A simple and general axiomatization of average utility maximization for infinite streams

CHEN LI & PETER P. WAKKER

C.li@ese.eur.nl; Wakker@ese.eur.nl

Erasmus School of Economics, Erasmus University Rotterdam

June 2023

ABSTRACT. This paper provides, first, the most general preference axiomatization of average utility (AU) maximization over infinite sequences presently available, reaching almost complete generality (only restriction: all periodic sequences should be contained in the domain). Here, infinite sequences may designate intertemporal outcomes streams where AU models patience, or welfare allocations where AU models fairness, or decision under ambiguity where AU models complete ignorance. Second, as a methodological contribution, this paper shows that infinite-dimensional representations can be simpler, rather than more complex, than finite-dimensional ones: infinite dimensions provide a richness that is convenient rather than cumbersome. In particular, (empirically problematic) continuity assumptions are not needed. Continuity is optional.

JEL-classification: D63, D81, C61, D90, C30

Keywords: discounted utility, fairness, expected utility, insufficient reason, generalized means; infinite populations, Cesàro mean

1. INTRODUCTION

Many authors have argued for “fair” average utility (AU) maximization, as a normative objection to impatience and discounted utility (da Volterra 1574; Elster 1986 pp. 10-11; Jevons 1871; Pigou 1920; Sidgwick 1874; Weinstein 1993). AU maximization has also gained popularity for fair social welfare evaluations, decisions under complete ignorance (Gravel, Marchant, & Sen 2018) or under the principle of insufficient reason (Laplace 1796), and in many other contexts. As an important and widely applicable decision model, AU has received numerous axiomatizations for the finite-dimensional case.[1]

This paper considers AU for infinitely many timepoints (dimensions). Here, mathematical and philosophical problems arise. Limits may diverge and choice paradoxes arise. For example, patience or fairness may be irreconcilable with strong Pareto optimality (Diamond 1965) for welfare evaluations. Equivalently, for uncertainty, stochastic dominance may be violated (Wakker 1993). The two-envelope paradox and Dubin’s paradox are among the paradoxes for uncertainty.[2] These problems have intrigued researchers for over a century, and many solutions have been discussed, often by restricting the domain of preference or relaxing completeness of preference.

This paper provides a preference axiomatization of AU maximization over infinite sequences of great, and almost complete, generality. For instance, unlike all preceding axiomatizations, we do not need the assumption of continuity, which is less innocuous than may seem as explained below. Similar to preceding work, we adopt the approach of domain restriction. Our only restrictive assumption is that all periodic sequences (defined later) should be contained in the preference domain. This is

---

[1] These axiomatizations were central in Blackorby et al. (1977), Gravel, Marchant, & Sen (2012), and Kothiyal, Spinu, & Wakker (2014). Every axiomatization of additive $\sum_{j=1}^{n} V_j(x_j)$ with fairness (symmetry) added gives AU (Debreu 1960; Gorman 1968; Krantz et al. 1971). Further, quasilinear means from mathematics are constant-equivalence functions of AUs, giving many more axiomatizations (Aczél 1966 pp. 151 & 240; Münnich, Maksa, & Mokken 2000).

[2] For the former, see Kraitchik (1953), Nalebuff (1989), and Yi (2013). For the latter, see Howson (2014). Both concern the (im)possibility to have a uniform probability distribution over the natural numbers, which underlies AU.

considerably less stringent than what has been assumed in all preceding axiomatizations of AU.[3] Other than this restriction, we obtain complete generality regarding preference relations, outcome sets, utility functions, and preference domains. We, thus, provide maximal possibilities for reconciling desirable properties and for detecting mathematical problems. Whereas we do not need continuity, we can easily incorporate it when desired. Thus, Observation 6 adds continuity and provides the most general axiomatization of continuous AU in the literature. To obtain our results, we will solve two problems, explained next.

It is commonly believed that infinite-dimensional representations are more complex than finite-dimensional ones. Thus, researchers usually first derive finite-dimensional representations, and then face the problem of how to extend those to infinite dimensions (discussed by Asheim 2010 §4.2 and Pivato 2014 p. 35). However, this seemingly natural route is in fact creating obstacles that unnecessarily complicate the analysis. This brings us to the second problem, occurring for finite dimensions, and discussed next.

Finite preference domains are usually too coarse to imply precise quantitative representations. This greatly complicates the mathematical analysis, to the extent that preference axiomatizations for finite domains are usually unknown. Researchers, therefore, commonly resort to the simplifying assumption that the preference domain is a continuum, as for instance throughout Adler (2019) and Blackorby, Bossert, & Donaldson (2005). Then quantitative representations can be precisely identified[4], and the mathematical analysis becomes tractable. Thus, Chambers, Echenique, & Lambert (2021) wrote: “continuity is a necessary regularity condition; without it, no meaningful inferences can be made with any finite amount of data.”

However, the assumption of continuity does not come without any empirical cost. To explain the problematic empirical status of continuity, we first note that, in isolation, continuity can never be verified or falsified by a finite number of observations. In this sense it has no empirical content. However, this is not so much a problem and rather is a pro. We know exactly what we are doing empirically when

---

[3] See Fishburn & Edwards (1997), Harvey (1986), Johnsson & Voorneveld (2018), Lauwers (1998), Marinacci (1998), Pivato (2022), and Rébillé (2007).

[4] It is understood that then the right scale type, such as an equivalence class of interval scales (defined later), is uniquely identified.

assuming continuity in isolation: nothing! Thus, several authors have argued that continuity is empirically harmless. For instance, Arrow (1971 p. 48) wrote: “The assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem”, a point supported by Drèze (1987 p. 12) and Thomson (2001 §4.1.3 p. 338). Unfortunately, those suggestions camouflage serious underlying problems. If some other axioms are assumed, then continuity is not empirically vacuous. It can then add empirical content to those other axioms. And the problem is that we do not know exactly what. We do not exactly know how to verify or falsify our axiom set including continuity from finitely many observations. That is, we lose track of what we are doing when continuity and other axioms are assumed together. Many authors have pointed out this problem.[5]

The aforementioned problems seem challenging. We provide a solution, extending a result for finite-length sequences by Kothiyal, Spinu, & Wakker (2014), presented in the Appendix. The problem of continuity can be avoided by taking an alternative path: making use of the richness provided by infinite dimensions (rather than avoiding them). This richness is a convenience rather than a complication, which enables us to obtain necessary and sufficient preference axioms without continuity. Theorem 3 presents this solution, indeed assuming infinitely many dimensions and not needing any continuity. Observation 4 further illustrates this point by showing how the infinite dimensions enable us to uniquely capture utility without needing continuity. Pivato (2014 p. 56) shared our observation that infinitely many dimensions may be convenient. This is key to our first contribution, being the general axiomatization of AU. Our second contribution is the general technique of exploiting the richness of infinite dimensions. It can serve to generalize other results in the literature.

Observation 5 shows that our new methodology can be used constructively to obtain a preference domain where AU is maximized. This way, we can readily identify any domain where AU maximization does not encounter any of the

[5] See Khan & Uyanik (2021), Kothiyal, Spinu, & Wakker (2014), Krantz et al. (1971 §9.1), Pfanzagl (1968 §6.6), Pivato (2014 p. 32), and Wakker (1988).

aforementioned mathematical problems or paradoxes. Thus, a reconciliation results for discrete outcome sets, a case not considered before in the literature.

2. Basic Definitions

Γ denotes a set of *outcomes*, with generic notation $\alpha, \beta, \ldots$ (or f_j, g_j; see later). Outcomes can be quantitative or not, and Γ can be finite or infinite. *Streams* are infinite sequences $f = (f_1, f_2, \ldots)$ of outcomes, with generic notation $f, g, \ldots$. They can be welfare allocations over individuals where average utility (AU) captures fairness, time profiles where AU captures patience, gambles on states of nature where AU captures extreme ambiguity (complete ignorance), commodity bundles, and so on. Our results can be applied to all these contexts. We interpret f_j as the *outcome* for *generation* j, combining welfare and intertemporal considerations. F, the *preference domain*, denotes a subset of the set of all streams, further specified later.

By $\succcurlyeq$ we denote a binary relation on F, the *preference relation*. We call $\succcurlyeq$ a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all $f, g \in F$) and transitive. The notation $\succ, \preccurlyeq, \prec, \sim$ is as usual. *Average utility* (*AU*) holds on a subset $F' \subset F$ if there exists a *utility function* $U: \Gamma \to \mathbb{R}$ such that the *average utility* $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j)$ $(= AU(f))$ exists[6] for every stream f in F' and *represents preference* on F', i.e., $f \succcurlyeq g \Leftrightarrow AU(f) \geq AU(g)$. Note that the ordering of natural numbers and indexes is essential in the definition of AU. AU and our analysis can, therefore, only be applied when such an ordering is naturally available. This occurs, for instance, for discrete timepoints and for consecutive generations, but not if the set of indexes were the set of rational numbers.

We identify outcomes α with constant streams $(\alpha, \alpha, \ldots)$. Hence, $\succcurlyeq$ also applies to outcomes. To avoid triviality, we assume throughout that Γ contains at least two nonindifferent outcomes. *Monotonicity* holds if $f \succcurlyeq g$ whenever $f_j \succcurlyeq g_j$ for all j.

---

[6] In our terminology, existence implies being real-valued. The average is sometimes called the Cesàro limit.

Strong Pareto optimality holds if $f \succ g$ whenever $f_j \succcurlyeq g_j$ for all j and $f_j \succ g_j$ for some j. *Weak Pareto optimality* holds if $f \succ g$ whenever $f_j \succ g_j$ for all j.

Fairness means that, for a given set of permutations of the indexes, none affects preference. For precise definitions, the set of permutations should be specified. Other names for fairness used in the literature include anonymity, patience, impartiality, (intergenerational) equity, intergenerational neutrality, or symmetry. Fairness is typically violated under discounting with impatience, with early generations privileged over later ones. By treating infinitely many generations all the same, mainly fairness (together with our other axioms) implies that any finite number of generations is negligible. Only the long run matters. Some authors have argued that intergenerational equity should consider nontrivial tradeoffs between present and future generations.[7] Then the complete neglect of the present generation would be a drawback of AU. The present generation can still play a role in generalized models when using non-Archimedean or infinite-valued AU representations. However, we will focus on standard finite-valued representations.

Fairness is often incompatible with strong Pareto optimality (Diamond 1965; Lauwers 1998; Neyman 2023 Fact 12; Weymark 1995). This has been the topic of many studies, where different strategies of reconciliation have been examined. Asheim (2010) and Petri (2019) provided surveys. Because our model can do without continuity, a new reconciliation possibility immediately emerges for weak Pareto optimality: if we have a discrete utility range and there exists $\varepsilon > 0$ such that any two non-indifferent outcomes have a utility distance of at least ε. Then under AU, indeed, $f \succ g$ whenever $f_j \succ g_j$ for all j. Another solution, widely studied, is to limit the set of preferences considered, e.g., by abandoning completeness (see §6) or, our approach, by considering particular subdomains (p-streams, defined later).

3. Average Utility for Periodic Streams

For a finite sequence $(x_1, \dots, x_n)$ of outcomes, called *generator*, the *periodic extension*, denoted $[x_1, \dots, x_n]$, is the stream $x = (x_1, \dots, x_n, x_1, \dots, x_n, x_1, \dots)$. That is,

[7] We thank an anonymous referee for pointing out this argument.

$x_{jn+i} = x_i$ for all $i \leq n$ and j. Periodic extensions are also called *periodic streams* or *p-streams*. We call n the *length* (of the periodic extension/stream or the generator), denoted $||x||$, bearing in mind that periodic streams are infinite sequences. For each outcome α, $[\alpha] = (\alpha, \alpha, \ldots)$ has length $||[\alpha]|| = 1$. F^p denotes the set of *all* periodic sequences, with generic notation $x, y, \ldots$. .

The "infinitistic" AU of a p-stream is the "finitistic" average utility of its generator. This suggests that p-streams can combine the intuitive convenience of finite-dimensional simplicity with the mathematical convenience of infinite-dimensional richness. This insight underlies the analysis of this paper. We impose conditions as much as possible only on F^p, where they are simple. We, accordingly, first derive our results on F^p. Observation 1 will in fact achieve complete generality of AU maximization, extending Kothiyal, Spinu, & Wakker (2014 Theorem 7) to infinite dimensions.

For every preference condition C of $\succcurlyeq$, *periodic C*, or *p-C* for short, refers to that condition when restricted to F^p, as in p-weak ordering, p-monotonicity, and so on. We define *periodic fairness*, or *p-fairness* for short, to imply that any permutation of the x_is in any p-stream $[x_1, \ldots, x_n]$ leaves the stream indifferent. This permutation, finitistic in spirit, does involve infinitely many generations. For example, interchanging $i = 1,2$ of the generator means interchanging every $j \times n + 1$ and $j \times n + 2$ of the p-stream.

P(eriodic)-independence holds if

$$[c_1, x_2, \ldots, x_n] \succcurlyeq [c_1, y_2, \ldots, y_n] \implies [d_1, x_2, \ldots, x_n] \succcurlyeq [d_1, y_2, \ldots, y_n] \qquad (1)$$

Note that this involves identical outcomes for generations $n+1$, $2n+1$, and so on. By fairness, the condition implies that preferences between p-streams of the same length are also independent of common 2^{nd}, 3^{rd}, …, and nth dimensions of the generator and, by repeated application, of any number of common dimensions. That is, the condition amounts to regular separability for the generators. Rébillé (2007) and Asheim & Zuber (2014) reconciled fairness and Pareto optimality by using rank-dependent weakenings of separability.

AUs, as do all real numbers, have to satisfy an Archimedean axiom. To prepare for the preference condition to capture this, a notation: for p-streams x, y, $[nx, my]$ denotes the periodic stream extending n times the (finite) generator of x followed by

m times the (finite) generator of y. For example, if $x = [x_1, x_2]$ and $y = [y_1, y_2]$, then $[2x, 3y] = [x_1, x_2, x_1, x_2, y_1, y_2, y_1, y_2, y_1, y_2]$.[8] We will suppress $n = 1$ and $m = 1$, as in $[x, 4y] = [1x, 4y]$. This paper will use the "for all" symbol ∀ and the "there exists" symbol ∃. The *p-Archimedean axiom* holds if:

$$\forall x, y, v, w \in F^p \text{ with } ||x|| = ||y||, ||v|| = ||w||, x \succ y, \exists n: [nx, v] \succcurlyeq [ny, w] \quad (2)$$

That is, no matter how disadvantageous $v \prec w$ is, it can be overcome by sufficiently many (n) advantages $x \succ y$.[9]

OBSERVATION 1. Assume that the preference domain F contains all periodic streams F^p. Then

(i) AU holds on F^p

$\Leftrightarrow$

(ii) $\succcurlyeq$ satisfies:

1. p-weak ordering;
2. p-Archimedeanity;
3. p-fairness;
4. p-independence.

□

In the above observation, p-monotonicity is implied by the other conditions, mainly p-independence and p-fairness. Strong Pareto optimality is also implied on F^p by the other conditions. Given the restricted domain of F^p, this is not very surprising,

---

[8] For a p-stream x, the streams $[mx]$ (using obvious notation) and x are formally identical in our model. We could take both $||x||$ and $m||x||$ as length. The choice of length (or of the corresponding generator) never matters in our analysis. Formally, for unique definitions, we could commit to minimal lengths and generators, with for instance (x_1, x_2) the generator of $[x_1, x_2, x_1, x_2]$, e.g., to obtain a unique notation $[nx, my]$. P-streams are isomorphic with the set of finite sequences of varying lengths modulo the formal identity just mentioned, corresponding with a condition for finite sequences sometimes called replication-invariance (Kothiyal, Spinu, & Wakker 2014; see the proof of our Observation 1).

[9] Parfit (1984) did not like this condition and used the term "repugnant" for it. Tännsjö (2002) criticized this term. Asheim & Zuber (2014) used rank dependence to avoid it.

and on large domains AU may violate it.[10] We will need the following result for later purposes. We present its proof in the main text because it illustrates how the richness of infinitely many timepoints brings richness of utility without requiring any continuity. This will be exploited in the next section.

LEMMA 2. $AU(F^p)$ is dense within its convex hull.

PROOF. Consider p-streams x, y with lengths $||x||$ and $||y||$. For every rational $0 < \mu = \frac{m}{m+n} < 1$ we have $AU([m||y||x, n||x||y]) = \mu AU(x) + (1 - \mu)AU(y)$. □

4. AVERAGE UTILITY FOR GENERAL STREAMS

We now turn to general, nonperiodic, streams. Pivato (2014) considered representations for such streams using nonstandard real numbers, but these are not widely known to researchers. He generalized Kothiyal, Spinu, & Wakker (2014), using intuitive axioms like those in our Observation 1. Several papers considered reconciliations of fairness and Pareto optimality using general orderings with multi-utility representations and incompleteness that cannot be represented by real numbers (Basu & Mitra 2007; Bossert, Sprumont, & Suzumura 2007; Khan & Stinchcombe 2018). We will stick to representations by standard real numbers and see how far we can go. Our decision-theoretic task is to obtain axiomatizations that use only conditions directly in terms of $\succcurlyeq$.

P-streams will be used to calibrate the other streams. F^p is rich enough to serve this purpose well. We therefore assume, implicitly throughout the text and explicitly in theorems, that F contains all p-streams. We next define the required preference conditions. To clarify them, we will claim several implications of AU on F^p. Those are all proved in the proof of Theorem 3.

We first rule out infinite AU values. To prepare, the outcome set Γ is *unbounded above* if

[10] We thank an anonymous referee for pointing this out.

$$\forall \gamma \succ \beta \in \Gamma, n \in \mathbb{N}\ \exists \delta:\ [n\beta, \delta] \succ [n\gamma, \gamma] \quad (3)$$

We here used the notation $[n\gamma, \gamma]$ instead of γ (which is the same) for clarification. The condition implies that, no matter how many times (n) we receive the drawback of β instead of γ, there is an outcome δ so good and exceeding γ so much that receiving it once instead of γ is enough to overcome those n drawbacks. Under AU on F^p it is necessary and sufficient for U to be unbounded above. Similarly, Γ is *unbounded below* if

$$\forall \gamma \succ \beta \in \Gamma, n \in \mathbb{N}, \exists \alpha:\ [n\gamma, \alpha] \prec [n\beta, \beta] \quad (4)$$

Under AU on F^p it is necessary and sufficient for U to be unbounded below. Stream f is *value-unbounded above*, or *v-unbounded above* for short, if Γ is unbounded above and

$$\forall \gamma \in \Gamma\ \exists n \in \mathbb{N}:\ [f_1, \dots, f_n] \succcurlyeq \gamma \quad (5)$$

Under AU on F^p, Eq. 5 holds if and only if $limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U(f_j) = \infty$. Stream f is *v-unbounded below* if Γ is unbounded below and

$$\forall \alpha \in \Gamma\ \exists n \in \mathbb{N}:\ [f_1, \dots, f_n] \preccurlyeq \alpha \quad (6)$$

Under AU on F^p, Eq. 6 holds if and only if $liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U(f_j) = -\infty$. Stream f is *v-bounded* if it is neither v-unbounded above nor below. Under AU on F^p, f is v-bounded if and only if $liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U(f_j)$ and $limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U(f_j)$ are finite. Thus, v-boundedness of f need not imply boundedness of f itself.

We next rule out v-bounded streams f whose AU is not well-defined. Stream f is *stable* if for all p-streams $x \succ y$ there exists $N \in \mathbb{N}$ such that $[f_1, \dots, f_n] \preccurlyeq x$ for all $n \geq N$ or $[f_1, \dots, f_n] \succcurlyeq y$ for all $n \geq N$. That is, we must be able to decide whether f is below x or above y, possibly both. Under AU on F^p, this holds if and only if $liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U(f_j) = limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U(f_j)$. That is, if the liminf is strictly smaller than the limsup, so that $\frac{1}{n}\sum_{j=1}^{n} U(f_j)$ keeps on fluctuating through the interval between them, then because of the denseness of $AU(F^p)$ we can find p-streams x, y within this interval that reveal this fluctuating character. For this revelation, we only use observable preferences between auxiliary p-streams.

If we have AU on F^p then, for v-bounded stable f, AU is well-defined and finite. We, finally, ensure that AU represents the preference relation. *P-denseness* holds if, for all streams f, g,

$$(f \succ g) \Longleftrightarrow (\exists x, y \in F^p, N \in \mathbb{N}: \forall n \geq N: [f_1, \ldots, f_n] \succ x \succ y \succ [g_1, \ldots, g_n]) \quad (7)$$

Under AU on F^p, the condition ensures that for every strict preference $f \succ g$, because of denseness of $AU(F^p)$, we can get a strict p-stream preference in between f and g. More precisely, we can get them in between the beginning n outcomes of f and g for all n far enough into the future. For this revelation, we again only use observable preferences between auxiliary p-streams. In the condition, getting $x \succ y$ in between precludes infinitesimal strict preferences $f \succ g$. In the following theorem, an *interval scale* is unique up to level and unit (positive affine transformations).

THEOREM 3. Assume that the preference domain F contains all periodic streams F^p. Then
(i) AU holds on F

$$\Longleftrightarrow$$

(ii) $\succcurlyeq$ satisfies:

1. weak ordering;
2. p-Archimedeanity;
3. p-fairness;
4. p-independence;
5. all $f \in F$ are v-bounded;
6. all $f \in F$ are stable;
7. p-denseness.

Further, if (i) holds, then U is an interval scale. □

In the theorem, monotonicity is again implied by the other conditions. Strong Pareto optimality need not always hold, and this depends on the domain F. Because our domain F is the most general for AU available in the literature as yet, it provides maximal possibilities of reconciliations through domain restriction.

5. FURTHER RESULTS

The following observation shows in a direct manner that the infinitely many timepoints provide enough richness to uniquely calibrate U. This further illustrates why we do not need continuity. The result is reminiscent of standard gamble utility measurements in expected utility. A proof of Observation 1 alternative to the one in this paper could have been obtained by defining a mixture operation with rational mixing weights on p-streams and then using mixture space techniques. The observation immediately follows from substitution.

OBSERVATION 4. Assume AU and $F^p \subset F$. Assume $\gamma \succ \beta \succ \alpha \in \Gamma$. Then the preference between $[m\gamma, n\alpha]$ and β corresponds exactly with the ordering of $\frac{m}{m+n}U(\gamma) + \frac{n}{m+n}U(\alpha)$ and $U(\beta)$. If we scale $U(\gamma) = 1$ and $U(\alpha) = 0$, then this uniquely determines $U(\beta)$. In general, it uniquely identifies the interval scale U. □

Our preference conditions facilitate a constructive definition of preference domains where AU can hold. We assume weak ordering in what follows. We start with the set F^p. We then seek to extend the AU representation step by step, each time verifying if streams to be added satisfy all required conditions. That is, they must be v-bounded, stable, and for every newly added preference $f \succ g$ the right-hand of Eq. 7 should hold. We developed our preference conditions so that only p-streams are invoked as auxiliary tools. This shows that p-streams provide a convenient calibration tool for constructing domains where mathematical problems and inconsistencies can be avoided. This, somewhat informal, result is displayed next.

OBSERVATION 5. The set of periodic streams offers sufficient calibration possibilities to constructively define any preference domain for AU. □

As explained, continuity of utility is optional in our approach. The following observation covers connected topological spaces Γ, which includes all intervals, all convex subsets of commodity spaces, and many mixture-closed sets of probability distributions over prizes. Fishburn & Edwards (1997) also assumed connected topological outcome spaces. All other references made more restrictive topological assumptions.

OBSERVATION 6. Assume AU on F which contains all periodic streams F^p with the outcome space Γ a connected topological space. Then U is continuous if and only if $\succcurlyeq$, restricted to p-streams $[f_1, f_2]$, is continuous when taken as a binary relation on $\Gamma \times \Gamma$ endowed with the product topology. □

Of course, continuity of preference above can be strengthened to hold for any p-streams of any length, and further by specifying proper infinite-dimensional continuities.

6. RELATED LITERATURE

Fishburn & Edwards (1997) axiomatized AU but only for pairs of streams that differ on no more than finitely many timepoints. Then the long run does not matter. The model is essentially finite-dimensional and too restrictive for most purposes. It contains no preferences between periodic streams.

Pivato (2022) is closest to us. He was he first to axiomatize AU for (truly) infinite sequences. His outcome set is a connected metric space and U is continuous. His preference domain contains (roughly) the closure of all "regular totally bounded" sequences. Regular means that limiting frequencies exist. This incorporates all our periodic sequences. All permutations on periodic streams that our fairness condition involves belong to his Lévy group and, hence, his Γ-invariance implies our fairness. His AU does not satisfy strong Pareto optimality.

We next consider some papers that did not exactly axiomatize AU for infinite sequences, but that considered close generalizations and/or modifications for infinite sequences. Harvey (1986, Theorem 2a and 2´a) considered the maximization of sums of utilities rather than averages. To have those sums finite, he specified a status quo outcome α^* with utility 0, and only considered streams that, roughly, converge to α^* so strongly that the sum of utilities is defined (his Definition 9). They all have $AU =$

0 and his model can be considered to maximize infinitesimal AU. But it shares most characteristics with AU. In particular, it satisfies the preference conditions in our Theorem 3 except p-denseness. On his domain, strong Pareto optimality (“strict increasingness”) and fairness (“time neutrality”) are reconciled. The main difference with our model is that his domain, basides not including periodic streams, is very restricted, with the long run never mattering, always ending up at α^*.

Lauwers’ (1998) theorems did not take a preference relation, but a representing functional, as primitive. His domain was the set of all bounded real-valued sequences. He assumed linear utility (($U(\alpha) = \alpha$), implied by linearity of the functional, and supnorm continuity. He also incorporated nonstable streams with representations between the limsup and the liminf of AU (his Theorem 2). His anonymity immediately implies p-fairness. It cannot be reconciled with strong Pareto optimality.

In Marinacci (1998), the outcome set consists of all simple probability distributions over a set of prizes. Expected utility is maximized over outcomes. That is, the outcome set is convex and utility is linear. The preference domain consists of all bounded streams. Marinacci assumed continuity with respect to probabilistic mixing of the outcomes and axiomatized liminf AU representations or, more generally, their Polya extensions, using axioms similar to axioms from multiple priors models in decision under ambiguity. He next added a time invariance axiom that implies AU maximization. His patience implies our p-fairness. He did not consider reconciliation with strong Pareto optimality.

Rébillé (2007) assumed real-valued outcomes and linear utility and considered the domain of all bounded sequences. He considered several generalizations of AU such as discounted rank-dependent forms, but he did not derive AU itself.

Neymann (2023), again, assumed real-valued outcomes and linear utility and considered only bounded sequences. His Theorem 2 characterized AU by, mainly, linearity and what he called, in deviation from common terminology, patience (roughly, linear patience restricted to constant streams).

Strong Pareto optimality and fairness can be, trivially, reconciled by excuding all problematic preference situations, leading to a relaxation of completeness. Several papers studied such relaxations, but with the purpose of obtaining nontrivial results with a rich and interesting domain of preference situations. These studies often incorporated generalizations of AU representations that then become possible. For instance, the overtaking criterion

$$f \succcurlyeq g \Leftrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\left(U(f_j) - U(g_j)\right) \geq 0 \quad (8)$$

extends AU by incorporating many preferences where f and g themselves have infinite or undefined AU values. Gale (1967), Johnsson & Voorneveld (2015), and Svensson (1980) provided further generalizations. Yet more general is Johnsson & Voorneveld's (2018) limit-discounted utilitarian criterion. It replaces the lim in Eq. 8 by a liminf and adds a positive discount factor tending to zero discounting. The main charactering preference condition is a compensation principle: postponing all utilities by one period, compensated by adding average utility at the beginning, leaves the stream indifferent. These papers all assumed linear utility or, equivalently, that outcomes are utils, and they focused on bounded sequences. Other papers considered fairness conditions so much weaker that they do not conflict with strong Pareto optimality, including Lauwer's (2012) fixed-step anonimity. Mitra & Basu (2007) showed that such fairness restrictions must satisfy cyclicity and group-operation conditions. Our p-fairness does satisfy those conditons and is weaker than the aforementioned fairness conditions. In return, the cited papers could handle more general representations than AU.

All aforementioned studies assumed continuous, and mostly even linear, utility. Whenever the long run was relevant, all p-streams were included in the domain. Focusing on AU representations as defined in §2, our result is uniformly most general in the sense that for every such representation in the literature our assumptions are satisfied but not the other way around. Besides (1) continuity (always more restrictive than in our Observation 6) all published results (2) focused on bounded streams whereas we can handle all unbounded streams as long as their AU is finite; (3) assumed rich domains of all streams that are bounded and satisfy some regularity assumptions, whereas we allow for almost any kind of subdomain; (4) used more permutations in their fairness that in our p-fairness.

Our result does not generalize the existing results in a logical sense. That is, the assumptions in existing results do not imply our assumptions in an elementary manner—to our best knowledge—and in this sense are not corollaries of our results. Further, most other studies extended the AU representation to streams for which AU is not defined or infinite, and studied other properties than considered in this paper. Thus, the existing results remain of independent interest.

7. DISCUSSION

Many papers in welfare theory take individual utilities ("utils" or “welfare”) as given or, equivalently, assume real-valued outcomes and linear utility for all individuals. However, in most applications individual utilities are subjective and not directly observable and utility is nonlinear. We took the more general approach of allowing for general utility, to be revealed from preferences. AU does assume the same utility function for different j. That is, the same outcome gives the same utility for different generations. For individual intertemporal (non)discounted utility this is a common assumption, but for welfare evaluations there is interest in generation-dependent utility. Generation independence may then be restored by requiring complete descriptions of outcomes, at the cost of more complexity there. Harvey (1986 Theorem 9) and Wakker & Zank (1999) provided models with generation-dependent utility, but they heavily used continuity. We leave such generalizations of our results to future work.

The Archimedean axiom is a technical axiom, like continuity. One could, accordingly, be concerned about a similarly problematic empirical status. In several situations it has been shown though that Archimedean axioms do not have such problems: finitely many observations verify or falsify a set of other axioms if and only if they do so with the Archimedean axiom added (Luce et al. 1990 Theorem 21.21). The axiom then has no empirical content and is innocuous. We do not know to what extent such a result holds for the theorems in this paper. However, these problems of empirical status are smaller than for the more restrictive continuity axioms and, further, they cannot be avoided anyhow because the Archimedean axiom is necessary for any AU representation.

The three axioms used to extend AU to non-periodic streams involve “there exist” quantifiers and their negations, and share the drawbacks of all axioms of this kind. Whereas it is common in the literature to use continuity axioms to ensure that integrals are well-defined and finite, axioms that bring extra restrictions, our three axioms are not only sufficient but also necessary for AU. That is, they cannot be avoided for real-valued representations. Pivato (2014) provided AU-type representations using nonstandard real numbers, avoiding both continuity and Archimedean axioms. For empirical and conceptual purposes, this approach is preferable to the use of standard real numbers as in this and most other papers. However, as a price to pay, most researchers are not familiar with nonstandard real numbers.

APPENDIX. POOFS

Our proofs are based on Theorem 7 of Kothiyal, Spinu, & Wakker (2014). Their domain of preference consisted of all finite sequences $x^* = (x_1, \dots, x_n)$ of any length n, and a preference relation $\succcurlyeq^*$ on it. Because there was no upperbound to the length n, their domain was essentially infinite-dimensional. It can be taken as isomorphic to our space of periodic sequences (details are provided in the proof of Observation 1). Their preference conditions are the natural analogs of our p-preference conditions. Their Theorem 7 showed that AU holds if and only preferences satisfy (1) weak ordering; (2) Archimedeanity; (3) fairness (called symmetry); (4) independence; and one further condition called *replication equivalence*, i.e.: $x^* \sim^* mx^*$ for all finite sequences x^*. Here mx^* again denotes the m-fold, finite, periodic replication of x^*.

PROOF OF OBSERVATION 1. It readily follows that Statement (i) implies Statement (ii). P-independence follows because common terms in the AU summations cancel. We next assume Statement (ii) and derive Statement (i). We consider the domain of Kothiyal, Spinu, & Wakker (2014) defined above. We define a preference relation $\succcurlyeq^*$ on this domain by $(x_1, \dots, x_n) \succcurlyeq^* (y_1, \dots, y_m)$ if $[x_1, .., x_n] \succcurlyeq [y_1, .., y_m]$. Given weak ordering, this definition implies replication equivalence of $\succcurlyeq^*$. Kothiyal et al.'s domain is isomorphic to our domain of periodic sequences if we identify all periodic extensions mx^* with x^* in their domain. All conditions of their Theorem 7 follow, implying an AU representation of $\succcurlyeq^*$ and, accordingly, one of $\succcurlyeq$ on F^p. □

PROOF OF THEOREM 3. We first assume Statement (i) and derive Statement (ii). The first four conditions, weak ordering, p-Archimedeanity, p-fairness, and p-independence, follow directly, and by Observation 1.

To derive v-boundedness of every act, we assume, for contradiction, that f is v-unbounded above. Then so is $U(\Gamma)$: because Γ is unbounded, Eq. 3 holds, implying $U(\delta) - U(\gamma) > n\big(U(\gamma) - U(\beta)\big)$. So, $U(\Gamma)$ is indeed unbounded above. Eq. 5 implies that $AU[f_1, \dots, f_n]$ exceeds any $U(\gamma)$ and, hence, any real number. This contradicts that $limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U\big(f_j\big)$ and $AU(f)$ are well-defined and finite. V-unboundedness below similarly leads to a contradiction and f must be v-bounded.

We next assume, for contradiction, that f is not stable. Then there exist p-streams $x \succ y$ with $[f_1, \ldots, f_n] \succ x$ for infinitely many n and $[f_1, \ldots, f_n] \prec y$ for infinitely many n. This would imply $limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j) \geq AU(x) > AU(y) \geq liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j)$, contradicting well-definedness of $AU(f)$.

For P-denseness, assume $f \succ g$. Then $AU(f) > AU(g)$, i.e., $AU(f) - \varepsilon > AU(g) + \varepsilon$ for some $\varepsilon > 0$. Then $\exists N: \forall n \geq N: AU[f_1, \ldots, f_n] > AU(f) - \varepsilon > AU(g) + \varepsilon > AU[g_1, \ldots, g_n]$. Because of denseness of $AU(F^p)$ (Lemma 2), there are p-streams x, y such that $AU[f_1, \ldots, f_n] > AU(f) - \varepsilon > AU(x) > AU(y) > AU(g) + \varepsilon > AU[g_1, \ldots, g_n]$ for all $n \geq N$. P-denseness holds and Statement (ii) has been proved.

We next assume Statement (ii) and derive Statement (i). We have AU on F^p by Observation 1, providing U.

Assume, for contradiction, $limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j) = \infty$. Then $U(\Gamma)$ is unbounded above, readily implying that Γ is unbounded above and so is f: a contradiction has resulted. Similarly, $liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j) = -\infty$ cannot be.

Next assume, for contradiction, $liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j) < limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j)$. Then we can find real numbers $\mu < \nu$ between these such that $\frac{1}{n} \sum_{j=1}^{n} U(f_j) < \mu$ for infinitely many n and $\nu < \frac{1}{n} \sum_{j=1}^{n} U(f_j)$ for infinitely many n. By denseness of $AU(F^p)$ (Lemma 2), we can find periodic x, y with $AU(y) < AU(x)$ between μ and ν, implying $\frac{1}{n} \sum_{j=1}^{n} U(f_j) < AU(y)$ for infinitely many n and $\frac{1}{n} \sum_{j=1}^{n} U(f_j) > AU(x)$ for infinitely many n. This contradicts stability of f.

We can conclude at this stage that $AU(f)$ is well-defined and finite for all f. We finally show that it is representing. Assume $f \succ g$. By p-denseness, we have the right-hand side of Eq. 7, which implies $AU(f) \geq AU(x) > AU(y) \geq AU(g)$, so that $AU(f) > AU(g)$. Conversely, assume $AU(f) > AU(g)$. By denseness of $AU(F^p)$ (Lemma 2), we can find $\varepsilon > 0$ and p-streams x, y such that $AU(f) - \varepsilon > AU(x) > AU(y) > AU(g) + \varepsilon$. There exists N such that for all $n > N$: $\frac{1}{n} \sum_{j=1}^{n} U(f_j) > AU(f) - \varepsilon > AU(x) > AU(y) > AU(g) + \varepsilon > \frac{1}{n} \sum_{j=1}^{n} U(g_j)$. By p-denseness, $f \succ g$. We have shown: $f \succ g \Leftrightarrow AU(f) > AU(g)$, i.e., AU represents the preference relation.

We, finally, establish that U is an interval scale. It is obvious that we can add any constant and multiply by any positive constant. It readily follows from Observation 4 that we do not have more liberty.

We make two further comments: for $U(\Gamma)$ to be unbounded above, it does not suffice to require that there is no maximal outcome, as for instance with $U(\Gamma) = (0,1)$. In that case, there can still exist a maximal stream f, for instance for $U(f_j) = 1 - 1/j$, but no maximal outcome. Further, for P-denseness, it is not enough to require that for every $f \succ g$ there exists a p-stream x with $f \succ x \succ g$. Then "infinitesimal" strict preferences could exist between different streams with the same AU value. □

PROOF OF OBSERVATION 6. Continuity of U immediately implies continuity of $\succcurlyeq$. Continuity of $\succcurlyeq$ implies continuity of U by Wakker (1988 Theorem 3.1). □

REFERENCES

Aczél, János (1966) "*Lectures on Functional Equations and Their Applications.*" Academic Press, New York.

Adler, Matthew D. (2019) "*Measuring Social Welfare: An Introduction*." Oxford University Press, New York.

Arrow, Kenneth J. (1971) "*Essays in the Theory of Risk Bearing.*" North-Holland, Amsterdam.

Asheim, Geir B. (2010) "Intergenerational Equity," *Annual Review of Economics* 2, 197–222.

Asheim, Geir B. & Stéphane Zuber (2014) "Escaping the Repugnant Conclusion: Rank-Discounted Utilitarianism with Variable Population," *Theoretical Economics* 9, 629–650.

Basu, Kaushik & Tapan Mitra (2007) "Utilitarianism for Infinite Utility Streams: A New Welfare Criterion and Its Axiomatic Characterization," *Journal of Economic Theory* 133, 350–373.

Blackorby, Charles, Walter Bossert, & David Donaldson (2005) "*Population Issues in Social Choice Theory, Welfare Economics and Ethics.*" Cambridge University Press, Cambridge, UK.

Blackorby, Charles, Russell Davidson, & David Donaldson (1977) "A Homiletic Exposition of the Expected Utility Hypothesis," *Economica* 44, 351–358.

Bossert, Walter, Yves Sprumont, & Kotaro Suzumura (2007) "Ordering Infinite Utility Streams," *Journal of Economic Theory* 135, 579–589.

Chambers, Christopher P., Federico Echenique, & Nicolas S. Lambert (2021) "Recovering Preferences from Finite Data," *Econometrica* 89, 1633–1664.

da Volterra, Gian Francesco Lottini (1574) "*Avvedimenti Civili*."

Debreu, Gérard (1960) "Topological Methods in Cardinal Utility Theory." *In* Kenneth J. Arrow, Samuel Karlin, & Patrick Suppes (1960, eds.) *Mathematical Methods in the Social Sciences*, 16–26, Stanford University Press, Stanford, CA.

Diamond, Peter A. (1965) "The Evaluation of Infinite Utility Streams," *Econometrica* 33, 170–177.

Drèze, Jacques H. (1987) "*Essays on Economic Decision under Uncertainty.*" Cambridge University Press, Cambridge.

Elster, John (1986) "Introduction." *In* John Elster (ed.) *Rational Choice,* 1–33, New York University Press, New York.Fishburn, Peter C. & Ward Edwards (1997) "Discount-Neutral Utility Models for Denumerable Time Streams," *Theory and Decision* 43, 139–166.

Fishburn, Peter C. & Ward Edwards (1997) "Discount-Neutral Utility Models for Denumerable Time Streams," *Theory and Decision* 43, 139–166.

Gale, David (1967) "An Optimal Development in a Multi-Sector Economy," *Review of Economic Studies* 34, 1–18.

Gorman, William M. (1968) "The Structure of Utility Functions," *Review of Economic Studies* 35, 367–390.

Gravel, Nicolas, Thierry Marchant, & Arunava Sen (2012) "Uniform Expected Utility Criteria for Decision Making under Ignorance or Objective Ambiguity," *Journal of Mathematical Psychology* 56, 297–315.

Gravel, Nicolas, Thierry Marchant, & Arunava Sen (2018) "Conditional Expected Utility Criteria for Decision Making under Ignorance or Objective Ambiguity," *Journal of Mathematical Economics* 78, 79–95.

Harvey, Charles M. (1986) "Value Functions for Infinite-Period Planning," *Management Science* 32, 1123–1139.

Howson, Colin (2014) "Finite Additivity, Another Lottery Paradox and Conditionalisation," *Synthese* 191, 989–1012.

Jevons, W. Stanley (1871) "*The Theory of Political Economy*." (5th edn. 1957, Kelley and MacMillan, New York; other edn. Penguin, 1970.)

Jonsson, Adam & Mark Voorneveld (2015) "Utilitarianism on Infinite Utility Streams: Summable Differences and Finite Averages," *Economic Theory Bulletin* 3, 19–31.

Jonsson, Adam & Mark Voorneveld (2018) "The Limit of Discounted Utilitarianism," *Theoretical Economics* 13, 19–37.

Khan, Urmee & Maxwell B. Stinchcombe (2018) "Planning for the Long Run: Programming with Patient, Pareto Responsive Preferences," *Journal of Economic Theory* 176, 444–478.

Khan, M. Ali & Metin Uyanik (2021) "Topological Connectedness and Behavioral Assumptions on Preferences: A Two-Way Relationship," *Economic Theory* 71, 411–460.

Kothiyal, Amit, Vitalie Spinu, & Peter P. Wakker (2014) "Average Utility Maximization: A Preference Foundation," *Operations Research* 62, 207–218.

Kraitchik Maurice B. (1953) "*Mathematical Recreations*." Dover, New York, NY.

Krantz, David H., R. Duncan Luce, Patrick Suppes, & Amos Tversky (1971) "*Foundations of Measurement, Vol. I (Additive and Polynomial Representations)*." Academic Press, New York. (2nd edn. 2007, Dover Publications, New York.)

Laplace, Pierre Simon de (1796) "*Essai Philosophique sur les Probabilités.*" Paris. (5th edn. 1825). Translated into English as "A Philosophical Essay on Probabilities," Dover Publications, New York, 1951.

Lauwers, Luc (1998) "Intertemporal Objective Functions: Strong Pareto versus Anonymity," *Mathematical Social Sciences* 35, 37–55.

Lauwers, Luc (2012) "Intergenerational Equity, Efficiency, and Constructability," *Economic Theory* 49, 227–242.

Luce, R. Duncan, David H. Krantz, Patrick Suppes, & Amos Tversky (1990) "*Foundations of Measurement, Vol. III. (Representation, Axiomatization, and Invariance).*" Academic Press, New York.

Marinacci, Massimo (1998) “An Axiomatic Approach to Complete Patience and Time Invariance,” *Journal of Economic Theory* 83, 105–144.

Mitra, Tapan & Kaushik Basu (2007) “On the Existence of Paretian Social Welfare Quasi-Orderings for Infinite Utility Streams with Extended Anonymity.” *In* John Roemer & Kotaro Suzumura (Eds.) *Intergenerational Equity and Sustainability*. Palgrave, London, 2007).

Münnich, Àkos, Gyula Maksa, & Robert J. Mokken (2000) “n-Variable Bisection,” *Journal of Mathematical Psychology* 44, 569–581.

Nalebuff, Barry J. (1989) “The Other Person´s Envelope is always Greener,” *Journal of Economic Perspectives* 3, 171–181.

Neyman, Abraham (2023) “Additive Valuations of Streams of Payoffs That Satisfy the Time Value of Money Principle: A Characterization and Robust Optimization,” *Theoretical Economics* 18, 303–340.

Parfit, Derek (1984) “*Reasons and Persons.*” Clarendon Press, Oxford, UK.

Petri, Henrik (2019) “Asymptotic Properties of Welfare Relations,” *Economic Theory* 67, 853–874.

Pfanzagl, Johann (1968) “*Theory of Measurement.*” Physica-Verlag, Vienna.

Pigou, Arthur C. (1920) “*The Economics of Welfare.*” (edn. 1952: MacMillan, London.)

Pivato, Marcus (2014) “Additive Representation of Separable Preferences over Infinite Products,” *Theory and Decision* 73, 31–83.

Pivato, Marcus (2022) “A Characterization of Cesàro Average Utility,” *Journal of Economic Theory* 201 105440.

Rébillé, Yan (2007) “Patience in some Non-Additive Models,” *Journal of Mathematical Economics* 43, 749–763.

Sidgwick, Henry (1874) “*The Methods of Ethics*.” 7th edn. 1907. MacMillan, London.

Svensson, Lars-Gunnar (1980) “Equity among Generations,” *Econometrica* 48, 1251–1256.

Tännsjö, Torbjörn (2002) “Why We Ought to Accept the Repugnant Conclusion,” *Utilitas* 14, 339–359.

Thomson, William (2001) “On the Axiomatic Method and Its Recent Applications to Game Theory and Resource Allocation,” *Social Choice and Welfare* 18, 327–386.

Wakker, Peter P. (1988) “The Algebraic versus the Topological Approach to Additive Representations,” *Journal of Mathematical Psychology* 32, 421–435.

Wakker, Peter P. (1993) “Savage’s Axioms Usually Imply Violation of Strict Stochastic Dominance,” *Review of Economic Studies* 60, 487–493.

Wakker, Peter P. & Horst Zank (1999) “State Dependent Expected Utility for Savage’s State Space; Or: Bayesian Statistics without Prior Probabilities,” *Mathematics of Operations Research* 24, 8–34.

Weinstein, Milton C. (1993) “Time-preference Studies in the Health Care Context,” *Medical Decision Making*, 218–219.

Weymark, John A. (1995) “Further Remarks on Harsanyi’s Social Aggregation Theorem and the Weak Pareto Principle,” *Social Choice and Welfare* 12, 87–92.

Yi, Byeong-Uk (2013) “Conditionals and a Two-envelope Paradox,” *The Journal of Philosophy* 110, 5, 233–257.