

# Exceedance probability of the integral of a stochastic process \*

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May 22, 2009

## Abstract

This paper studies the tail distribution of  $\int_S X(s)ds$ , where  $\{X(s)\}_{s \in S}$  is an almost surely continuous stochastic process defined on some compact subset  $S$  of  $R^d$ . We discuss how to estimate the tail probability  $p = P(\int_S X(s)ds > x)$  for some high value  $x$ . The paper has two main purposes: first to formalize and justify the results of Coles and Tawn (1996); further we treat the problem in a non-parametric way as opposed to their fully parametric methods. We prove consistency of the proposed estimator of  $p$ . Our method is applied to the total rainfall in the North Holland area.

Keywords: extreme value theory, max-stable processes, tail probability estimation

## 1 Introduction and limit result

Let  $X := \{X(s)\}_{s \in S}$  be an almost surely continuous stochastic process defined on some compact set  $S$  of  $R^d$  ( $d \geq 1$ ). We study the tail property of  $\int_S X(s)ds$ , particularly, how to estimate the tail probability  $p = P(\int_S X(s)ds > x)$  for some high value  $x$ .

One may find applications of such a problem with  $d = 2$  for instance when monitoring rainfall, where  $X(s)$  might represent the daily rainfall at each point

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\*Research partially supported by FCT, Project PTDC/MAT/64924/2006, Portugal.

$s$  of the space  $S$ . Then  $\int_S X(s)ds$  represents the total daily rainfall over the whole area  $S$ .

Let  $C(S)$  be the space of continuous functions on  $S$ , equipped with the supremum norm,  $|f|_\infty = \sup_{s \in S} |f(s)|$ . The stochastic process  $X$  is assumed to be on  $C(S)$ , with non-degenerate marginals. Denote by  $X_1, X_2, \dots$ , independent and identically distributed copies of  $X$ . Suppose there are continuous functions (i.e. for each  $n$ ),  $a_s(n)$  positive and  $b_s(n)$  real, such that, for some limiting stochastic process  $Y := \{Y(s)\}_{s \in S}$ ,

$$\left\{ \max_{1 \leq i \leq n} \frac{X_i(s) - b_s(n)}{a_s(n)} \right\}_{s \in S} \xrightarrow{d} \{Y(s)\}_{s \in S}, \quad \text{as } n \rightarrow \infty, \quad (1.1)$$

in  $C(S)$ . Condition (1.1) means that  $X$  is in the domain of attraction of some max-stable process  $Y$ . For convenience, and w.l.o.g., the normalizing functions are chosen such that the marginal distributions of  $Y$  have a standard form:  $-\log P(Y(s) \leq x) = (1 + \gamma(s)x)^{-1/\gamma(s)}$  for all  $x$  with  $1 + \gamma(s)x > 0$ .

Then (cf. de Haan and Ferreira (2006), Sections 9.2, 9.3 and 9.5), there is a continuous real function  $\gamma(s)$ , called the index function, and a measure  $\nu$  on the space,

$$C^+(S) := \{f \in C(S) : f \geq 0\}$$

such that, for each Borel subset  $E$  of  $C^+(S)$  with  $\inf\{|f|_\infty : f \in E\} > 0$  and  $\nu(\partial E) = 0$ ,

$$\lim_{t \rightarrow \infty} tP \left( \left\{ \left( 1 + \gamma(s) \frac{X(s) - b_s(t)}{a_s(t)} \right)^{1/\gamma(s)} \right\}_{s \in S} \in E \right) = \nu(E) \quad (1.2)$$

is finite. A characterizing property of the exponent measure is its homogeneity:  $\nu(cE) = c^{-1}\nu(E)$ , for all  $c > 0$ .

Moreover (cf. de Haan and Ferreira (2006), Section 9.4), there exists a finite measure  $\rho$  on

$$\bar{C}_1^+(S) := \{g \in C^+(S) : |g|_\infty = 1\},$$

such that

$$\nu(E) = \int \int_{r g \in E} \frac{dr}{r^2} d\rho(g). \quad (1.3)$$

This measure  $\rho$  is called the spectral measure and satisfies the side conditions

$$\int_{\bar{C}_1^+(S)} g(s) d\rho(g) = 1, \quad \text{for all } s \in S. \quad (1.4)$$

Our main theorem on the tail distribution of the integral is given as follows.

**Theorem 1.1.** *Let the stochastic process  $X$  in  $C(S)$  satisfy the basic convergence (1.1), with  $a_s(t)$  such that*

$$\sup_{s \in S} \left| \frac{a_s(t)}{a(t)} - A(s) \right| \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad (1.5)$$

for some functions  $a(t)$  positive and  $A(s)$  non-negative. Without loss of generality we may (and do) take  $a(t) := \int_S a_s(t) ds$  which implies  $\int_S A(s) ds = 1$ . Suppose that (1.2) holds with a constant index function  $\gamma(s) \equiv \gamma \in \mathbb{R}$ . For  $\gamma \leq 0$  we also require that  $A$  is strictly positive and

$$\rho\{g \in \bar{C}_1^+(S) : \inf_{s \in S} g(s) = 0\} = 0. \quad (1.6)$$

Then,

$$\lim_{t \rightarrow \infty} tP \left( \frac{\int_S X(s) ds - \int_S b_s(t) ds}{a(t)} > x \right) = \theta_\gamma (1 + \gamma x)^{-1/\gamma}, \quad 1 + \gamma x > 0, \quad (1.7)$$

with

$$\theta_\gamma := \int_{\bar{C}_1^+(S)} \left( \int_S A(s) g^\gamma(s) ds \right)^{1/\gamma} d\rho(g); \quad (1.8)$$

for  $\gamma = 0$  the right-hand side of (1.7) should be read as  $\theta_0 e^{-x}$  and the right-hand side of (1.8) as  $\int_{\bar{C}_1^+(S)} \exp \left( \int_S A(s) \log g(s) ds \right) d\rho(g)$ . With this definition the right-hand side of (1.8) is continuous in  $\gamma$ . (cf. Appendix A)

*Proof.* From (1.1) it follows that

$$\begin{aligned} & \lim_{t \rightarrow \infty} tP \left( \int_S \frac{X(s) - b_s(t)}{a_s(t)} A(s) ds > x \right) \\ &= \lim_{t \rightarrow \infty} tP \left( \int_S \frac{\left\{ \left( 1 + \gamma \frac{X(s) - b_s(t)}{a_s(t)} \right)^{1/\gamma} \right\}^\gamma - 1}{\gamma} A(s) ds > x \right) \\ &= \nu \left( \left\{ f \in C^+(S) : \int_S \frac{f^\gamma(s) - 1}{\gamma} A(s) ds > x \right\} \right) \end{aligned} \quad (1.9)$$

provided the later set is a Borel measurable  $\nu$ -continuity set. This follows from Proposition A.1.

Next we calculate the right-hand side of (1.9) by applying the spectral measure. We start with the case  $\gamma > 0$ . Note that

$$\begin{aligned} \int_S A(s) \frac{f^\gamma(s) - 1}{\gamma} ds > x &\Leftrightarrow \int_S A(s) f^\gamma(s) ds > 1 + \gamma x \\ &\Leftrightarrow |f|_\infty^\gamma \int_S A(s) \left( \frac{f(s)}{|f|_\infty} \right)^\gamma ds > 1 + \gamma x \Leftrightarrow |f|_\infty^\gamma > \frac{1 + \gamma x}{\int_S A(s) \left( \frac{f(s)}{|f|_\infty} \right)^\gamma ds}. \end{aligned}$$

Hence,

$$\begin{aligned}
& \nu \left( \left\{ f \in C^+(S) : \int_S A(s) \frac{f^\gamma(s) - 1}{\gamma} ds > x \right\} \right) \\
&= \int_{\bar{C}_1^+(S)} \int_{\left(\frac{1+\gamma x}{\int_S A(s) g^\gamma(s) ds}\right)^{1/\gamma}}^\infty \frac{dr}{r^2} d\rho(g) \\
&= (1 + \gamma x)^{-1/\gamma} \int_{\bar{C}_1^+(S)} \left( \int_S A(s) g^\gamma(s) ds \right)^{1/\gamma} d\rho(g). \quad (1.10)
\end{aligned}$$

For  $\gamma < 0$  and  $1 + \gamma x > 0$  the calculations are similar, with the same result as for  $\gamma > 0$ . For  $\gamma = 0$ ,

$$\int_S A(s) \log f(s) ds > x \Leftrightarrow \log |f|_\infty > x - \int_S A(s) \log \left( \frac{f(s)}{|f|_\infty} \right) ds$$

hence,

$$\begin{aligned}
& \nu \left( \left\{ f \in C^+(S) : \int_S A(s) \log f(s) ds > x \right\} \right) \\
&= e^{-x} \int_{\bar{C}_1^+(S)} \exp \left( \int_S A(s) \log g(s) ds \right) d\rho(g). \quad (1.11)
\end{aligned}$$

Finally, we verify that

$$\lim_{t \rightarrow \infty} tP \left( \left| \frac{\int_S X(s) ds - \int_S b_s(t) ds}{a(t)} - \int_S \frac{X(s) - b_s(t)}{a_s(t)} A(s) ds \right| > \varepsilon \right) = 0, \quad (1.12)$$

for all  $\varepsilon > 0$ . By (1.5),

$$\left| \frac{\int_S X(s) ds - \int_S b_s(t) ds}{a(t)} - \int_S \frac{X(s) - b_s(t)}{a_s(t)} A(s) ds \right| \leq c_t \int_S \frac{X(s) - b_s(t)}{a_s(t)} ds,$$

with

$$c_t := \sup_{s \in S} \left| \frac{a_s(t)}{a(t)} - A(s) \right| \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

That is,

$$\begin{aligned}
& P \left( \left| \frac{\int_S X(s) ds - \int_S b_s(t) ds}{a(t)} - \int_S \frac{X(s) - b_s(t)}{a_s(t)} A(s) ds \right| > \varepsilon \right) \\
&\leq P \left( c_t \int_S \frac{X(s) - b_s(t)}{a_s(t)} ds > \varepsilon \right)
\end{aligned}$$

for all  $\varepsilon > 0$ . Then, it is easy to see (e.g. from (1.9)–(1.11) with  $A(s) \equiv 1$ ) that,

$$\lim_{t \rightarrow \infty} tP \left( c_t \int_S \frac{X(s) - b_s(t)}{a_s(t)} ds > \varepsilon \right) = 0$$

for all  $\varepsilon > 0$ , hence (1.12).

Combining (1.9)–(1.11) and (1.12), the result follows.  $\square$

**Corollary 1.1.** *Let  $I_1, I_2, \dots$  be independent and identically distributed copies of  $\int_S X(s)ds$ . Then, under the conditions of Theorem 1.1,*

$$\lim_{n \rightarrow \infty} P \left( \max_{1 \leq i \leq n} \frac{I_i - \int_S b_s(n)ds}{a(n)} \leq x \right) = \exp \left( -\theta_\gamma (1 + \gamma x)^{-1/\gamma} \right), \quad 1 + \gamma x > 0, \quad (1.13)$$

with  $\theta_\gamma$  as before.

**Corollary 1.2.** *Let the stochastic process  $X$  in  $C(S)$  satisfy (1.1). Assume that the index function is constant and (1.6) holds. Then,*

$$\lim_{t \rightarrow \infty} tP \left( \int_S \frac{X(s) - b_s(t)}{a_s(t)} ds > x \right) = \theta_\gamma^* (1 + \gamma x)^{-1/\gamma}, \quad 1 + \gamma x > 0, \quad (1.14)$$

with

$$\theta_\gamma^* := \int_{\bar{C}_1^+(S)} \left( \int_S g^\gamma(s) ds \right)^{1/\gamma} d\rho(g); \quad (1.15)$$

for  $\gamma = 0$  apply the same considerations as in Theorem 1.1.

It is well known in one dimensional extreme value theory that the maximum domain of attraction condition can be stated as,

$$\lim_{t \rightarrow \infty} tP \left( \frac{Y - b(t)}{a(t)} > y \right) = (1 + \gamma y)^{-1/\gamma}, \quad \text{for all } y \text{ with } 1 + \gamma y > 0, \quad \gamma \in \mathbb{R}, \quad (1.16)$$

with  $a(t) > 0$  and  $b(t)$  real normalizing functions. Hence note the resemblance between (1.7) and (1.16), the former with the extra factor  $\theta_\gamma$ . In Coles and Tawn (1996), this quantity  $\theta_\gamma$  was named the *areal coefficient* and interpreted as the effect of spatial dependence. They gave some bounds for it. Since their proof is difficult to follow we provide a different proof.

**Proposition 1.1.** *Under the conditions of Theorem 1.1,*

1.  $0 < \theta_\gamma \leq 1$ , for  $\gamma \leq 1$ ,
2.  $1 \leq \theta_\gamma \leq \rho(\bar{C}_1^+(S))$ , for  $\gamma \geq 1$ .

*In particular,  $\theta_1 = 1$  (hence spatial dependence does not play a role).*

*Proof.* By (1.4) and using  $\int_S A(s) ds = 1$ ,

$$\theta_1 = \int_S A(s) ds \int_{\bar{C}_1^+(S)} g(s) d\rho(g) = 1.$$

Since  $\theta_\gamma$  is non-decreasing in  $\gamma$  (cf. Appendix A, Proposition A.1),  $\theta_\gamma \leq 1$  for  $\gamma \leq 1$ , and  $\theta_\gamma \geq 1$  for  $\gamma \geq 1$ . Furthermore, since  $\sup_{s \in S} g(s) = 1$  for any  $g \in \bar{C}_1^+(S)$ ,

$$\theta_\gamma \leq \int_{\bar{C}_1^+(S)} \left( \int_S A(s) ds \right)^{1/\gamma} d\rho(g) = \rho(\bar{C}_1^+(S)),$$

for  $\gamma \neq 0$ . The case  $\gamma = 0$  is similar.

Finally, we prove that  $\theta_\gamma > 0$ . Since  $(\int_S g^\gamma(s)A(s)ds)^{1/\gamma}$  is monotone in  $g$  for any  $\gamma \in \mathbb{R}$  (Proposition A.1),

$$\begin{aligned}\theta_\gamma &\geq \int_{\bar{C}_1^+(S)} \left( \int_S A(s) \left( \inf_{s \in S} g(s) \right)^\gamma ds \right)^{1/\gamma} d\rho(g) \\ &= \left( \int_{\bar{C}_1^+(S)} \inf_{s \in S} g(s) d\rho(g) \right) \left( \int_S A(s) ds \right)^{1/\gamma} = \int_{\bar{C}_1^+(S)} \inf_{s \in S} g(s) d\rho(g) > 0.\end{aligned}$$

□

**Remark 1.1.** If (1.6) is not imposed,  $\theta_\gamma$  may be zero for some  $\gamma \leq 0$ . A spectral measure that assigns positive measure to the subset of functions  $g$  that are zero on a set of positive Lebesgue measure in  $S$  can give rise to  $\theta_\gamma = 0$ , for all  $\gamma \leq 0$ .

**Remark 1.2.** Note that  $\rho(\bar{C}_1^+(S)) = \int_{\bar{C}_1^+(S)} d\rho(g)$  is the limit of  $\theta_\gamma$ , as  $\gamma \rightarrow \infty$ . (cf. Appendix A, Proposition A.1)

## 2 Estimation of exceedance probability

Recall that we want to estimate the tail probability  $p$  defined as

$$p = P \left( \int_S X(s) ds > x \right) \quad (2.1)$$

for some high value  $x$ . Since we intend to estimate an extreme event, we have to assume that  $p = p_n$  and  $\lim_{n \rightarrow \infty} p_n = 0$ , where  $n$  is the number of available observations. It then follows that  $x = x_n$  and  $x_n$  converges to the right endpoint of the distribution of  $\int_S X(s) ds$ .

The proposed estimator is motivated by (1.7), where  $t$  is replaced by  $n/k$ , with  $k = k(n)$  an intermediate sequence, i.e.  $k \rightarrow \infty$  and  $k/n \rightarrow 0$ , as  $n \rightarrow \infty$ . Our estimator for exceedance probability is defined as follows,

$$\hat{p}_n = \frac{k}{n} \hat{\theta} \left( 1 + \hat{\gamma} \frac{x_n - \int_S \hat{b}_s(\frac{n}{k}) ds}{\hat{a}(\frac{n}{k})} \right)^{-1/\hat{\gamma}} \quad (2.2)$$

with

$$\hat{\theta} = \int_{\bar{C}_1^+(S)} \left( \int_S \hat{A}(s) g^{\hat{\gamma}}(s) ds \right)^{1/\hat{\gamma}} d\hat{\rho}(g). \quad (2.3)$$

We wish to prove that  $\hat{p}_n/p_n \xrightarrow{P} 1$  as  $n \rightarrow \infty$ . For this we need estimators for  $\gamma(s)$ ,  $a_s(\frac{n}{k})$  and  $b_s(\frac{n}{k})$  converging at a certain rate. In order to obtain such rates, generally second order conditions are assumed. This is the subject we discuss first.

### Second Order Condition I

The basic relation (1.1) implies convergence of the marginals which, in terms of the function  $U_s$  can be written as, with  $a_s(t)$  satisfying (1.5),

$$\lim_{t \rightarrow \infty} \frac{U_s(tx) - U_s(t)}{a_s(t)} = \frac{x^\gamma - 1}{\gamma} \quad \text{uniformly in } s \in S, \text{ and for } x > 0. \quad (2.4)$$

The natural second order condition in this context is: there exists a function  $\alpha_s(t)$  positive or negative with  $|\alpha_s(\cdot)|$  regularly varying of index  $\tilde{\rho} \leq 0$  and  $\lim_{t \rightarrow \infty} \alpha_s(t) = 0$  uniformly such that

$$\lim_{t \rightarrow \infty} \frac{\frac{U_s(tx) - U_s(t)}{a_s(t)} - \frac{x^\gamma - 1}{\gamma}}{\alpha_s(t)}, \quad \text{exists for } x > 0 \quad (2.5)$$

holds uniformly for  $s \in S$ .

In the literature on extremal processes (cf. de Haan and Ferreira (2006)) estimators  $\hat{\gamma}(s)$ ,  $\hat{a}_s(\frac{n}{k})$  and  $\hat{b}_s(\frac{n}{k})$  are known such that under (2.5),

$$\sqrt{k} \sup_{s \in S} \left( |\hat{\gamma}(s) - \gamma(s)| + \left| \frac{\hat{a}_s(\frac{n}{k})}{a_s(\frac{n}{k})} - 1 \right| + \left| \frac{\hat{b}_s(\frac{n}{k}) - b_s(\frac{n}{k})}{a_s(\frac{n}{k})} \right| \right) = O_p(1). \quad (2.6)$$

Any set of estimators that satisfy (2.6) would do.

Next we discuss how these provide estimators for  $\gamma$ ,  $a(\frac{n}{k})$  and  $\int_S b_s(\frac{n}{k}) ds$  with similar properties. Define

$$\begin{aligned} \hat{\gamma} &:= \int_S \hat{\gamma}(s) ds / |S| \\ \hat{a}(\frac{n}{k}) &:= \int_S \hat{a}_s(\frac{n}{k}) ds \\ \hat{A}(s) &:= \frac{\hat{a}_s(\frac{n}{k})}{\int_S \hat{a}_s(\frac{n}{k}) ds}. \end{aligned} \quad (2.7)$$

Then clearly, as  $n \rightarrow \infty$

$$\sqrt{k}(\hat{\gamma} - \gamma) = \int_S \sqrt{k}(\hat{\gamma}(s) - \gamma) ds / |S| = O_p(1), \quad (2.8)$$

$$\sqrt{k} \left( \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1 \right) = \int_S \sqrt{k} \left( \frac{\hat{a}_s(\frac{n}{k})}{a_s(\frac{n}{k})} - 1 \right) \frac{a_s(\frac{n}{k})}{a(\frac{n}{k})} ds = O_p(1) \quad (2.9)$$

$$\sqrt{k} \frac{\int_S \hat{b}_s(\frac{n}{k}) ds - \int_S b_s(\frac{n}{k}) ds}{a(\frac{n}{k})} = \int_S \sqrt{k} \frac{\hat{b}_s(\frac{n}{k}) - b_s(\frac{n}{k})}{a_s(\frac{n}{k})} \frac{a_s(\frac{n}{k})}{a(\frac{n}{k})} ds = O_p(1). \quad (2.10)$$

Moreover, uniformly in  $s$ ,

$$\begin{aligned} \sqrt{k} \left( \hat{A}(s) - A(s) \right) &= \sqrt{k} \left( \frac{\hat{a}_s(\frac{n}{k})}{\hat{a}(\frac{n}{k})} - \frac{a_s(\frac{n}{k})}{a(\frac{n}{k})} \right) \\ &= \sqrt{k} \left( \left\{ \frac{\hat{a}_s(\frac{n}{k})}{a_s(\frac{n}{k})} - 1 \right\} - \left\{ \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1 \right\} \right) \frac{a_s(\frac{n}{k})}{a(\frac{n}{k})} \frac{a(\frac{n}{k})}{\hat{a}(\frac{n}{k})} \\ &= O_p(1). \end{aligned} \quad (2.11)$$

Also an estimator of the measure  $\rho$  is known such that for the statistic  $\hat{\theta}$ , we get  $\hat{\theta} \xrightarrow{P} \theta_\gamma$  as  $n \rightarrow \infty$ . This will be proved in Section 2.1 and 2.2.

Next we shall also need a somewhat different second order condition.

**Second Order Condition II**

Note firstly that by simple inversion relation (1.7) implies for  $x > 0$

$$\lim_{t \rightarrow \infty} \frac{U(tx) - \int_S b_s(t) ds}{a(t)} = \frac{(\theta_\gamma x)^\gamma - 1}{\gamma} \quad \text{exists,} \quad (2.12)$$

where  $U$  is the inverse function of  $1/P(\int_S X(s) ds > x)$ . This means that  $U(t)$ , the inverse of the distribution of the integral, is asymptotically close to  $\int_S U_s(t) ds$ , the integral of the inverse of the distribution. We require a second order condition related to (2.12): there exists a function  $\alpha(t)$ , positive or negative with  $\alpha(\cdot)$  regularly varying of index  $\tilde{\rho} \leq 0$ , or  $\tilde{\rho} = 0$  if  $\gamma < 0$ , and  $\lim_{t \rightarrow \infty} \alpha(t) = 0$  such that

$$\lim_{t \rightarrow \infty} \frac{\frac{U(tx) - \int_S b_s(t) ds}{a(t)} - \frac{(\theta_\gamma x)^\gamma - 1}{\gamma}}{\alpha(t)} \quad \text{exists for } x > 0. \quad (2.13)$$

The following theorem gives the consistency of our estimator  $\hat{p}_n$  of the exceedance probability  $p_n$ .

**Theorem 2.1.** Assume that the basic first order condition (1.1) holds with  $\gamma(s) \equiv \gamma > -1/2$ , and moreover that the second order condition II holds. We also assume that the following conditions (familiar from the one-dimensional case, cf. de Haan and Ferreira, Section 4.4) hold: as  $n \rightarrow \infty$ ,

- 1)  $d_n := k/(np_n) \rightarrow \infty$ ;
- 2)  $w_\gamma(d_n)/\sqrt{k} \rightarrow 0$ , where  $w_\gamma(t) := t^{-\gamma} \int_1^t s^{\gamma-1} \log s ds$ ;
- 3)  $\sqrt{k}\alpha(\frac{n}{k}) \rightarrow \lambda$ , finite for some  $k = k(n) \rightarrow \infty$ ,  $k/n \rightarrow 0$ .

Further we use estimators for  $\hat{\gamma}$ ,  $\hat{a}(\frac{n}{k})$  and  $\int_S \hat{b}_s(\frac{n}{k}) ds$  such that (2.6) holds and an estimator  $\hat{\rho}$  for spectral measure  $\rho$  such that  $\hat{\rho} \xrightarrow{P} \rho$  in the space of finite measures on  $\bar{C}_1^+(S)$ . Then for the tail probability estimator  $\hat{p}_n$  of (2.2) and (2.3), we have that

$$\frac{\hat{p}_n}{p_n} \xrightarrow{P} 1 \quad \text{as } n \rightarrow \infty.$$

*Proof.* We write

$$\begin{aligned} \frac{\hat{p}_n}{p_n} &= \frac{k}{np_n} \hat{\theta} \left( 1 + \hat{\gamma} \frac{x_n - \int_S \hat{b}_s(\frac{n}{k}) ds}{\hat{a}(\frac{n}{k})} \right)^{-1/\hat{\gamma}} \\ &= \frac{\hat{\theta}}{\theta_\gamma} \left\{ d_n \theta_\gamma \left( 1 + \hat{\gamma} \frac{x_n - \int_S \hat{b}_s(\frac{n}{k}) ds}{\hat{a}(\frac{n}{k})} \right)^{-1/\hat{\gamma}} \right\} \end{aligned}$$

and we consider the two factors separately.

For the first factor, in Section 2.1, it is proved that  $\hat{\theta} \rightarrow^P \theta_\gamma$ .

For the second factor, we follow the line of proof that is usual in one-dimensional estimation (de Haan and Ferreira, Section 4.4). It can be written as

$$\begin{aligned} & \left( 1 + \frac{\hat{\gamma}}{\tilde{d}_n^\gamma} \left( \frac{x_n - \int_S \hat{b}_s(\frac{n}{k}) ds}{\hat{a}(\frac{n}{k})} + \frac{1 - \tilde{d}_n^\gamma}{\hat{\gamma}} \right) \right)^{-1/\hat{\gamma}} \\ &= \left( 1 - \hat{\gamma} \frac{\tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)}{\tilde{d}_n^\gamma \sqrt{k}} \frac{a(\frac{n}{k})}{\hat{a}(\frac{n}{k})} \sqrt{k} \frac{\tilde{x}_n - x_n}{a(\frac{n}{k}) \tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} \right)^{-1/\hat{\gamma}} \end{aligned}$$

with  $\tilde{d}_n := \theta_\gamma d_n$  and  $\tilde{x}_n = \int_S \hat{b}_s(\frac{n}{k}) ds + \hat{a}(\frac{n}{k})(\tilde{d}_n^\gamma - 1)/\hat{\gamma}$ .

First note the fact that as  $t \rightarrow \infty$ ,

$$t^\gamma w_\gamma(t) \sim \begin{cases} \frac{1}{\gamma} t^\gamma \log t, & \gamma > 0 \\ \frac{1}{2} (\log t)^2, & \gamma = 0 \\ 1/\gamma^2, & \gamma < 0. \end{cases} \quad (2.14)$$

Thus, condition 2) implies that  $\log d_n = o(\sqrt{k})$ . Together with (2.8), we get that

$$\frac{\tilde{d}_n^\gamma}{\tilde{d}_n^{\hat{\gamma}}} = \exp \left\{ -\sqrt{k}(\hat{\gamma} - \gamma) \frac{\log d_n + \log \theta_\gamma}{\sqrt{k}} \right\}$$

converges to 1.

Next, by condition 2),

$$\frac{w_\gamma(\tilde{d}_n)}{\sqrt{k}} \rightarrow 0,$$

and by (2.9)

$$\frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} \rightarrow 1,$$

as  $n \rightarrow \infty$ . It remains to prove that

$$\sqrt{k} \frac{\tilde{x}_n - x_n}{a(\frac{n}{k}) \tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} = O_P(1).$$

This is proved in the following proposition. Hence the result follows.  $\square$

**Proposition 2.1.** *Assume the conditions of Theorem 2.1. Then, with  $\tilde{x}_n := \int_S \hat{b}_s(\frac{n}{k}) ds + \hat{a}(\frac{n}{k})(\tilde{d}_n^\gamma - 1)/\hat{\gamma}$  and  $\tilde{d}_n := \theta_\gamma d_n$ , we have that*

$$\sqrt{k} \frac{\tilde{x}_n - x_n}{a(\frac{n}{k}) \tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} = O_P(1).$$

*Proof.* The proof runs as in the estimation of high quantiles in one-dimensional extreme value theory. (cf. de Haan and Ferreira (2006), Section 4.3) Write,

$$\begin{aligned}
& \sqrt{k} \frac{\tilde{x}_n - x_n}{a(\frac{n}{k}) \tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} \\
&= \sqrt{k} \left( \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} - 1 \right) \frac{\tilde{d}_n^\gamma - 1}{\gamma \tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} \\
&\quad + \frac{\sqrt{k}}{\tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} \frac{\int_S \hat{b}_s(\frac{n}{k}) ds - \int_S b_s(\frac{n}{k}) ds}{a(\frac{n}{k})} \\
&\quad + \frac{\hat{a}(\frac{n}{k})}{a(\frac{n}{k})} \frac{\sqrt{k}}{\tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} \left( \frac{\tilde{d}_n^\gamma - 1}{\hat{\gamma}} - \frac{\tilde{d}_n^\gamma - 1}{\gamma} \right) \\
&\quad - \frac{\sqrt{k}}{\tilde{d}_n^\gamma w_\gamma(\tilde{d}_n)} \left( \frac{x_n - \int_S b_s(\frac{n}{k}) ds}{a(\frac{n}{k})} - \frac{\tilde{d}_n^\gamma - 1}{\gamma} \right) \\
&= I + II + III + IV,
\end{aligned}$$

and we deal with each term separately. For the first two terms, apply (2.9), (2.10) and (2.14) to obtain that they are both  $O_P(1)$ . The third term goes exactly as in the finite dimensional case (cf. de Haan and Ferreira (2006), pp.136–137), and it is also  $O_P(1)$ .

For term *IV*, note that relation (2.13) implies

$$U(t) - \int_S b_s(t\theta_\gamma) ds = o(a(t)\alpha(t)), \quad t \rightarrow \infty.$$

Further, (2.13) implies

$$\lim_{t \rightarrow \infty} \frac{\frac{U(tx) - U(t)}{\theta_\gamma^\gamma a(t)} - \frac{x^\gamma - 1}{\gamma}}{\alpha(t)}$$

exists for  $x > 0$ . Hence by relation (B.3.4) in de Haan and Ferreira (2006),

$$\lim_{t \rightarrow \infty} \frac{\frac{a(tx)}{a(t)} - x^\gamma}{\alpha(t)}$$

exists for  $x > 0$ , and consequently,

$$\lim_{t \rightarrow \infty} \frac{\frac{U(tx) - U(t)}{a(\theta_\gamma t)} - \frac{x^\gamma - 1}{\gamma}}{\alpha(t)}$$

exists for  $x > 0$ , which implies, by Lemma 4.3.5 in de Haan and Ferreira (2006), that

$$\lim_{\substack{t \rightarrow \infty \\ x = x(t) \rightarrow \infty}} \frac{\frac{U(tx) - U(t)}{a(t\theta_\gamma)} \frac{\gamma}{x^\gamma - 1} - 1}{\alpha(t)} \text{ exists.}$$

It then follows that

$$\lim_{\substack{t \rightarrow \infty \\ x = x(t) \rightarrow \infty}} \frac{\frac{U(tx) - \int_S b_s(t\theta_\gamma) ds}{a(t\theta_\gamma)} \frac{\gamma}{x^\gamma - 1} - 1}{\alpha(t)} \text{ exists,}$$

and hence

$$\lim_{\substack{t \rightarrow \infty \\ x = x(t) \rightarrow \infty}} \frac{\frac{U(tx) - \int_S b_s(t) ds}{a(t)} \frac{\gamma}{(x\theta_\gamma)^\gamma - 1} - 1}{\alpha(t)} \text{ exists.}$$

Substituting  $\frac{n}{k}$  for  $t$  and  $d_n$  for  $x$  in this relation we get in virtue of the second condition of the theorem that part *IV* has a finite limit.  $\square$

## 2.1 Consistency of $\hat{\theta}$

The proof of  $\hat{\theta} \xrightarrow{P} \theta_\gamma$  is given in a sequence of results. Recall that

$$\hat{\theta} = \int_{\bar{C}_1^+(s)} \left( \int_S \hat{A}(s) g^{\hat{\gamma}}(s) ds \right)^{1/\hat{\gamma}} d\hat{\rho}(g).$$

We assume the conditions of Theorem 2.1 throughout.

**Proposition 2.2.** *For each  $\varepsilon > 0$ , the probability*

$$\begin{aligned} (1 - \varepsilon) \int_{\bar{C}_1^+(s)} \left( \int_S A(s) g^{\hat{\gamma}}(s) ds \right)^{1/\hat{\gamma}} d\rho(g) \\ < \int_{\bar{C}_1^+(s)} \left( \int_S \hat{A}(s) g^{\hat{\gamma}}(s) ds \right)^{1/\hat{\gamma}} d\rho(g) \\ < (1 + \varepsilon) \int_{\bar{C}_1^+(s)} \left( \int_S A(s) g^{\hat{\gamma}}(s) ds \right)^{1/\hat{\gamma}} d\rho(g) \end{aligned}$$

converges to one, as  $n \rightarrow \infty$ .

*Proof.* By (2.11) and the positivity of  $A$

$$\hat{A}(s) = A(s)(1 + o_P(1)) \quad \text{uniformly in } s.$$

The result follows.  $\square$

**Proposition 2.3.** *For each  $\varepsilon > 0$ , the probability that*

$$\begin{aligned} \int_{\bar{C}_1^+(s)} \left( \int_S A(s) g^{\gamma - \varepsilon}(s) ds \right)^{1/(\gamma - \varepsilon)} d\hat{\rho}(g) \\ \leq \int_{\bar{C}_1^+(s)} \left( \int_S A(s) g^{\hat{\gamma}}(s) ds \right)^{1/\hat{\gamma}} d\hat{\rho}(g) \\ \leq \int_{\bar{C}_1^+(s)} \left( \int_S A(s) g^{\gamma + \varepsilon}(s) ds \right)^{1/(\gamma + \varepsilon)} d\hat{\rho}(g) \end{aligned}$$

converges to one, as  $n \rightarrow \infty$ .

*Proof.* Follows from  $\hat{\gamma} \xrightarrow{P} \gamma$  and the fact that  $(\int_S A(s) g^\gamma(s) ds)^{1/\gamma}$  is monotone in  $\gamma$  (Proposition A.1).  $\square$

**Proposition 2.4.** *For either sign (+ or -)*

$$\begin{aligned} \int_{\bar{C}_1^+(S)} \left( \int_S A(s) g^{\gamma \pm \varepsilon}(s) ds \right)^{1/(\gamma \pm \varepsilon)} d\hat{\rho}(g) \\ \xrightarrow{P} \int_{\bar{C}_1^+(S)} \left( \int_S A(s) g^{\gamma \pm \varepsilon}(s) ds \right)^{1/(\gamma \pm \varepsilon)} d\rho(g), \end{aligned}$$

as  $n \rightarrow \infty$ .

*Proof.* Since  $(\int_S A(s) g^{\gamma \pm \varepsilon}(s) ds)^{1/(\gamma \pm \varepsilon)}$  is a continuous function of  $g$  by Proposition A.1, the convergence follows directly from the weak convergence of  $\hat{\rho}$  to  $\rho$ .  $\square$

**Proposition 2.5.** *For either sign (+ or -)*

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\bar{C}_1^+(S)} \left( \int_S A(s) g^{\gamma \pm \varepsilon}(s) ds \right)^{1/(\gamma \pm \varepsilon)} d\rho(g) \\ = \int_{\bar{C}_1^+(S)} \left( \int_S A(s) g^\gamma(s) ds \right)^{1/\gamma} d\rho(g) = \theta_\gamma. \end{aligned}$$

*Proof.* Lebesgue's theorem on dominated convergence.  $\square$

## 2.2 Estimation of the spectral measure

If the process  $X$  on  $S$  is in the domain of attraction of a max-stable process, i.e. (1.1) holds, then, with  $\xi(s) := 1/(1 - F_s(X(s)))$  for  $s \in S$ ,

$$\lim_{t \rightarrow \infty} tP(t^{-1}\xi \in E) = \nu(E), \quad (2.15)$$

for every Borel set  $E$  of  $C^+(S)$  such that  $\inf\{|f|_\infty : f \in E\} > 0$  and  $\nu(\partial E) = 0$  (Theorem 9.3.1 in de Haan and Ferreira (2006)).

Now let  $A$  be a Borel set of  $\bar{C}_1^+(S)$  and for  $r > 0$  define  $B_{r,A} := (r, \infty) \times A$ . Then  $B_{r,A} = rB_{1,A}$  and (cf. (1.3))

$$\nu(B_{r,A}) = r^{-1}\nu(B_{1,A}) = r^{-1}\rho(A).$$

Hence, if  $\rho(\partial A) = 0$ ,

$$\lim_{t \rightarrow \infty} tP(t^{-1}\xi \in B_{1,A}) = \rho(A). \quad (2.16)$$

This limit relation motivates an estimator of the spectral measure  $\rho$ . In (2.16) replace  $P$  by its empirical measure and  $t$  by  $n/k$ , with  $n$  denoting the sample

size and  $k = k(n)$  is an intermediate sequence of integers, i.e.  $k/n \rightarrow 0$  and  $k \rightarrow \infty$ , as  $n \rightarrow \infty$ . Then, the left-hand side of (2.16) reads

$$\frac{n}{k} \frac{1}{n} \sum_{i=1}^n \mathbf{1} \left\{ \frac{k}{n} |\xi_i|_\infty > 1 \text{ and } \left\{ \frac{\xi_i(s)}{|\xi_i|_\infty} \right\}_{s \in S} \in A \right\},$$

with  $\xi_i(s) := 1/(1 - F_s(X_i(s)))$ , for  $s \in S$ .

Next, replace  $\{\xi_i(s)\}_{s \in S}$  by its empirical version  $\hat{\xi}_i(s) := n/(n+1 - R(X_i(s)))$  ( $s \in S$ ), where  $R(X_i(s))$  is the rank of  $X_i(s)$  among  $(X_1(s), \dots, X_n(s))$ . Then one gets

$$\hat{\rho}(A) := \frac{1}{k} \sum_{i=1}^n \mathbf{1} \left\{ \sup_{s \in S} R(X_i(s)) > n+1-k \text{ and } \left\{ \frac{n+1 - \sup_{s \in S} R(X_i(s))}{n+1 - R(X_i(s))} \right\}_{s \in S} \in A \right\}. \quad (2.17)$$

Theorem 10.3.1 of de Haan and Ferreira (2006) now implies,

**Theorem 2.2.** *Let  $X, X_1, X_2, \dots$  be independent and identically distributed stochastic processes in  $C(S)$  and assume that their distribution is in the domain of attraction of some max-stable process in  $C(S)$ , i.e. that (1.1) holds. Then,  $\hat{\rho} \xrightarrow{P} \rho$ , in the space of finite measures on  $\bar{C}_1^+(S)$ , with  $k = k(n) \rightarrow \infty$ ,  $k/n \rightarrow 0$ , as  $n \rightarrow \infty$ .*

### 3 Application

We apply our estimator to evaluate the extreme rainfall in a low-lying flat area in the northwest of the Netherlands, indicated as North Holland (see Figure 1, total area equals to 2009.58 km<sup>2</sup>). Daily rainfall data is available at 32 monitoring stations in this area for 30-year period 1971-2000. The same data set has been employed in Buishand, de Haan and Zhou (2008) to answer the question: what is the amount of rain on one day that is exceeded once in 100 year? In other words, what is the 100-year quantile of the total rainfall in this area? Since only the fall season, September, October and November, is considered, there are 91 observations per year. Therefore, a 100-year quantile of the daily rainfall distribution corresponds to a tail probability 1/9100.

In order to model the spatial dependence, Buishand, de Haan and Zhou (2008) choose a specific max-stable process. The estimated quantile of the total rainfall averaged by the total area is 58.8 mm. We try to justify this estimation by estimating the exceedance probability above such a level. Notice that our approach allows a non-parametric approach on the spatial dependence structure.

Our analysis departs from estimating marginal extreme value index, marginal scale and shift functions. On each monitoring station, they are estimated from 1-dimensional extreme value analysis. We use the moment estimator as mentioned in Section 2. In Theorem 1.1, it is assumed that the extreme value index is constant across the area. This is confirmed from the estimated extreme value indices on the stations as shown in Figure 2. We take their average as the estimates of the constant extreme value index across the area.

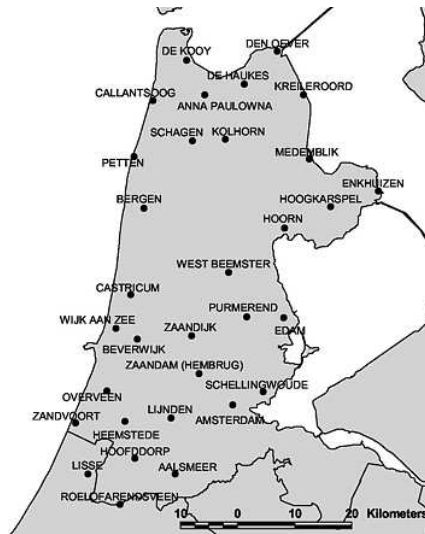


Figure 1: North Holland area with data available at 32 monitoring stations

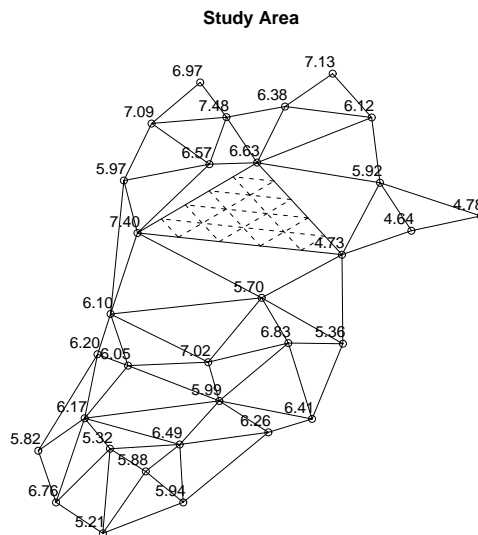


Figure 2: Estimated extreme value indices on the stations in the North Holland area

Since we only have observations on monitoring stations, it is necessary to extrapolate the estimated scale and shift functions from the stations to other points in the area. We divided the area as in Buishand, de Haan and Zhou (2008), see Figure 2. We call all the stations as Vertices, the lines connecting the stations as Edges. With the Edges, the area is divided into Triangles. We assume that the scale and shift functions are linear within each Triangles. From the estimates on the Vertices and the division of the area, one could get marginal estimates at any point in the area. Then we use the linear assumption in each Triangle to integrate the scale function, which results in  $\hat{a}(n/k)$  and thus we can calculate  $\hat{A}(s)$  as in (2.7).

Secondly, we estimate the spectral measure  $\rho$  as in (2.17). The estimate in (2.17) assumes that we have observations on each  $s \in S$ . In fact, we only have observations on the stations. We again use the linear assumption when estimating the spectral measure. Therefore, our estimated spectral measure will concentrate on functions that are linear within each Triangle. More precisely, our estimated spectral measure concentrates on functions in the set

$$D := \{f \text{ is linear within each Triangles} | f(s_j) = \frac{n+1 - \sup_{1 \leq j \leq 32} R(X_i(s_j))}{n+1 - R(X_i(s_j))}, \\ \text{for all } i \text{ s.t. } \sup_{1 \leq j \leq 32} R(X_i(s_j)) > n+1-k\}.$$

For each  $f \in D$ , we assign  $\rho(f) = 1/k$ , where  $k$  is the number of high order statistics used in the estimation.

After estimating both the marginal information and the spectral measure, we apply (2.3) to obtain the estimate on  $\theta$ . When integrating over the entire area, we use numerical integration. In order to do that we divide each Triangle into triangles as in Buishand, de Haan and Zhou (2008), see Figure 2. We call the vertices of the triangles, ‘‘vertices’’, in order to distinguish them from the Vertices. By calculating the estimates of  $\hat{A}$  and functions in  $D$  at each vertices, we can numerically integrate them to obtain  $\theta$  estimate,  $\hat{\theta}$ .

The last step is to estimate the exceedance probability for the average rainfall above 58.8 mm. We recall the tail probability estimator as in (2.2),

$$\hat{p}_n = \frac{k}{n} \hat{\theta} \left( 1 + \hat{\gamma} \frac{x_n - \int_S \hat{b}_s(\frac{n}{k}) ds}{\hat{a}(\frac{n}{k})} \right)^{-1/\hat{\gamma}}$$

where  $x_n = 58.8 * TotalArea$ . The integral of the shift function is again calculated by the linear connection of the shift function estimates on the Vertices according to the Triangles division.

The last issue in the estimation procedure is the choice of the number of upper order statistics  $k$ . We take the value of  $k$  varying from 50 to 550<sup>1</sup> and perform the above procedure for each  $k$ . We make the plot of the estimated exceedance probability against  $k$  as shown in Figure 3. From Figure 3 we choose

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<sup>1</sup>The total number of observations at each station is 2730.

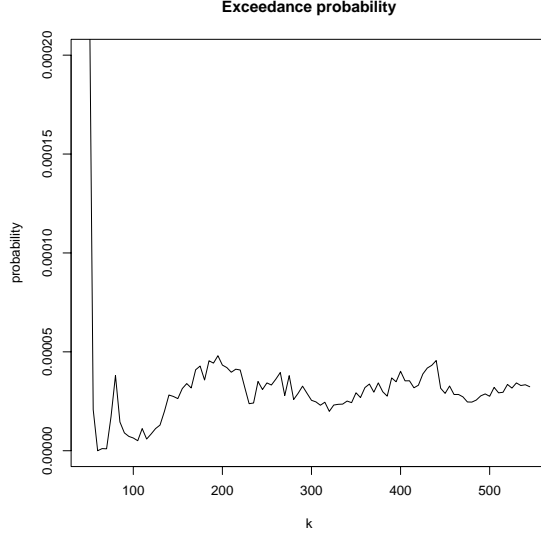


Figure 3: Estimated quantile against the number of upper order statistics

$k = 200$  to obtain the estimated probability of the average rainfall exceeding 58.8 mm as  $4.3 \times 10^{-5}$ . Compared to the tail probability of "once in 100 year",  $1/9100 = 1.1 \times 10^{-4}$ , our estimated probability is lower (around half). To have a better view, we calculate the corresponding frequency of our estimated tail probability as  $1/\hat{p}_n/91 = 253.8$  year. Hence according to our non-parametric analysis, the average rainfall exceeding 58.8 mm occurs once per 250 years.

The different results between these two approaches may owe to the fact that in the non-parametric approach we use the linearization of the functions in the spectral measure estimation, whereas Buishand, de Haan and Zhou (2008) uses specific max-stable process which allows more variation.

## A Appendix - Properties of $L_p(g)$

Define for  $p \in \bar{\mathbb{R}} =: \mathbb{R} \cup \{-\infty, +\infty\}$  and  $g \in \{f \in C(S) : f > 0, |f|_\infty = 1\}$

$$L_p(g) := \begin{cases} (\int_S g^p(s) A(s) ds)^{1/p}, & p \neq 0 \\ \exp(\int_S (\log g(s)) A(s) ds), & p = 0 \\ \sup_s g(s) = 1, & p = +\infty \\ \inf_s g(s), & p = -\infty \end{cases} \quad (\text{A.1})$$

where  $A > 0$  satisfies  $\int_S A(s) ds = 1$ .

**Proposition A.1.** *Properties:*

1.  $L_p(g)$  is continuous and non-decreasing in  $g$  for all  $p$ .

2.  $L_p(g)$  is continuous and non-decreasing in  $p$  for all  $g$ .

The proof follows from a series of lemmas.

**Lemma A.1.** Obviously  $L_p(g)$  is non-decreasing in  $g$  for all  $p$ .

**Corollary A.1.**  $0 < L_p(g) \leq 1$  for all  $p$  and  $g$ .

*Proof.*  $0 < \inf_u g(u) \leq g(s) \leq \sup_u g(u) = 1$ , for  $s \in S$ . Hence the result.  $\square$

**Lemma A.2.**  $L_p(g)$  is non-decreasing in  $p$  for  $p \in \mathbb{R}$  and  $p \neq 0$ .

*Proof.* For  $p > 0$  this is Lyapunov's inequality. For  $p < 0$  it follows also from Lyapunov's inequality (with  $g^* = 1/g$  and  $p^* = -p$ ).  $\square$

**Lemma A.3.**  $L_p(g)$  is continuous in  $p \in \mathbb{R}$  for all  $g$ .

*Proof.* We write for  $p \in \mathbb{R}$

$$L_p(g) = \left( 1 + p \left( \int_S \frac{g^p(s) - 1}{p} A(s) ds \right) \right)^{1/p}$$

where  $\int_S p^{-1} (g^p(s) - 1) A(s) ds$  should be read  $\int_S \log g(s) ds$  for  $p = 0$  and  $(1 + py)^{1/p}$  should be read  $\exp y$  for  $p = 0$ . The integral  $\int_S p^{-1} (g^p(s) - 1) A(s) ds$  is continuous for  $p \in \mathbb{R}$  by Lebesgue's theorem on dominated convergence. Obviously the function  $(1 + py)^{1/p}$  is continuous in  $p$  for  $p \in \mathbb{R}$ .  $\square$

**Lemma A.4.**  $L_p(g)$  is continuous at  $p = \pm\infty$  for all  $g$ .

*Proof.* We only prove for  $p = +\infty$ . The proof for  $p = -\infty$  is similar.

On the one hand, for all  $p \in \mathbb{R}$ ,  $L_p(g) \leq 1$ . On the other hand, for any given  $g$  and  $\varepsilon > 0$ , define

$$A_\varepsilon = \{s \in S | g(s) > 1 - \varepsilon\}.$$

Since  $\sup_s g(s) = 1$  and  $g$  is a continuous function, we get that  $\mu(A_\varepsilon) > 0$ , where  $\mu$  is the Lebesgue measure on  $S$ . Then, for  $p > 0$ ,

$$\begin{aligned} L_p(g) &= \left( \int_S g^p(s) A(s) ds \right)^{1/p} \\ &\geq \left( \int_{A_\varepsilon} g^p(s) A(s) ds \right)^{1/p} \\ &\geq (1 - \varepsilon) \left( \int_{A_\varepsilon} A(s) ds \right)^{1/p}. \end{aligned}$$

By taking  $p \rightarrow +\infty$ , we have that  $\liminf_{p \rightarrow \infty} L_p(g) \geq 1 - \varepsilon$ . By taking  $\varepsilon \rightarrow 0$  we proved that

$$\lim_{p \rightarrow +\infty} L_p(g) = 1 = L_{+\infty}(g).$$

$\square$

**Corollary A.2.**  $L_p(g)$  is non-decreasing in  $p \in \bar{\mathbb{R}}$  for all  $g$ .

**Lemma A.5.**  $L_p(g)$  is continuous in  $g$  for all  $p \in \bar{\mathbb{R}}$ .

*Proof.* If  $\lim_{n \rightarrow \infty} \sup_{s \in S} |g_n(s) - g(s)| = 0$ , for all  $\varepsilon > 0$  and sufficiently large  $n$

$$(1 - \varepsilon)g(s) \leq g_n(s) \leq (1 + \varepsilon)g(s),$$

since  $g > 0$ . □

**Remark A.1.** For  $0 < p < +\infty$ , Proposition A.1 also holds in case  $g \in \{f \in C(S) : f \geq 0, |f|_\infty = 1\}$  and  $A \geq 0$ .

**Remark A.2.** Hardy, Littlewood and Pólya (1951, Section 2.9) state about a discrete version of our  $L_p(g)$ : “ We restrict the parameters to be positive, the complications introduced by negative or zero values being hardly worth pursuing systematically”.

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