interesting to apply them to Ellsberg urns. These have been widely studied and the emotions

[20] Schmeidler (1989 p. 574) used the term uncertainty premium for index b.

relevant there are well understood. They can, therefore, serve well for comparisons with other sources of uncertainty. Example 11 is a variation of the well-known three-color Ellsberg urn, where two different sources of uncertainty come together. Studying such situations is yet another interesting topic for future research, both empirical and theoretical. Cappelli et al. (2018) give some theoretical suggestions.

Baillon et al. (2018) is the empirical precedent of this paper. Those authors considered a full design with three disjoint events and then all unions (see our §3). This did not reveal the central new concept of this paper–belief hedges, whose requirements are trivially satisfied in their design. Further, Baillon et al.'s indexes (our Eq. 9) did not reveal the relevant general indexes (our Eqs. 4 and 7). In this sense, those authors provided the first special case, whereas the current paper introduces the relevant general concepts and lays the theoretical foundations. In particular, we show that the ambiguity indexes properly capture ambiguity attitudes if and only if the measurement design satisfies belief hedging. Thus, we can solve some practical problems. In most practical situations, more than three disjoint events are relevant, and we should be able to handle such richer state spaces. Then, taking any three-fold partition (where Baillon et al.'s formulas can still be used) ignores some of the relevant uncertainties, which is unsatisfactory. Taking full designs of all subsets is also unsatisfactory because then the size of the measurement design grows exponentially. Belief hedges enable practitioners to study all relevant uncertainties while maintaining a feasible design size, and they provide flexibility when control over empirical data is limited (§3). In particular, we can use several, different, designs, and compare and combine them. Baillon et al. (2018) did not provide theoretical justifications for their design and indexes but referred forward to this paper for those. Our sections 5-6 have shown that our indexes indeed capture the proper general components of ambiguity and unify virtually all existing indexes in the literature.

8. CONCLUSION

We have introduced belief hedges which are necessary and sufficient for measuring ambiguity attitudes when beliefs are unknown. Belief hedges extend the hedging concept from finance, where it provides protection against unknown outcomes, to ambiguity theory where it provides protection against unknown probabilities. Using belief hedges and some econometric concepts, we introduce two new indexes of ambiguity. For more than half a century, ambiguity measurements were confined to artificial events such as secretized urns, because researchers did not know how to control for unknown beliefs. Our indexes, to the contrary, can directly be measured for application-relevant events, increasing their validity, because the required control for unknown beliefs is provided by belief hedges. Our indexes do not require expected utility for risk, which is desirable for empirical purposes (Starmer 2000; Bruhin, Fehr-Duda, and Epper 2010), and they can accommodate the empirically prevailing ambiguity seeking for unlikely events. Unlike their predecessors in the literature, they use no theoretical constructs so that they are directly observable. They generalize and unify many existing indexes in the literature and are valid under virtually all ambiguity theories.

APPENDIX A. PROOFS OF §2

For $f:\mathcal{H} \to \Re$, $\overline{f} = \frac{\sum_{E\in\mathcal{H}} f(E)}{|\mathcal{H}|}$; $Var(f) = \frac{\sum_{E\in\mathcal{H}}\left(f(E)-\overline{f}\right)^2}{|\mathcal{H}|}$; $Cov(f,g) = \frac{\sum_{E\in\mathcal{H}}\left(f(E)-\overline{f}\right)\times\left(g(E)-\overline{g}\right)}{|\mathcal{H}|}$.[21]

PROOF OF EQ. 7. We assume Assumptions 1, 2, and 7, and belief hedging. We further assume a relationship

$$m(E) = w_a\big(P(E)\big) + \varepsilon \tag{20}$$

with P capturing a-neutral beliefs, w_a the deviations from ambiguity neutrality, and ε noise. That is, for a given decision maker, we take all factors of the events impacting m other than P as noise ("uniformity"). The following three statements are equivalent:

(i) v-hedging; (ii) $E(1_s \times \nu)$ is the same for each s; (iii)[22] $Cov(1_s, \nu)$ is the same for each s. (21)

Writing 1_{E_i} for the probability measure on the atoms assigning probability 1 to E_i, we have, for all s, i, P:

$$\frac{Cov(1_s,\nu)}{Var(\nu)} = \frac{Cov\left(1_{E_i},\nu\right)}{Var(\nu)} = \frac{Cov(P,\nu)}{Var(\nu)} = \frac{Cov(\nu,\nu)}{Var(\nu)} = 1. \tag{22}$$

By Eq. 21.(iii), the first fraction is the same for all s. The first equality now follows because $1_s = 1_{E_i}$ on $\mathcal{H}$ for each $s \in E_i$. The second equality follows because every probability measure P on $\mathcal{H}$ is a convex combination of measures $1_{E_i}(.)$, and sensitivity and covariance are compatible with convex combinations.[23] The third equality follows because ν is a special case of a probability measure, and the last equality is by definition. The obtained equality $\frac{Cov(P,\nu)}{Var(\nu)} = 1$ means that, on average, a

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[21] We use population statistics. If one interprets $\mathcal{H}$ as a sample, small relative to $|S|$, then one may prefer sample statistics, with denominators $|\mathcal{H}| - 1$ instead of $\mathcal{H}$. However, those always give the same indexes and same results throughout our paper because the denominator cancels from all equations.

[22] $Cov(1_s, \nu) = \big(E(1_s \times \nu) - E(1_s) \times E(\nu)\big) = E(1_s \times \nu) - \frac{1}{4}$.

[23] That is, the sensitivity (or covariance) of a convex combination of functions with respect to some other variable (ν in our case) is the convex combination of their sensitivities (or covariances).

change of one unit of ν generates one unit change of P which, by Eq. 20, generates an average change $\frac{Cov(m,P)}{Var(P)}$ in m. Under reasonable assumptions on ε (e.g., that ν only works through P), the first-order approximation in Eq. 7 is best (Stock and Watson 2015 §12.1 and Eq. 12.7). In general, if our econometric assumptions are not perfectly well satisfied, then we still propose our indexes as good pragmatic estimates. □

PROOF OF THEOREM 9. Under ambiguity neutrality, m is a probability measure on $\mathcal{H}$ and its atoms. By Eq. 2, $\overline{m} = 0.5$ and $b = 0$. By Eq. 22, $\frac{Cov(m,\nu)}{Var(\nu)} = 1$ and $a = 0$. Conversely, assume $b = 0$ for all probability measures $P = m$. Then $\overline{m} = 0.5$ for all $m = 1_s$, which is l-hedging. Similarly, if $a = 0$ for all probability measures $P = m$ then it is so for all $m = 1_s$, implying $\frac{Cov(1_s,\nu)}{Var(\nu)} = 1$ for all s which, by Eq. 21, implies v-hedging.

We, finally, turn to the supremum values of the indexes. b tends to its suppremum 1 as $\bar{m}$ tends to its minimum 0. a tends to its supremum 1 as $Cov(m,\nu)$ tends to its infimum 0 (by monotonicity, it cannot be negative), which occurs when m tends to a constant function. □

PROOF OF THEOREM 10. We take our data set m as a vector in $\mathbb{R}^{|\mathcal{H}|}$. Index b is a normalization of the inner product of m with the *aversion vector* $(1,\ldots,1)$. Index a is a normalization of the inner product of m with the *insensitivity vector* $\left(\nu(E) - \frac{1}{2}\right)_{E\in|\mathcal{H}|}$.[24] The aversion and insensitivity vectors are orthogonal because their inner product is $\sum\left(\nu(E) - \frac{1}{2}\right) = 0$. □

APPENDIX B. PROOF OF THEOREM 13

That (i) implies (ii) in Theorem 13 follows because Eq. 11 holds for every $\mathcal{H}$. From now on, we assume (ii) and derive (i). To prepare, we first prove that, if our indexes

[24] $|\mathcal{H}|Cov(m,\nu) = \sum(m(E) - \overline{m})\left(\nu(E) - \frac{1}{2}\right) = \sum m(E)\left(\nu(E) - \frac{1}{2}\right)$.

fit perfectly, then we must have probabilistic sophistication within our source S. That is, we must have uniformity in the terminology of Abdellaoui et al. (2011), ruling out Example 11.

OBSERVATION 19. Under Assumptions 2 and 7, if our indexes are the same for every $\mathcal{H}\{E_1, E_2, E_3\}$, and fineness and event-continuity hold, then $m(.) = w_a(P(.))$ for a strictly increasing w_a and a fine (atomless) countable additive probability measure P.

PROOF. The proof uses Lemmas 20-24.

LEMMA 20. We cannot have $A_1 \succ B_1, A_2 \succcurlyeq B_2, A_3 \succcurlyeq B_3$ for two threefold partitions $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ of S containing nonnull events.

PROOF. Consider $\mathcal{H}\{A_1, A_2, A_3\}$ and $\mathcal{H}\{B_1, B_2, B_3\}$. They have the same aversion index b and, hence, the same average $\overline{m}$. Because $\overline{m_s}$ of the former exceeds $\overline{m_s}$ of the latter, for $\overline{m_c}$ it must be opposite. But then (Eq. 9) a is smaller for the former than for the latter, contradicting perfect fit. QED

We next derive implications of event continuity, similar to Villegas (1964 p. 1790) but we do not have what he called monotonicity ($\approx$ additivity)—this is also the reason that we need two event continuity conditions, whereas for Villegas one is equivalent to the other.

LEMMA 21. If $D \succ B \succ \emptyset$, then there exist $C \subset D, A \subset D,$ with $D \succ C \succ B \succ A \succ \emptyset$.

PROOF. There exists $H \subset D$ such that $D \succ H \succ \emptyset$. $D - H$ is nonnull and, by monotonicity, $\succ \emptyset$. We have partitioned D into two nonnull events that we now denote D_1, S_1, where we assume $D_1 \succcurlyeq S_1$. We can similarly partition the smaller of these two, S_1, into two nonnull events $D_2 \succcurlyeq S_2$, and inductively continue to obtain an infinite decreasing (in terms of $\succcurlyeq$) sequence of disjoint nonnull subevents $D_j \subset D$.

Assume, for contradiction, that $D_j \succcurlyeq B$ for all j, which can be interpreted as a violation of Archimedeanity. Whereas $\cup_{i=j}^{\infty} D_i$ decreases to the empty set for $j \to \infty$, every union is $\succ B \succ \emptyset$, violating event continuity. Hence, an $A = D_j$ as required exists. This also implies that $S_\infty := D - \bigcup_{i=1}^{\infty} D_i$ is null. Otherwise, with S_∞ in the role of B, $D_j \prec S_\infty$ should occur for some j as we just showed, contradicting $D_j \succcurlyeq S_j$. We can, therefore, replace D_1 by $D_1 \cup S_\infty$ and every S_j by $S_j - S_\infty$, without affecting preference. That is, $\bigcup_{i=1}^{\infty} D_i = D$. By event continuity, $C := \bigcup_{i=1}^{J} D_j \succ B$ for J large enough. □

LEMMA 22. We cannot have $A_1 \succ B_1, A_2 \succcurlyeq B_2$ for two twofold partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ of S.

PROOF. Assume, for contradiction, events as in the lemma. By Lemma 21, there exists $A_1{'} \subset A_1$ such that $A_1 \succ A_1{'} \succ B_1$. We define $A_1{''} = A_1 - A_1{'} \succ \emptyset$ (by monotonicity). Again by Lemma 21, there exists $B_1{''} \subset B_1$ with $\emptyset \prec B_1{''} \prec A_1{''}$. We define $B_1{'} = B_1 - B_1{''}$. We have two partitions $\{A_1{'}, A_1{''}, A_2\}$ and $\{B_1{'}, B_1{''}, B_2\}$ that violate Lemma 20. QED

LEMMA 23. If $A \cap C = B \cap C = \emptyset$, then $A \succcurlyeq B \Leftrightarrow A \cup C \succcurlyeq B \cup C$.

PROOF. Assume $A \succcurlyeq B$. Consider partitions $\{A, C, S - A - C\}$ and $\{B, C, S - B - C\}$. By Lemma 20, $S - A - C \preccurlyeq S - B - C$. By Lemma 22, $A \cup C \succcurlyeq B \cup C$. The same reasoning holds with strict preferences. QED

Villegas used the following implication.

LEMMA 24. Assume $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$. Then $A_1 \succcurlyeq B_1$ & $A_2 \succcurlyeq B_2 \Rightarrow A_1 \cup A_2 \succcurlyeq B_1 \cup B_2$, with strict preference if at least one of the two premises is strict.

PROOF. By Lemma 23, and Abdellaoui and Wakker (2018). QED

Observation 19 now follows from Villegas (1964, Theorem 4.3). □

OBSERVATION 25. m is neo-additive.

PROOF. By perfect fit, each belief hedge $\mathcal{H}$ imposes two equalities on $m(.) = w_a(P(.))$, one for each index. We know that there exists at least one w_a satisfying all those equalities, being the neo-additive function corresponding with the values b, a found (Eq. 11). It, hence, suffices to show that $w_a(p)$ is uniquely determined for each p. Consider $\mathcal{H}\{E_1, E_2, E_3\}$ with $P(E_j) = \frac{1}{3}$ for each j. By fineness and countable additivity, such E_js exist. Here, b determines the average of $\overline{m_s} = w_a(\frac{1}{3})$ and $\overline{m_c} = w_a(\frac{2}{3})$ and a determines their difference. This uniquely determines $w_a(\frac{1}{3})$ and $w_a(\frac{2}{3})$ as the neo-additive values.

Next assume, for induction w.r.t. $k \geq 0$, that w_a takes the neo-additive values at all $p = \frac{i}{3\times2^k}$. Consider $\frac{j}{3\times2^{k+1}}$ (< ½) for an odd $j < 3 \times 2^k$, and a threefold partition $\{E_1, E_2, E_3\}$ with $P(E_1) = P(E_2) = \frac{j}{3\times2^{k+1}}$, so that $P(E_3) = \frac{3\times2^k - j}{3\times2^k}$. For $\mathcal{H}\{E_1, E_2, E_3\}$'s m values, there are only two unknowns: $w_a(\frac{j}{3\times2^{k+1}})$ (for E_1 and E_2) and $w_a(1 - \frac{j}{3\times2^{k+1}})$ (for $E_1 \cup E_3$ and $E_2 \cup E_3$). Again, Eq. 9 uniquely determines the average and the difference of the two unknowns, so that they are both uniquely determined and must be the neo-additive values. This way, w_a takes the neo-additive values at all $p = \frac{j}{3\times2^{k+1}}$, both below and above $\frac{1}{2}$. By induction, it does so for all k. These values lie dense in (0,1), so that the nondecreasing (by monotonicity it is even strictly increasing) function w_a is the neo-additive function everywhere. □

The following observation follows from the above proof because we only used the designs mentioned.

OBSERVATION 26. Perfect fit in Statement (ii) in Theorem 13 can be restricted to designs $\mathcal{H}\{E_1, E_2, E_3\}$. □

APPENDIX C. PROOFS FOR §4

We fix $\gamma \succ \theta$ and analyze Eq. 1 under the smooth model of ambiguity, explaining notation later:

$$\int_{\Delta(S)} \varphi\big(Q(E)\big) d\mu = \varphi\big(m(E)\big). \tag{23}$$

The smooth model assumes expected utility for risk with utility function u, which we normalize at $u(\gamma) = 1$ and $u(\theta) = 0$. $\Delta(S)$ denotes the set of (first-order) probability measures over S, and μ is a second-order probability distribution over $\Delta(S)$ interpreted as perception of ambiguity. To evaluate $\gamma_E\theta$ (through the integral in Eq. 23), we take the second-order μ-weighted expectation of $Q(E)$, the Q-expected utility, but transformed by a function φ. Concavity of φ captures ambiguity aversion, linearity captures ambiguity neutrality, and convexity captures ambiguity seeking. The right prospect in Eq. 1, $\gamma_{m(E)}\theta$, is evaluated by the right-hand side of Eq. 23, the first-order probability of receiving γ being certain to be $m(E)$. $p = P(E) = \int_{\Delta(S)} Q(E) d\mu$ denotes the a-neutral probability of E. The variance of $Q(E)$ with respect to μ is $\sigma^2 = \int_{\Delta(S)} (Q(E) - P(E))^2 d\mu$.

PROOF OF EQ. 12. Pratt (1964 Eqs. 4-6) studied local risk premiums by letting lotteries converge to a riskless lottery/outcome x, with expectation kept fixed and variance tending to 0. We similarly study local ambiguity premiums by letting acts converge to an unambiguous act/lottery $\gamma_p\theta$, with the ambiguity-neutral part kept fixed and ambiguity σ^2 tending to 0, as follows.

We assume in this Appendix C that all functions are sufficiently smooth with all required derivatives existing and all O and o terms uniform. We follow Klibanoff, Marinacci, and Mukerji (2005) and assume a compound state space $S = S' \times (0,1]$, providing an Anscombe-Aumann mixture structure. Here S' captures the uncertainty of interest and [0,1] is only auxiliary. For example, F' is the event of the AEX index going up by more than 0.2%, and $F = F' \times [0,1]$ is the event of that happening and the result of our randomizing machine just being anything. F and F' can be identified for many purposes. In what follows, we keep some F and the corresponding F' fixed, with fixed a-neutral probability p (μ-averaged $Q(F)$) and fixed μ-variance of $Q(F)$, denoted τ^2. We consider mixtures $\alpha\gamma_F\theta + (1-\alpha)\gamma_p\theta$ comprising an α ambiguous and a $1-\alpha$ unambiguous part, with $\alpha \downarrow 0$. This mixture can be obtained by receiving γ under the disjoint union of an ambiguous and unambiguous event:

$$(F' \times (1-\alpha, 1]) \ \cup \ (S' \times (0, (1-\alpha)p]); \qquad (24)$$

and θ otherwise. The event in Eq. 24 plays the role of event E in the main text. The limit of E tending to an ambiguity neutral event in the main text is achieved by letting

α tend to 0 in Eq. 24. The corresponding ambiguity-neutral probability is $\alpha p + (1-\alpha)p = p$ for all α.

The matching probability m_α is defined by the indifference

$\gamma_{(F' \times (1-\alpha,1]) \cup (S' \times (0,(1-\alpha)p])}\theta \sim \gamma_{m_\alpha}\theta$.

Writing $q = Q(F)$,

$$\int_{\Delta(S)} \varphi(\alpha q + (1-\alpha)p) d\mu = \varphi(m_\alpha)\,. \tag{25}$$

Substituting Taylor series of φ for $\alpha \downarrow 0$ in the right-hand side:

$$\varphi(m_\alpha) = \varphi(p) + (m_\alpha - p)\varphi'(p) + O((m_\alpha - p)^2) \tag{26}$$

and for the integrand of the left-hand side:

$$\varphi(\alpha q + (1-\alpha)p) = \varphi(p) + \alpha(q-p)\varphi'(p) + \frac{1}{2}\alpha^2(q-p)^2\varphi''(p) + o(\alpha^2)\,.$$

Hence the left-hand side of Eq. 25 is:

$$\varphi(p) + \alpha\varphi'(p)\int_{\Delta(S)}(q-p)d\mu + \tfrac{1}{2}\alpha^2\varphi''(p)\int_{\Delta(S)}(q-p)^2 d\mu + o(\alpha^2) =$$

$$\varphi(p) + \tfrac{1}{2}\alpha^2\varphi''(p)\tau^2 + o(\alpha^2) \tag{27}$$

(the term with φ' drops). Because of Eq. 25, we can equate Eqs. 26 and 27:

$$(m_\alpha - p)\varphi'(p) + O((m_\alpha - p)^2) = \tfrac{1}{2}\alpha^2\varphi''(p)\tau^2 + o(\alpha^2)\,.$$

Dividing by $\varphi'(p)$, which does not affect O or o:

$$(m_\alpha - p)\big(1 + O(m_\alpha - p)\big) = \tfrac{1}{2}\alpha^2\frac{\varphi''(p)}{\varphi'(p)}\tau^2 + o(\alpha^2)\,.$$

$$m_\alpha - p = \frac{-\frac{1}{2}\alpha^2 A(p)\tau^2 + o(\alpha^2)}{\big(1 + O(m_\alpha - p)\big)}$$

$$= -\frac{1}{2}\alpha^2 A(p)\tau^2 + \frac{O(m_\alpha - p)\frac{1}{2}\alpha^2 A(p)\tau^2 + o(\alpha^2)}{\big(1 + O(m_\alpha - p)\big)}$$

$$= -\frac{1}{2}A(p)\alpha^2\tau^2 + o(\alpha^2) = -\frac{1}{2}A(p)\alpha^2\tau^2 + o(\alpha^2\tau^2)\,.$$

$\alpha^2\tau^2$ here is the variance of the event in Eq. 24, i.e., it is denoted σ^2 in Eq. 12, which now follows. □

By Eq. 12, $a = 1 - \frac{Cov(p,v)}{Var(v)} + \frac{Cov\left(\frac{1}{2}\sigma^2 A(p) + o(\sigma^2), v\right)}{Var(v)}$. Because $Cov(p,v) = Var(v)$ (Eq. 22), Eq. (14) follows.

APPENDIX D. PROOFS AND FURTHER COMMENTS FOR §§5 AND 6

The representation in Eq. 17 is unique up to a positive factor for U and a joint power for the three functions. All results in our analysis, starting with the matching probability in Eq. 1, are invariant under those transformations and, hence, meaningful. We specify the various models considered only for acts and lotteries with only one outcome different from θ because this is all that is needed for our analysis. General definitions of these models are in the papers cited.

D.1. Qualitative Ambiguity Orderings

Our indexes are compatible with most definitions of ambiguity neutrality in the literature, and orderings of ambiguity aversion, in the sense of extending them. We first discuss neutrality. Ghirardato and Marinacci (2002), and virtually all other papers in the literature, assumed expected utility for risk and then equated ambiguity neutrality with subjective expected utility maximization. Epstein (1999), more generally, allowed for violations of expected utility under risk and equated ambiguity aversion with probabilistic sophistication. In general, his definition may be debatable but, following Dean and Ortoleva (2017 Footnote 31), we agree with it if objective probabilities are included in the domain, and this is our case. In all cases of ambiguity neutrality mentioned in the literature, matching probabilities are additive probabilities P and both our indexes are 0. That is, our indexes are compatible with the common definitions of ambiguity neutrality. The sign of b then is compatible with ambiguity aversion/seeking.

In Epstein (1999), and virtually all other papers in the literature, $\succcurlyeq^1$ is defined to be more ambiguity averse than $\succcurlyeq^2$ if $f \succcurlyeq^1 r \Rightarrow f \succcurlyeq^2 r$ where f is a general, possibly ambiguous act and r is an unambiguous act (risky, with known probabilities).[25] This

[25] This condition implies identical risk attitudes, restricting its applicability. Under same risk attitudes and continuity, it suffices to restrict to constant acts r, which is what Ghirardato and Marinacci (2002) did. The definition was used by Chateauneuf and Faro (2009 p. 541), Dean and Ortoleva (2017 Definition 5), Epstein (1999 Eq. 2.3), Ghirardato and Marinacci (2002 Definitions 4 & 7), Giraud (2014 Definition 7), Gul and Pesendorfer (2014 Corollary 1) and Gul and Pesendorfer (2015

implies that $\succcurlyeq^1$ has lower matching probabilities and, hence, a larger b index. That is, our index b is compatible with this definition. In particular, it is compatible with the pointwise ordering of the uncertainty aversion function G in Cerreia et al.'s (2011 Proposition 6) general uncertainty aversion model.

Some papers considered qualitative orderings of insensitivity or, relatedly, ambiguity perception. In multiple priors models, set-inclusions of sets of priors have been considered (Ghirardato, Maccheroni, and Marinacci 2004 Proposition 6) that, for some tractable subcases, agree with our insensitivity index (Observation [18]). Tversky and Wakker (1995) considered comparative subadditivity for general weighting functions W. If applied to matching probabilities, they correspond with the indexes of Baillon and Bleichrodt (2015) and, therefore, this comparative subadditivity is compatible with our index a. Similarly, Tversky and Wakker's (1995) source preference conditions are compatible with b.

D.2. The Source Method

Abdellaoui et al.'s (2011) tested indexes in their source method, a specification of Choquet expected utility and prospect theory (Tversky and Kahneman 1992). It adds Chew and Sagi's (2008) preference conditions, implying

$$W(E) = w_{So}(P(E)). \tag{28}$$

w_{So} is strictly increasing with $w_S(0) = 0$ and $w_S(1) = 1$. An ambiguity neutral decision maker would have $w_S = w$, treating the subjective *a-neutral probabilities P* as objective probabilities. Chateuneuf, Grant, and Eichberger's (2007) neo-additive model is the special case where w_S is neo-additive. The subscript So expresses dependence on the source of uncertainty. Abdellaoui et al. (2011) call a source So *uniform* if Eq. 28 is satisfied. It was satisfied for five of the six sources that they tested. Gul and Pesendorfer (2015) incorporate uniformity in their definition of source. We focus here on one uniform source So of ambiguity—besides risk with known probabilities.

Abdellaoui et al. (2011) and Dimmock, Kouwenberg, and Wakker (2016), abbreviated AD here, used the best neo-additive approximation of a function (w_S and

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Propositions 3 and 4) with ideal events instead of risk, Klibanoff, Marinacci, and Mukerji (2005 Definition 5), Qu (2015).

$m(E)$, respectively) on the open interval $(0,1)$ by minimizing quadratic distance as in regular regressions. They then derived their indexes from this. We here do so for the function $m(E)$ (Eq. 10), where $\sigma \geq 0$ and τ are chosen to minimize the distance.

AD needed a-neutral probabilities $P(E)$ specified beforehand, by classical Ellsberg symmetry assumptions (Dimmock, Kouwenberg, and Wakker 2016) or by separate measurements (Abdellaoui et al. 2011). Our approach does not need them. We consider some cases of the AD approach, and see how much our approach deviates, depending on the probabilities specified by AD. With τ and σ the best-fitting neo-additive parameters, AD defined (as in Eq. 11)

$$b' := 1 - 2\tau - \sigma,\ \ a' := 1 - \sigma \ \ (\textit{AD indexes}). \tag{29}$$

Our and AD's aversion indexes always completely agree.

OBSERVATION 27. Under Assumptions 2 and 7, our index b (Eq. 4) is identical to the AD index b' (Eq. 29).

PROOF. The distance to be minimized in the AD approach is

$$\sum_{E\in\mathcal{H}}\big(m(E) - \tau - \sigma P(E)\big)^2. \tag{30}$$

The first order condition of Eq. 30 with respect to τ, divided by -2, gives $\sum_{E\in\mathcal{H}}\big(m(E) - \tau - \sigma P(E)\big) = 0$. Thus, using Eq. 2,

$$\tau = \overline{m} - \sigma/2. \tag{31}$$

In words, the best-fitting line passes through the center of gravity of the data points, being $(\frac{1}{2}, \overline{m})$. Now $b' = 1 - 2\tau - \sigma = 1 - 2\overline{m} = b$. It is independent of P. □

For the insensitivity index a, we first consider the most plausible case of P in the sense that it is chosen to best fit the matching probability data.

OBSERVATION 28. Assume Assumptions 2 and 7 Eq. 28, where not only τ and σ, but also P results from minimizing quadratic distance (Eq. 30).Then our insensitivity index a (Eq. 8) is identical to AD's index a' (Eq. 29).

PROOF. We define the additive measure $Q(E) := \sigma P(E)$ and $q_i := Q(E_i) = \sigma P(E_i)$ and find the optimally fitting q_i. We optimize over all $q_i \in \mathbb{R}$, later verifying that they are all positive. We impose first-order optimality conditions on the q_i. Substituting Eq. 31 in Eq. 30, the distance to be minimized becomes

$$\sum_{E\in\mathcal{H}}\big((m(E)-\overline{m})-(Q(E)-\sigma/2)\big)^2. \tag{32}$$

The first-order condition with respect to q_i is

$$\sum_{E\supset E_i}\big((m(E)-\overline{m})-(Q(E)-\sigma/2)\big)=0. \tag{33}$$

Summing over i:

$$\sum_i\sum_{E\supset E_i}\big((m(E)-\overline{m})-(Q(E)-\sigma/2)\big)=0. \tag{34}$$

$$\sum_{E\in\mathcal{H}}\left((m(E)-\overline{m})-\sigma\left(P(E)-\tfrac{1}{2}\right)\right)\nu(E)=0. \tag{35}$$

$$\sum_{E\in\mathcal{H}}\left((m(E)-\overline{m})-\sigma\left(P(E)-\tfrac{1}{2}\right)\right)(\nu(E)-\tfrac{1}{2})=0. \tag{36}$$

$$\sigma\ =\ \frac{\sum_{E\in\mathcal{H}}(m(E)-\overline{m})(\nu(E)-\frac{1}{2})}{\sum_{E\in\mathcal{H}}\left(P(E)-\frac{1}{2}\right)(\nu(E)-\frac{1}{2})}=\frac{|\mathcal{H}|}{|\mathcal{H}|}\frac{Cov(m,\nu)}{Cov(P,\nu)}=\text{(by Eq. 22)}\ \frac{Cov(m,\nu)}{Var(\nu)}\,. \tag{37}$$

As for the first equality in Eq. 37, by monotonicity, the numerator in Eq. 37, the covariance of m and ν, is positive and, by Eq. 22, so is the denominator. This implies $\sigma>0$ and, hence, $q_i=\sigma p_i>0$ for all i. Then $a'=1-\sigma=a$. □

Observation 28—and also Eq. 11—imply complete agreement between AD and us in the neo-additive case. Observation 28 also includes the case most studied in the literature, the classical Ellsberg symmetry case (Dimmock et al. 2015, 2016; Dimmock, Kouwenberg, and Wakker 2016; and most other studies—Camerer and Weber 1992 p. 361), because the best-fitting P then is the symmetric $P=\nu$.

In general, the parameters a and a' are close and diverge only in anomalous cases where subjects extensively violate monotonicity. We used the data of Baillon et al. (2018) to compare a with a' and b with b'. Of course, the estimates of b and b' always completely agreed. The average absolute difference $|a'-a|$ was 0.006. In 95% of the cases, the difference between a' and a was less than 0.01. The remaining 5% all concerned subjects who violated monotonicity, with maximal absolute difference $|a'-a|=0.27$ for a highly erratic subject. We conclude that our and AD's indexes are the same for all practical purposes.

D.3. Choquet Expected Utility

OBSERVATION 29. Under Assumptions 2 and 7, expected utility for risk, and complementation-closedness of $\mathcal{H}$, our ambiguity aversion index b is the average of Eq. 18. In Schmeidler's (1989) model, ambiguity aversion[26] implies $b > 0$, ambiguity neutrality implies $b = 0$, and ambiguity seeking implies $b < 0$.

PROOF. Under expected utility for risk, matching probabilities are equal to event weights; i.e., $m(E) = W(E)$. Hence, the average of $W(E) + W(E^c)$ is $2\overline{m}$, and our b is the average of the values $1 - W(E) - W(E^c)$. Schmeidler defined ambiguity aversion [neutrality; seeking] as quasiconvexity [linearity; quasiconcavity] of preference with respect to outcome (2nd stage probabilities) mixing, which implies positivity [nullness; negativity] of Eq. 18 for all E_i and, hence, of our b. □

The Observation does not need belief hedging of $\mathcal{H}$ or Assumption 7, and complementation-closedness of $\mathcal{H}$ (implying l-hedging) is enough. For statistical reliability, one does not use only one measurement $1 - W(E) - W(E^c)$, but one uses several such and then commonly averages. We do so for our index b, following the restrictions on our measurement domain imposed by belief hedges.

D.4. Multiple Priors

PROOF OF THEOREM 17. $m(E^c) = \alpha P_*(E^c) + (1-\alpha)P^*(E^c) = \alpha(1 - P^*(E)) + (1-\alpha)(1 - P_*(E))$. Further, $m(E) + m(E^c) = \alpha P_*(E) + (1-\alpha)P^*(E) + \alpha(1 - P^*(E)) + (1-\alpha)(1 - P_*(E)) = 1 - (2\alpha - 1)\big(P^*(E) - P_*(E)\big)$. Finally, $b = 1 - \overline{2m(E)} = 1 - \overline{m(E) - m(E^c)} = (2\alpha - 1)\big(\overline{P^* - P_*}\big)$. □

The following result is like Observation 18, with $\varepsilon = 1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$.

OBSERVATION 30. Assume Assumptions 2 and 7, $\mathcal{H}(E_1, E_2, E_3\}$, and the α-maxmin model with C of the form $\{P(E_1) \geq \varepsilon_1, P(E_2) \geq \varepsilon_2, P(E_3) \geq \varepsilon_3\}$ where the ε_j are nonnegative and add to less than 1. Then $a = 1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ and $b = (2\alpha - 1)a$.

[26] Schmeidler used the term uncertainty aversion.

PROOF. By Observation 18, defining $Q(E_j) = \frac{\varepsilon_j}{\varepsilon_1+\varepsilon_2+\varepsilon_3}$ and $\varepsilon = 1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$. □

D.5. Further Remarks

Some ambiguity models have been proposed that do not involve an explicit specification of subjective degrees of belief or objective probabilities (Gilboa 1987; Tversky and Kahneman 1992; Wakker 2010 Part III). These models all are special cases of the biseparable model, and do not use Anscombe and Aumann's (1963) framework. They can analyze general uncertainty attitudes but, in the absence of neutrality calibration, cannot identify ambiguity attitudes.

Some papers defined ambiguity indexes, and orderings, using premiums in monetary units rather than in our probability units (Izhakian and Brenner 2011; Jewitt and Mukerji 2017; Lang 2017; l'Haridon et al. 2018; Maccheroni, Marinacci, and Ruffino 2013; Montesano and Giovannoni 1996). These indexes depend on the utility function, are outcome-oriented, and are not directly related to our indexes.

ERASMUS SCHOOL OF ECONOMICS, ERASMUS UNIVERSITY ROTTERDAM

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Online Appendix of
"Belief Hedges: Applying Ambiguity Measurements to All Events and All Ambiguity Models"

Aurélien Baillon, Han Bleichrodt, Chen Li, & Peter P. Wakker
Erasmus School of Economics, Erasmus University Rotterdam, Rotterdam, the Netherlands, Baillon@ese.eur.nl, c.li@ese.eur.nl, Wakker@ese.eur.nl

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PROOF OF EQ. 9. The case of index b is clear. As for index a,

$|\mathcal{H}|Cov(m,v) = \sum_{i=1}^{3}(m(E_i) - \overline{m})\left(\frac{1}{3} - \frac{1}{2}\right) + \sum_{i=1}^{3}(m(E_i^c) - \overline{m})\left(\frac{2}{3} - \frac{1}{2}\right) =$

$3(\overline{m_s} - \overline{m})\left(-\frac{1}{6}\right) + 3(\overline{m_c} - \overline{m})\left(\frac{1}{6}\right) = \frac{\overline{m_c} - \overline{m_s}}{2}$;

$|\mathcal{H}|Var(v) = \sum_{i=1}^{3}\left(\frac{1}{3} - \frac{1}{2}\right)^2 + \sum_{i=1}^{3}\left(\frac{2}{3} - \frac{1}{2}\right)^2 = \frac{1}{6}$.

$\frac{Cov(m,v)}{Var(v)} = 3(\overline{m_c} - \overline{m_s})$. □

PROOF OF EQ. 11. Because $\emptyset$ and S are not in $\mathcal{H}$, and all atoms are nonnull, $0 < P(E) < 1$ for all $E \in \mathcal{H}$. Hence , $\overline{m} = \tau + \sigma\,\overline{P}$ = (by Eq. 2) $\tau + \sigma/2$ and the result for b follows. As regards a, because m is an affine function of P with slope σ on $\mathcal{H}$, $\frac{Cov(m,v)}{var(v)} = \sigma\frac{Cov(P,v)}{var(v)}$ = (by Eq. 22) σ. □

COMPARISON OF a WITH a' IN THE EXPERIMENT OF BAILLON ET AL. (2018)

The experiment of Baillon et al (2018) consisted of two treatments (control vs. time pressure) and two parts. Each scatter plot displays a' as a function of a. Points that are not on the diagonal are all due to violations of monotonicity.

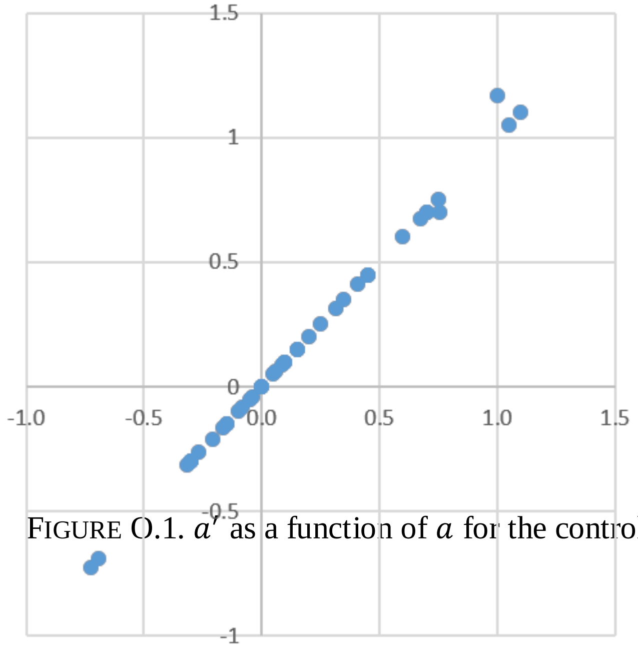

FIGURE O.1. a' as a function of a for the control treatment – Part 1

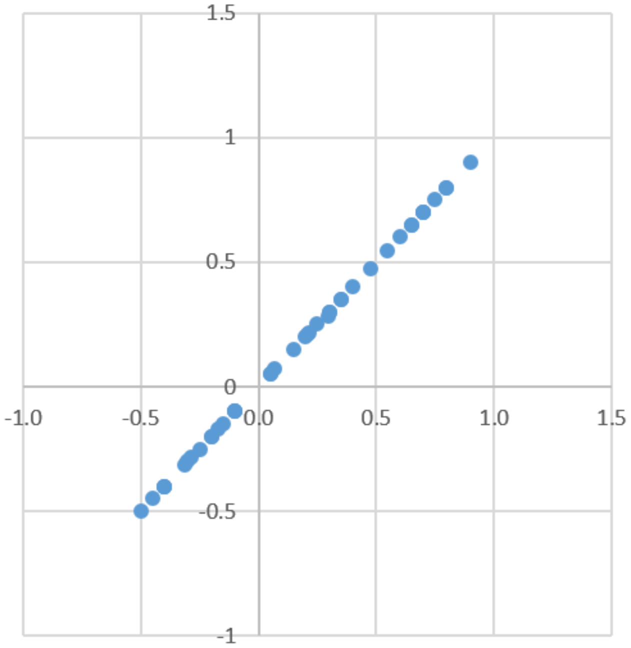

FIGURE O.2. a' as a function of a for the control treatment – Part 2

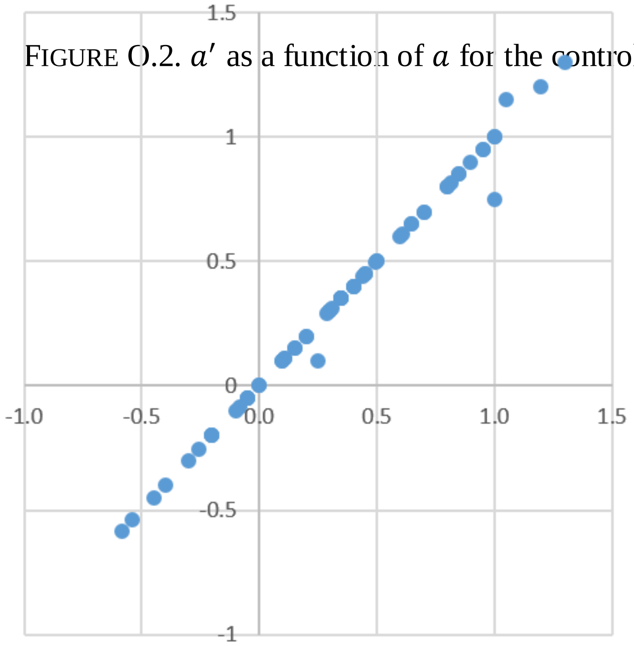

FIGURE O.3. Figure O.3. a' as a function of a for the time pressure treatment – Part 1

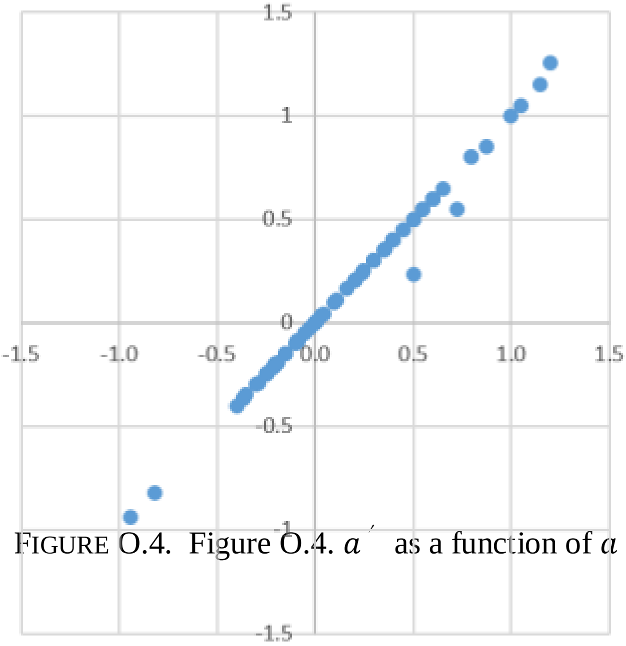

FIGURE O.4. Figure O.4. a' as a function of a for the time pressure treatment – Part 2